Quantum Schur algebras and their affine and super counterparts

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via
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1. Introduction—the Schur–Weyl Duality

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- By permuting the tensor factors, the symmetric group \( \mathfrak{S}_r \) in \( r \) letter acts on \( T_{n,r} \). This action commutes with the action of \( U(\mathfrak{gl}_n) \), giving \( T_{n,r} \) a bimodule structure.
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  - The realisation and presentation problems.
Issai Schur – A pioneer of representation theory

1875–1941

28 students
2467+ descendants

"I feel like I am somehow moving through outer space. A particular idea leads me to a nearby star on which I decide to land. Upon my arrival, I realize that somebody already lives there. Am I disappointed? Of course not. The inhabitant and I are cordially welcoming each other, and we are happy about our common discovery."

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From the article *A story about father* by Hilda Abelin-Schur, in “Studies in Memory of Issai Schur”, Progress in Math. 210.
Mathematics Genealogy Project
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Ferdinand G. Frobenius

|                |

Issai Schur

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Richard Brauer
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Shih-Hua Tsao
(Xi-hua Cao)

Jiachen Ye, Jianpan Wang, Jie Du, Nanhua Xi
J.A. Green and his book

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“The pioneering achievements of Schur was one of the main inspirations for Hermann Weyl’s monumental researches on the representation theory of semi-simple Lie groups. ... Weyl publicized the method of Schur’s 1927 paper, with its attractive use of the ‘double centraliser property’, in his influential book *The Classical Groups*.”
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- Lie algebras and algebraic groups
  - Resolution of Kazhdan–Lusztig conjecture;
  - Lusztig conjecture (large $p$ proof, counterexamples).

- Coxeter groups and Hecke algebras (canonical bases ...).
- Deligne–Lusztig's work on characters of finite groups of Lie type (character sheaves ...).
- Representations of (f.d.) algebras.
  - Gabriel's theorem and its generalisation by Donovan–Freislich, Dlab–Ringel;
  - Kac's generalization to infinite types.
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Examples

(1) The Hecke algebra $H$ associated with the symmetric group $S_r$ is the algebra over $\mathbb{Z}[[q]]$ with generators $T_i$, $i \in \{1, 2, \ldots, r-1\}$, and relations $T_i T_j = T_j T_i$ for $|i - j| > 1$, $T_i T_j T_i = T_j T_i T_j$ for $|i - j| = 1$, and $T_2^i = (q-1)T_i + q$.

(2) The quantum linear group is the quantum enveloping algebra $U_\hbar(\mathfrak{gl}_n)$ defined over $\mathbb{Q}(\hbar)$ with generators: $K_a$, $K_{-1}a$, $E_h$, $F_h$, $a, h \in [1, n]$, $h \neq n$ and relations:

(QG1) $K_a K_{-1}a = 1$, $K_a K_b = K_b K_a$;

(QG2) $K_a E_h = \hbar e^a (e^h - e^h + 1) E_h K_a$, $K_a F_h = (\hbar - e^a) (e^h - e^h + 1) F_h K_a$;

(QG3) $[E_h, F_k] = \delta_{hk} K_h K_{-1}h + 1 - K_{-1}h K_h + \hbar (\hbar - 1) K_h$;

(QG4) $E_h E_k = E_k E_h$, $F_h F_k = F_k F_h$, if $|k - h| > 1$;

(QG5) $E_2^i E_k - (\hbar + \hbar - 1) E_h E_k E^i + E_k E_2^i = 0$ and $F_2^i F_k - (\hbar + \hbar - 1) F_h E_k F^i + F_k F_2^i = 0$, if $|k - h| = 1$. 


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(QG4) $E_h E_k = E_k E_h, F_h F_k = F_k F_h$, if $|k - h| > 1$;

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Quantum Schur-Weyl duality

The introduction of quantum groups also lifts the Schur–Weyl duality to the quantum level.

