> Twists and braids for general threefold flops

If a complex surface contains a (-2)-curve, this curve corresponds to a spherical object in the derived category of coherent sheaves on the surface. For certain arrangements of such curves, Seidel and Thomas used these objects to establish a braid group action on the derived category. I explain joint work with Michael Wemyss giving a generalisation to curves on threefolds: this uses braid-type groups associated to hyperplane arrangements, and relative spherical objects over noncommutative base rings.

Twists & braids for gen^{l} 3-fold flops.

T = alg. surface with ADE sing, type Γ

$\uparrow$ min. resolution

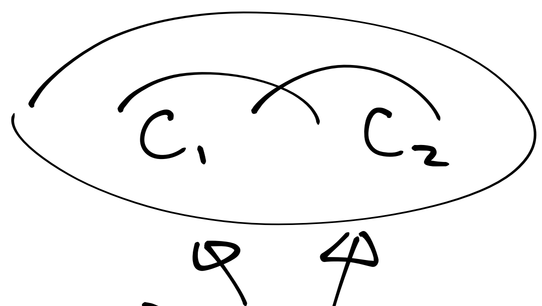
S

$$\boxed{Br_\Gamma \hookrightarrow D(S) \text{ faithfully via spherical twists}}$$

[Seidel–Thomas '01, Bridgeland '09, Brav–Thomas '11]

eg $T = (xy - z^3)$ $\quad \Gamma = A_2$

S



(-2)-curves

Br_Γ = 3-strand braid group

$$\begin{tikzcd} Y \\ X \arrow[u, "f"] \end{tikzcd}$$

Y = 3-fold with Gorenstein terminal sing.

f flopping contraction, type Γ'

$$\boxed{\text{braid-type group } G_{\Gamma'} \curvearrowright D(X) \text{ via noncommutative spherical twists}}$$

[D-Wemyss '15]

Derived category $D(S)$, smooth S

objects : complexes $(\dots \to E^{-1} \to E^{0} \to E^{1} \to \dots)$
vector bundles

$$\boxed{\begin{array}{l} \iota^{p} : \mathrm{Coh}(S) \hookrightarrow D(S) \\ \quad F \longmapsto (\underbrace{\dots \to E^{p-1} \to E^{p}} \to 0 \to \dots) \\ \text{resolution of } F, \text{ i.e. } \mathbb{H}^{j} = \begin{cases} F & j = p \\ 0 & \text{otherwise} \end{cases} \end{array}}$$

morphisms : chain maps $/ \sim$

s.t. different resolutions $\rightsquigarrow$ isomorphic obj

Def F coherent sheaf on S surface,

F **spherical** if $\left\{\begin{array}{l} \hom(F,F) = 1 \\ \operatorname{ext}^1(F,F) = 0 \\ \operatorname{ext}^2(F,F) = 1 \end{array}\right\}$ and $F \otimes \Omega^2 \cong F$

$$\boxed{\begin{array}{l} \textbf{Thm} \quad F \text{ sph} \implies \exists \text{ sph twist } T \in \operatorname{Aut} D(S) \text{ determined by} \\ \mathbb{R}\mathrm{Hom}(F,E) \overset{\mathbb{L}}{\otimes} F \to E \to T(E) \xrightarrow{[1]} \end{array}}$$

eg $F = \mathcal{O}_C$, for $C \cong \mathbb{P}^1$, $\mathcal{N}_C \cong \mathcal{O}_{\mathbb{P}^1}(-2)$

Thm ① F sph $\Longrightarrow$ $\exists$ sph twist $T \in \operatorname{Aut} D(S)$ determined by

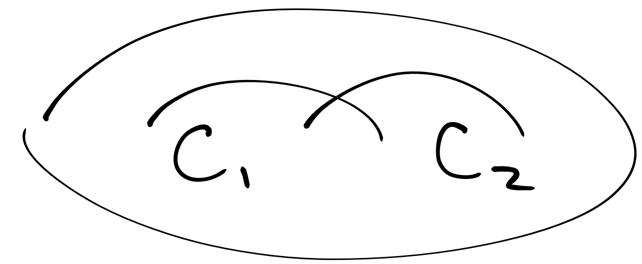
$$\mathbb{R}\mathrm{Hom}(F,E) \overset{\mathbb{L}}{\otimes} F \to E \to T(E) \xrightarrow{[1]}$$

