# Quantum Schur algebras and their affine and super counterparts

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  - $\mathrm{im}(\phi) = \mathrm{End}_{\mathfrak{S}_r}(T_{n,r}) = S(n,r)$, the Schur algebra, and $\mathrm{im}(\psi) = \mathrm{End}_{U(\mathfrak{gl}_n)}(T_{n,r})$;

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  - The realisation and presentation problems.

# Issai Schur – A pioneer of representation theory

**1875–1941**

28 students
$2467^{+}$ descendants

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[1] From the article *A story about father* by Hilda Abelin-Schur, in "Studies in Memory of Issai Schur", Progress in Math. 210.

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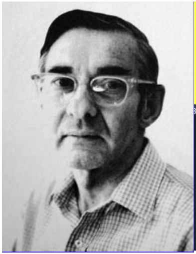
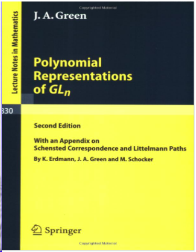
/ | ... \

Jiachen Ye, Jianpan Wang, Jie Du, Nanhua Xi

# J.A. Green and his book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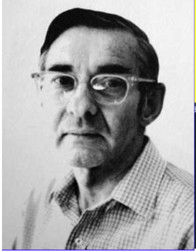

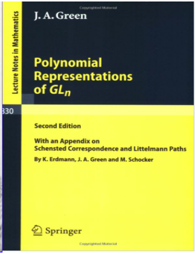


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"The pioneering achievements of Schur was one of the main inspirations for Hermann Weyl's monumental researches on the representation theory of semi-simple Lie groups. ... Weyl publicized the method of Schur's 1927 paper, with its attractive use of the 'double centraliser property', in his influential book *The Classical Groups*".

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- Representations of (f.d.) algebras.
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Drinfeld's 1986 ICM address
Drinfeld–Jimbo presentation

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(1) The Hecke algebra $\mathcal{H}$ associated with the symmetric group $\mathfrak{S}_r$ is the algebra over $\mathbb{Z}[q]$ with generators $T_i$, $i \in \{1, 2, \ldots, r-1\}$, and relations

$$T_i T_j = T_j T_i \text{ for } |i-j| > 1, \quad T_i T_j T_i = T_j T_i T_j \text{ for } |i-j| = 1,$$

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(2) The quantum linear group is the quantum enveloping algebra $\mathbf{U}_\upsilon(\mathfrak{gl}_n)$ defined over $\mathbb{Q}(\upsilon)$ with generators:

$$\mathsf{K}_a, \mathsf{K}_a^{-1}, \mathsf{E}_h, \mathsf{F}_h, \; a, h \in [1, n], h \neq n$$

and relations:

(QG1) $\mathsf{K}_a\mathsf{K}_a^{-1} = 1, \mathsf{K}_a\mathsf{K}_b = \mathsf{K}_b\mathsf{K}_a$;

(QG2) $\mathsf{K}_a\mathsf{E}_h = \upsilon^{\mathbf{e}_a \centerdot (\mathbf{e}_h - \mathbf{e}_{h+1})}\mathsf{E}_h\mathsf{K}_a, \mathsf{K}_a\mathsf{F}_h = \upsilon^{-\mathbf{e}_a \centerdot (\mathbf{e}_h - \mathbf{e}_{h+1})}\mathsf{F}_h\mathsf{K}_a$;

(QG3) $[\mathsf{E}_h, \mathsf{F}_k] = \delta_{h,k}\frac{\mathsf{K}_h\mathsf{K}_{h+1}^{-1} - \mathsf{K}_h^{-1}\mathsf{K}_{h+1}}{\upsilon_h - \upsilon_h^{-1}}$;

(QG4) $\mathsf{E}_h\mathsf{E}_k = \mathsf{E}_k\mathsf{E}_h, \mathsf{F}_h\mathsf{F}_k = \mathsf{F}_k\mathsf{F}_h$, if $|k-h| > 1$;