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Quantumization

- Let $\mathcal{P} \subseteq \mathbb{Z}$ be an infinite collection of prime powers $q = p^d$.
- For every $q \in \mathcal{P}$, suppose $A(q)$ is an algebra over $\mathbb{Z}$ with a basis $\{b_x(q)\}_{x \in X}$, where $X$ is independent of $q \in \mathcal{P}$.
- For $x, y \in X$ and $q \in \mathcal{P}$, structure constants $c_{x,y,z}(q) \in \mathbb{Z}$ for $A(q)$ are defined by $b_x(q)b_y(q) = \sum_{z \in X} c_{x,y,z}(q)b_z(q)$.
- Now assume that there exist $\phi_{x,y,z}$ in the polynomial ring over integers $\mathcal{R} := \mathbb{Z}[q]$ which, upon specialization to any $q \in \mathcal{P}$, satisfy $\phi_{x,y,z}(q) = c_{x,y,z}(q)$.
- A multiplication can be defined on the free $\mathcal{R}$-module $A$ with basis $\{b_x\}_{x \in X}$ by setting, for $x, y \in X$, $b_x b_y = \sum_{z \in X} \phi_{x,y,z} b_z$ and then extending it to all of $A$ by linearity.
- The $\mathcal{R}$-algebra $A$ is called the quantumization of the family $\{A(q)\}_{q \in \mathcal{P}}$ of algebras.
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Examples

Quantum Schur algebras and Ringel–Hall algebras
Theorem (BLM, 1990)

The quantum group $U_\nu(gl_n)$ has a basis

$$\{A(j) \mid A \in M_n(\mathbb{N})^\pm, j \in \mathbb{Z}^n\}$$

with the following multiplication rules:

1. $K_a \cdot A(j) = \nu^{ro(A) \cdot e_a} A(j + e_a)$, $A(j)K_a = \nu^{co(A) \cdot e_a}; A(j + e_a)$;

2. $E_h \cdot A(j) = \nu^{f(h+1)+j_{h+1}[a_{h,h+1} + 1]} (A + E_{h,h+1})(j)$
   $$+ \nu^{f(h)-j_{h-1}} \frac{(A - E_{h+1,h})(j + \alpha_h) - (A - E_{h+1,h})(j + \beta_h)}{1 - \nu^{-2}}$$
   $$+ \sum_{k < h, a_{h+1,k} \geq 1} \nu^{f(k)[a_{h,k} + 1]} (A + E_{h,k} - E_{h+1,k})(j + \alpha_h)$$
   $$+ \sum_{k > h+1, a_{h+1,k} \geq 1} \nu^{f(k)[a_{h,k} + 1]} (A + E_{h,k} - E_{h+1,k})(j);$$

3. $F_h \cdot A(j) = \cdots$. 

Application: quantum Schur–Weyl duality at the integral level and hence, at the root-of-unity level.
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Interactions between Lie theory and reps of algebras

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Almost at the same time, Ringel himself introduced the notion of Ringel–Hall algebras and proved that they are isomorphic to the ±-part of the corresponding quantum groups.
A 3-in-1 Book

Finite dimensional algebras and quantum groups

Bangming Deng, Jie Du, Brian Parshall and Jianpan Wang
Mathematical Surveys and Monographs, Volume 150
The American Mathematical Society, 2008 (759+ pages)
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3. The affine case

Soon after BLM’s work, Ginzburg and Vasserot extended to geometric approach to the affine case.


However, this paper made a wrong statement which was pointed out by Lusztig.


He wrote in the introduction: "The analogous geometrically defined algebras in the affine case are still receiving homomorphisms from quantum affine $gl_n$ with parameter $q$, but this time the homomorphisms are not surjective, contrary to what is asserted in [GV, Sec.9]."
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Lusztig used the aperiodicity of quantum affine $\mathfrak{sl}_n$ to show that it is impossible to map the quantum loop algebra of $\mathfrak{sl}_n$ onto the affine quantum Schur algebras.

However, Vasserot did give a detailed proof for the surjective map from the quantum loop algebra of $\mathfrak{gl}_n$ onto the affine quantum Schur algebras, but didn't point out the wrong statement in their previous paper.


We would like to algebraically understand these works and to develop an algebraic approach like the non-affine case. In this approach, we may use these surjective maps to extend the BLM construction to the affine case.