② (F_i) sph, **A_n-chain** $\Longrightarrow$

$$\mathrm{Br}_{n+1} \hookrightarrow \operatorname{Aut} D(S)$$

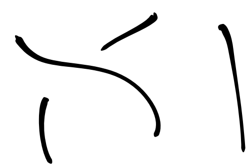
gen $\sigma_i \longmapsto T_i$

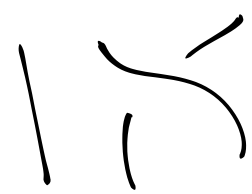
Rem $\operatorname{Aut} S \not\subseteq \operatorname{Aut} D(S)$

Rem cf symplectic Dehn twists, under mirror symm eg $F_i = \mathcal{O}_{C_i}$ S 

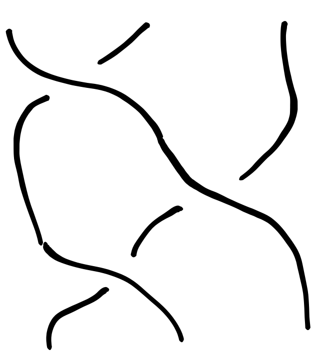
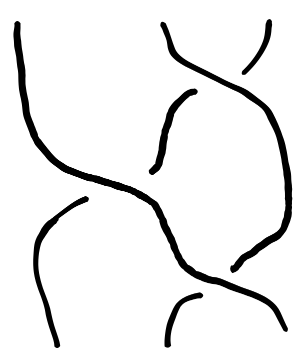
$$Br_3 \hookrightarrow \mathrm{Aut}\, D(S)$$