(QG5) $\mathsf{E}_h^2\mathsf{E}_k - (\upsilon + \upsilon^{-1})\mathsf{E}_h\mathsf{E}_k\mathsf{E}_h + \mathsf{E}_k\mathsf{E}_h^2 = 0$ and
$\mathsf{F}_h^2\mathsf{F}_k - (\upsilon + \upsilon^{-1})\mathsf{F}_h\mathsf{E}_k\mathsf{F}_h + \mathsf{F}_k\mathsf{F}_h^2 = 0$, if $|k-h| = 1$.

# Quantum Schur-Weyl duality

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A.A. Beilinson, G. Lusztig and R. MacPherson, *A geometric setting for the quantum deformation of $GL_n$*, Duke Math.J. **61** (1990), 655-677.

# Quantumization

- Let $\mathscr{P} \subseteq \mathbb{Z}$ be an infinite collection of prime powers $q = p^d$.
- For every $q \in \mathscr{P}$, suppose $A(q)$ is an algebra over $\mathbb{Z}$ with a basis $\{b_x(q)\}_{x \in X}$, where $X$ is independent of $q \in \mathscr{P}$.
- For $x, y \in X$ and $q \in \mathscr{P}$, structure constants $c_{x,y,z}(q) \in \mathbb{Z}$ for $A(q)$ are defined by $b_x(q)b_y(q) = \sum_{z \in X} c_{x,y,z}(q)b_z(q)$.
- Now assume that there exist $\phi_{x,y,z}$ in the polynomial ring over integers $\mathcal{R} := \mathbb{Z}[\mathbf{q}]$ which, upon specialization to any $q \in \mathscr{P}$, satisfy $\phi_{x,y,z}(q) = c_{x,y,z}(q)$.
- A multiplication can be defined on the free $\mathcal{R}$-module $A$ with basis $\{b_x\}_{x \in X}$ by setting, for $x, y \in X$, $b_x b_y = \sum_{z \in X} \phi_{x,y,z} b_z$ and then extending it to all of $A$ by linearity.
- The $\mathcal{R}$-algebra $A$ is called the *quantumization* of the family $\{A(q)\}_{q \in \mathscr{P}}$ of algebras.

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## Examples

Quantum Schur algebras and Ringel–Hall algebras

# Theorem (BLM, 1990)

The quantum group $\mathbf{U}_v(\mathfrak{gl}_n)$ has a basis

$$\{A(\mathbf{j}) \mid A \in M_n(\mathbb{N})^{\pm}, \mathbf{j} \in \mathbb{Z}^n\}$$

with the following multiplication rules:

(1) $\mathsf{K}_a \cdot A(\mathbf{j}) = v^{\mathrm{ro}(A).\mathbf{e}_a} A(\mathbf{j} + \mathbf{e}_a), \quad A(\mathbf{j})\mathsf{K}_a = v^{\mathrm{co}(A).\mathbf{e}_a}; A(\mathbf{j} + \mathbf{e}_a);$

(2)
$$\begin{aligned}
\mathsf{E}_h \cdot A(\mathbf{j}) &= v^{f(h+1)+j_{h+1}} \overline{[\![a_{h,h+1}+1]\!]}(A + E_{h,h+1})(\mathbf{j}) \\
&+ v^{f(h)-j_h-1} \frac{(A - E_{h+1,h})(\mathbf{j}+\alpha_h) - (A - E_{h+1,h})(\mathbf{j}+\beta_h)}{1 - v^{-2}} \\
&+ \sum_{k<h, a_{h+1,k}\geq 1} v^{f(k)} \overline{[\![a_{h,k}+1]\!]}(A + E_{h,k} - E_{h+1,k})(\mathbf{j}+\alpha_h) \\
&+ \sum_{k>h+1, a_{h+1,k}\geq 1} v^{f(k)} \overline{[\![a_{h,k}+1]\!]}(A + E_{h,k} - E_{h+1,k})(\mathbf{j});
\end{aligned}$$