Supported by ARC, we started the project in mid 2006. Since the aperiodicity has a natural interpretation in representations of cyclic quivers. We aim at the double Ringel–Hall algebra construction of cyclic quivers. Preliminary computations were done in 2007-8 and significant progress was made in 2009 and 2010. This resulted in a second research monograph:
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Supported by ARC, we started the project in mid 2006. Since the aperiodicity has a natural interpretation in representations of cyclic quivers. We aim at the double Ringel–Hall algebra construction of cyclic quivers. Preliminary computations were done in 2007-8 and significant progress was made in 2009 and 2010. This resulted in a second research monograph:
A double Hall algebra approach to affine quantum Schur–Weyl theory

Bangming Deng, Jie Du and Qiang Fu
London Mathematical Society Lecture Note Series, Volume 401
Cambridge University Press, 2012
207+ pages

We dedicate the book to our teachers:
Peter Gabriel
Shaoxue Liu
Leonard Scott
Jianpan Wang
Some conjectures in the book

Connections with various existing works by Lusztig, Schiffmann, Varagnolo–Vasserot, Hubery, Chari–Pressley, Frenkel–Mukhin and others are also discussed throughout the book.

There are several conjectures:

▶ The classification conjecture for simple $S △ (n, r)$-modules (the $n ≤ r$ case); [Done in 2013 by Deng-D. and Fu]

▶ The realisation conjecture; [Done in 2014 by D.-Fu]

▶ The Lusztig form conjecture; [Done in 2014 by D.-Fu]

▶ The second centraliser property conjecture.
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There are several conjectures:

- The classification conjecture for simple $S_\triangle(n, r)$-modules (the $n \leq r$ case); [Done in 2013 by Deng-D. and Fu]
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- The Lusztig form conjecture; [Done in 2014 by D.-Fu]
- The second centraliser property conjecture.
Theorem (D.-Fu, 2013)

The quantum loop algebra $U_\nu(\hat{\mathfrak{gl}}_n)$ is the $\mathbb{Q}(\nu)$-algebra which is spanned by the basis $\{ A(j) \mid A \in \Theta_\Delta^\pm(n), j \in \mathbb{Z}_\Delta^n \}$ and generated by $0(j)$, $S_\alpha(0)$ and $tS_\alpha(0)$ for all $j \in \mathbb{Z}_\Delta^n$ and $\alpha \in \mathbb{N}^n$, where $S_\alpha = \sum_{1 \leq i \leq n} \alpha_i E_{i,i+1}^\Delta$ and $tS_\alpha$ is the transpose of $S_\alpha$, and whose multiplication rules are given by:

1. $0(j')A(j) = \nu^{j' \cdot \text{ro}(A)} A(j' + j)$ and $A(j)0(j') = \nu^{j' \cdot \text{co}(A)} A(j' + j)$.

2. $S_\alpha(0)A(j) =$

\[
\sum_{T \in \Theta_\Delta(n) \atop \text{ro}(T) = \alpha} \nu^{f_{A,T}} \prod_{1 \leq i \leq n \atop j \in \mathbb{Z}, j \neq i} \begin{bmatrix} a_{i,j} + t_{i,j} - t_{i-1,j} \\ t_{i,j} \end{bmatrix} (A + T^\pm - \tilde{T}^\pm)(j_T, \delta_T).
\]

3. $tS_\alpha(0)A(j) =$

\[
\sum_{T \in \Theta_\Delta(n) \atop \text{ro}(T) = \alpha} \nu^{f'_{A,T}} \prod_{1 \leq i \leq n \atop j \in \mathbb{Z}, j \neq i} \begin{bmatrix} a_{i,j} - t_{i,j} + t_{i-1,j} \\ t_{i-1,j} \end{bmatrix} (A - T^\pm + \tilde{T}^\pm)(j'_T, \delta_{\tilde{T}}).
\]
References

- J. Du and Q. Fu, *The integral quantum loop algebra of $\mathfrak{gl}_n$*, preprint.
References

- J. Du and Q. Fu, *The integral quantum loop algebra of \( \mathfrak{gl}_n \)*, preprint.
4. The super case

There is a super version of Wedderburn’s Theorem: A finite dimensional simple superalgebras (i.e., a \( \mathbb{Z}_2 \)-graded algebra) over \( \mathbb{C} \) is isomorphic to

- either a (full) matrix superalgebra \( \mathcal{M} = M_{n+m}(\mathbb{C}) \) with

  \[
  \mathcal{M}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_n(\mathbb{C}), B \in M_m(\mathbb{C}) \right\}, \quad \mathcal{M}_1 = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \in M_{n,m}(\mathbb{C}), D \in M_{m,n}(\mathbb{C}) \right\},
  \]