gens

 $\longmapsto T_1$

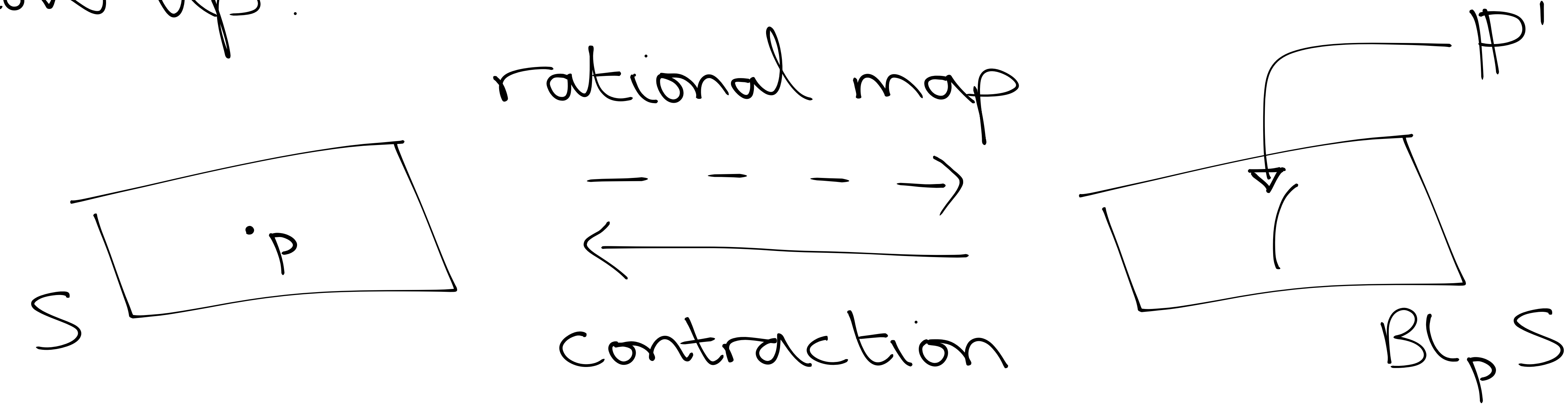
 $\longmapsto T_2$

rels $\rightsquigarrow$ iso of functors

 $\simeq$ 

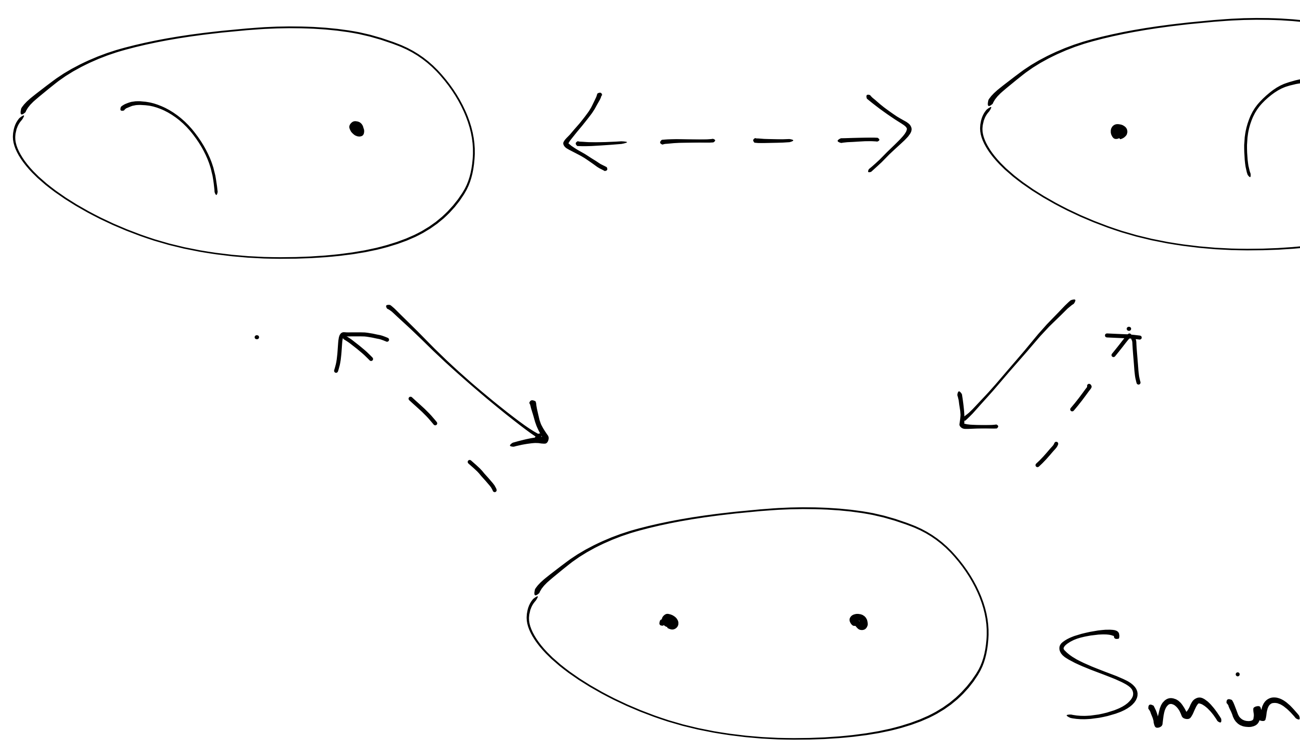
$$T_1 T_2 T_1 \cong T_2 T_1 T_2$$

Blow up:



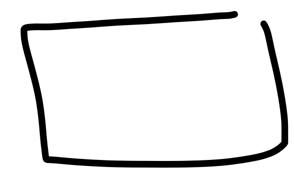
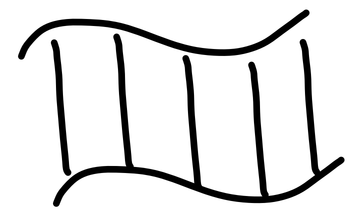
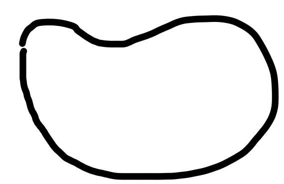
Q: understand surfaces / rat maps

eg



Thm S smooth proj alg surface

$\exists$ contraction $S \to S' =$

1. $\mathbb{P}^2$ 
2. or ruled surface 
3. or <u>unique</u> min model 

Def S min model $\iff K_S \cdot C \geq 0$ $\forall$ curves C

dim 2 $\exists$ unique min models

dim 3 Y sing 3-fold : **multiple** min resolutions related by ...

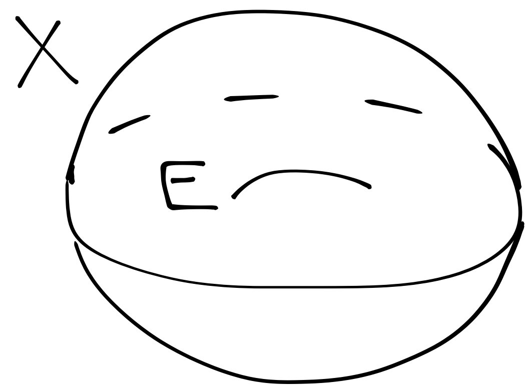
flop

$$\begin{tikzcd} X \arrow[rr, dashed, "\psi"] \arrow[dr, "f"'] & & X' \arrow[dl, "f'"] \\ & Y & \end{tikzcd}$$