(3) $\mathsf{F}_h \cdot A(\mathbf{j}) = \cdots.$

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The quantum group $\mathbf{U}_\upsilon(\mathfrak{gl}_n)$ has a basis

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with the following multiplication rules:

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(2)
$$\mathsf{E}_h \cdot A(\mathbf{j}) = \upsilon^{f(h+1)+j_{h+1}} \overline{[\![a_{h,h+1} + 1]\!]} (A + E_{h,h+1})(\mathbf{j})$$
$$+ \upsilon^{f(h)-j_h-1} \frac{(A - E_{h+1,h})(\mathbf{j} + \alpha_h) - (A - E_{h+1,h})(\mathbf{j} + \beta_h)}{1 - \upsilon^{-2}}$$
$$+ \sum_{k<h, a_{h+1,k} \geq 1} \upsilon^{f(k)} \overline{[\![a_{h,k} + 1]\!]} (A + E_{h,k} - E_{h+1,k})(\mathbf{j} + \alpha_h)$$
$$+ \sum_{k>h+1, a_{h+1,k} \geq 1} \upsilon^{f(k)} \overline{[\![a_{h,k} + 1]\!]} (A + E_{h,k} - E_{h+1,k})(\mathbf{j});$$

(3) $\mathsf{F}_h \cdot A(\mathbf{j}) = \cdots.$

Application: quantum Schur–Weyl duality at the integral level and hence, at the root-of-unity level.

# Interactions between Lie theory and reps of algebras

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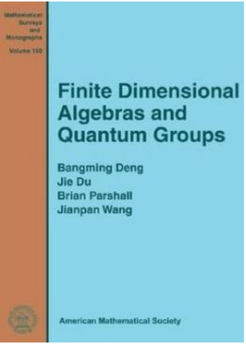
Almost at the same time, Ringel himself introduced the notion of **Ringel–Hall algebras** and proved that they are isomorphic to the $\pm$-part of the corresponding quantum groups.

# A 3-in-1 Book

## Finite dimensional algebras and quantum groups

Bangming Deng, Jie Du, Brian Parshall and Jianpan Wang
Mathematical Surveys and Monographs, Volume 150
The American Mathematical Society, 2008 (759+ pages)



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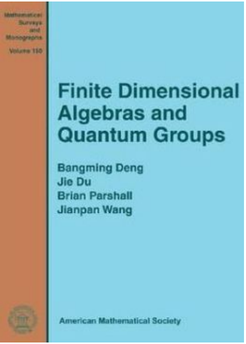


# Other types

1. J. Du, B. Parshall, and L. Scott, *Stratifying endomorphism algebras associated to Hecke algebras,* J. Algebra **203** (1998), 169–210.
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3. J. Du, B. Jensen, and X. Su, *Presenting Hecke endomorphism algebras by Hasse quivers with relations,* J. Pure Appl. Algebra (to appear).

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4. H. Bao, J. Kujawa, Y. Li and W. Wang, *Geometric Schur duality of classical type*, arXiv:1404.4000v3.
5. Z. Fan and Y. Li, *Geometric Schur Duality of Classical Type, II*, Trans. AMS, Ser. B 2 (2015), 51–92.

# 3. The affine case

Soon after BLM's work, Ginzburg and Vasserot extended to geometric approach to the affine case.

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He wrote in the introduction:
"The analogous geometrically defined algebras in the affine case are still receiving homomorphisms from quantum affine $\mathfrak{gl}_n$ with parameter $q$, but this time the homomorphisms are not surjective, contrary to what is asserted in [GV, Sec.9]."

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We would like to *algebraically* understand these works and to develop an algebraic approach like the non-affine case. In this approach, we may use these surjective maps to extend the BLM construction to the affine case.