- or a queer (or strange) matrix superalgebra \( Q = \{ (A B) \mid A, B \in M_{n}(\mathbb{C}) \} \) with

  \[
  Q_0 = \{ (A 0) \mid A \in M_n(\mathbb{C}) \}, \quad Q_1 = \{ (0 B) \mid B \in M_m(\mathbb{C}) \}.
  \]
4. The super case

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  \]

- or a queer (or strange) matrix superalgebra
  
  \[
  Q = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in M_n(\mathbb{C}) \right\}
  \]

  with $Q_0 = \{ (A^0_0) \}$ and $Q_{\tilde{1}} = \{ (0_B) \}$. 
There is a super version of Wedderburn’s Theorem: A finite dimensional simple superalgebras (i.e., a $\mathbb{Z}_2$-graded algebra) over $\mathbb{C}$ is isomorphic to

- either a (full) matrix superalgebra $\mathcal{M} = M_{n+m}(\mathbb{C})$ with $\mathcal{M}_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} \mid A \in M_n(\mathbb{C}), B \in M_m(\mathbb{C}) \right\}$, $\mathcal{M}_\bar{1} = \left\{ \begin{pmatrix} 0 & C \\ D & 0 \end{pmatrix} \mid C \in M_{n,m}(\mathbb{C}), D \in M_{m,n}(\mathbb{C}) \right\}$,

- or a queer (or strange) matrix superalgebra $Q = \left\{ \begin{pmatrix} A & B \\ B & A \end{pmatrix} \mid A, B \in M_n(\mathbb{C}) \right\}$

with $Q_0 = \left\{ \begin{pmatrix} A \bar{0} \\ 0 \bar{A} \end{pmatrix} \right\}$ and $Q_\bar{1} = \left\{ \begin{pmatrix} 0 \bar{B} \\ B \bar{0} \end{pmatrix} \right\}$.

Equipped a superalgebra $\mathcal{A}$ with the super commutator defined by $[x, y] := xy - (-1)^{\hat{x}\cdot\hat{y}}yx$, where $x, y \in \mathcal{A}$ are homogeneous elements and $\hat{z} = i$ if $z \in \mathcal{A}_i$, the two series simple superalgebras $\mathcal{M}$ and $Q$ give rise to, respectively, two series Lie superalgebras: $\mathfrak{gl}_{n|m}$, the general linear Lie superalgebra, and $\mathfrak{q}_n$, the queer Lie superalgebra.
Quantum Schur superalgebras

If $V$ denotes the natural representation of $\mathfrak{gl}_{n|m}$ (resp., $q_n$), then the tensor product $V \otimes r$ is a representation of the universal enveloping algebra $U(\mathfrak{gl}_{n|m})$ (resp., $U(q_n)$). The image $S(n|m, r)$ (resp., $Q(n, r)$) of $U(\mathfrak{gl}_{n|m})$ (resp., $U(q_n)$) in $\text{End}(V \otimes r)$ is called the Schur superalgebra, known as of type $M$, (resp. queer Schur superalgebra, known as of type $Q$).
Quantum Schur superalgebras

- If \( V \) denotes the natural representation of \( \mathfrak{gl}_{n|m} \) (resp., \( q_n \)), then the tensor product \( V \otimes r \) is a representation of the universal enveloping algebra \( U(\mathfrak{gl}_{n|m}) \) (resp., \( U(q_n) \)). The image \( S(n|m, r) \) (resp., \( Q(n, r) \)) of \( U(\mathfrak{gl}_{n|m}) \) (resp., \( U(q_n) \)) in \( \text{End}(V \otimes r) \) is called the Schur superalgebra, known as of type \( M \), (resp. queer Schur superalgebra, known as of type \( Q \)).

- Their quantum analogs \( U_\psi(\mathfrak{gl}_{n|m}) \), \( U_\psi(q_n) \) and \( S_\psi(n|m, r) \), \( Q_\psi(n, r) \) are called respectively the quantum linear supergroup, the quantum queer supergroup, a quantum Schur superalgebra and a queer quantum Schur superalgebra (or a quantum Schur superalgebras of type \( Q \)).
References

References


THANK YOU!