Supported by ARC, we started the project in mid 2006. Since the aperiodicity has a natural interpretation in representations of cyclic quivers. We aim at the double Ringel–Hall algebra construction of cyclic quivers. Preliminary computations were done in 2007-8 and significant progress was made in 2009 and 2010. This resulted in a second research monograph:

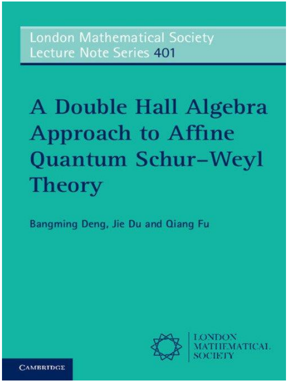
# A double Hall algebra approach to affine quantum Schur–Weyl theory

Bangming Deng, Jie Du and Qiang Fu

London Mathematical Society Lecture Note Series, Volume 401

Cambridge University Press, 2012

207+ pages



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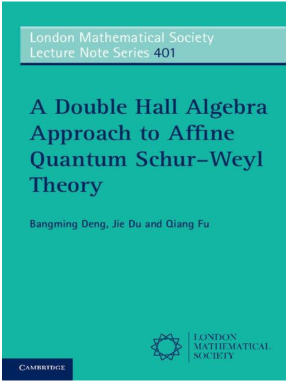


*We dedicate the book to our teachers:*

*Peter Gabriel*

*Shaoxue Liu*

*Leonard Scott*

*Jianpan Wang*

# Some conjectures in the book

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Connections with various existing works by Lusztig, Schiffmann, Varagnolo–Vasserot, Hubery, Chari–Pressley, Frenkel–Mukhin and others are also discussed throughout the book.

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Connections with various existing works by Lusztig, Schiffmann, Varagnolo–Vasserot, Hubery, Chari–Pressley, Frenkel–Mukhin and others are also discussed throughout the book.

There are several conjectures:

- The classification conjecture for simple $\mathcal{S}_{\vartriangle}(n, r)$-modules (the $n \leq r$ case); [Done in 2013 by Deng-D. and Fu]
- The realisation conjecture; [Done in 2014 by D.-Fu]
- The Lusztig form conjecture; [Done in 2014 by D.-Fu]
- The second centraliser property conjecture.

## Theorem (D.-Fu, 2013)

*The quantum loop algebra* $\mathbf{U}_\upsilon(\widehat{\mathfrak{gl}_n})$ *is the* $\mathbb{Q}(\upsilon)$*-algebra which is spanned by the basis* $\{A(\mathbf{j}) \mid A \in \Theta_\triangle^\pm(n), \mathbf{j} \in \mathbb{Z}_\triangle^n\}$ *and generated by* $0(\mathbf{j})$, $S_\alpha(\mathbf{0})$ *and* ${}^tS_\alpha(\mathbf{0})$ *for all* $\mathbf{j} \in \mathbb{Z}_\triangle^n$ *and* $\alpha \in \mathbb{N}^n$*, where* $S_\alpha = \sum_{1\le i\le n} \alpha_i E^\triangle_{i,i+1}$ *and* ${}^tS_\alpha$ *is the transpose of* $S_\alpha$*, and whose multiplication rules are given by:*

(1) $0(\mathbf{j}')A(\mathbf{j}) = \upsilon^{\mathbf{j}'\cdot ro(A)}A(\mathbf{j}'+\mathbf{j})$ *and* $A(\mathbf{j})0(\mathbf{j}') = \upsilon^{\mathbf{j}'\cdot co(A)}A(\mathbf{j}'+\mathbf{j})$.

(2) $S_\alpha(\mathbf{0})A(\mathbf{j}) =$

$$\sum_{\substack{T\in\Theta_\triangle(n)\\ ro(T)=\alpha}} \upsilon^{f_{A,T}} \overline{\prod_{\substack{1\le i\le n\\ j\in\mathbb{Z},\, j\ne i}} \left[\!\!\left[ \begin{matrix} a_{i,j}+t_{i,j}-t_{i-1,j} \\ t_{i,j} \end{matrix} \right]\!\!\right]} (A+T^\pm-\tilde{T}^\pm)(\mathbf{j}_T, \delta_T).$$

(3) ${}^tS_\alpha(\mathbf{0})A(\mathbf{j}) =$

$$\sum_{\substack{T\in\Theta_\triangle(n)\\ ro(T)=\alpha}} \upsilon^{f'_{A,T}} \overline{\prod_{\substack{1\le i\le n\\ j\in\mathbb{Z},\, j\ne i}} \left[\!\!\left[ \begin{matrix} a_{i,j}-t_{i,j}+t_{i-1,j} \\ t_{i-1,j} \end{matrix} \right]\!\!\right]} (A-T^\pm+\tilde{T}^\pm)(\mathbf{j}'_T, \delta_{\tilde{T}}).$$

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## 4. The super case

There is a super version of Wedderburn's Theorem: A finite dimensional simple superalgebras (i.e., a $\mathbb{Z}_2$-graded algebra) over $\mathbb{C}$ is isomorphic to

- either a (full) matrix superalgebra $\mathcal{M} = M_{n+m}(\mathbb{C})$ with

$$\mathcal{M}_{\bar{0}} = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \middle| \begin{array}{l} A \in M_n(\mathbb{C}) \\ B \in M_m(\mathbb{C}) \end{array} \right\}, \ \mathcal{M}_{\bar{1}} = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \middle| \begin{array}{l} C \in M_{n,m}(\mathbb{C}) \\ D \in M_{m,n}(\mathbb{C}) \end{array} \right\},$$

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- or a queer (or strange) matrix superalgebra

$$\mathcal{Q} = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in M_n(\mathbb{C}) \right\}$$

with $\mathcal{Q}_{\bar{0}} = \left\{ \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right) \right\}$ and $\mathcal{Q}_{\bar{1}} = \left\{ \left(\begin{smallmatrix} 0 & B \\ B & 0 \end{smallmatrix}\right) \right\}$.

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Equipped a superalgebra $\mathcal{A}$ with the super commutator defined by $[x, y] := xy - (-1)^{\hat{x}\cdot\hat{y}} yx$, where $x, y \in \mathcal{A}$ are homogeneous elements and $\hat{z} = i$ if $z \in \mathcal{A}_i$, the two series simple superalgebras $\mathcal{M}$ and $\mathcal{Q}$ give rise to, respectively, two series Lie superalgebras: $\mathfrak{gl}_{n|m}$, the **general linear Lie superalgebra**, and $\mathfrak{q}_n$, the **queer Lie superalgebra**.

## Quantum Schur superalgebras

- If $V$ denotes the natural representation of $\mathfrak{gl}_{n|m}$ (resp., $\mathfrak{q}_n$), then the tensor product $V^{\otimes r}$ is a representation of the universal enveloping algebra $U(\mathfrak{gl}_{n|m})$ (resp., $U(\mathfrak{q}_n)$). The image $S(n|m,r)$ (resp., $Q(n,r)$) of $U(\mathfrak{gl}_{n|m})$ (resp., $U(\mathfrak{q}_n)$) in $\mathrm{End}(V^{\otimes r})$ is called the **Schur superalgebra**, known as of type $M$, (resp. **queer Schur superalgebra**, known as of type Q).

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- Their quantum analogs $U_\upsilon(\mathfrak{gl}_{n|m})$, $U_\upsilon(\mathfrak{q}_n)$ and $S_\upsilon(n|m,r)$, $Q_\upsilon(n,r)$ are called respectively the quantum linear supergroup, the quantum queer supergroup, a quantum Schur superalgebra and a queer quantum Schur superalgebra (or a quantum Schur superalgebras of type $Q$).

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THANK YOU!