indeterminacy locus of ψ
= exceptional locus of f
= curve E

eg Atiyah flop

$E = (-1,-1)$-curve



$E \cong \mathbb{P}^1$

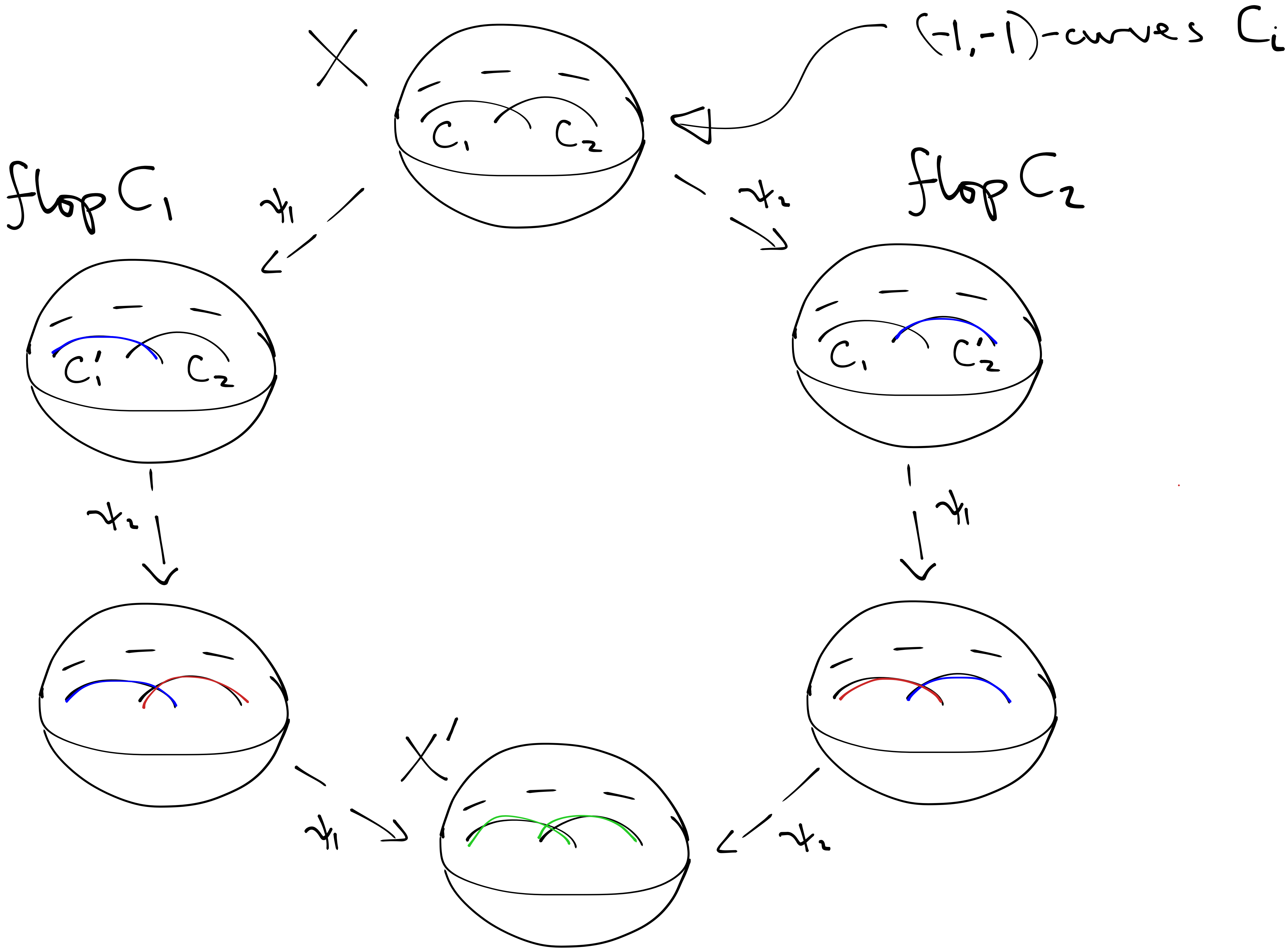
$\mathcal{N}_E \cong \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus 2}$

Thm [Bridgeland, Chen]

flop ψ, Gorenstein terminal

$\exists$ equivalence $D(X) \xrightarrow{\Psi} D(X')$

eg 1 X generic 1-param deformation of S



Thm [Reid] $f: X \to Y$ flopping contraction
$E \mapsto p$

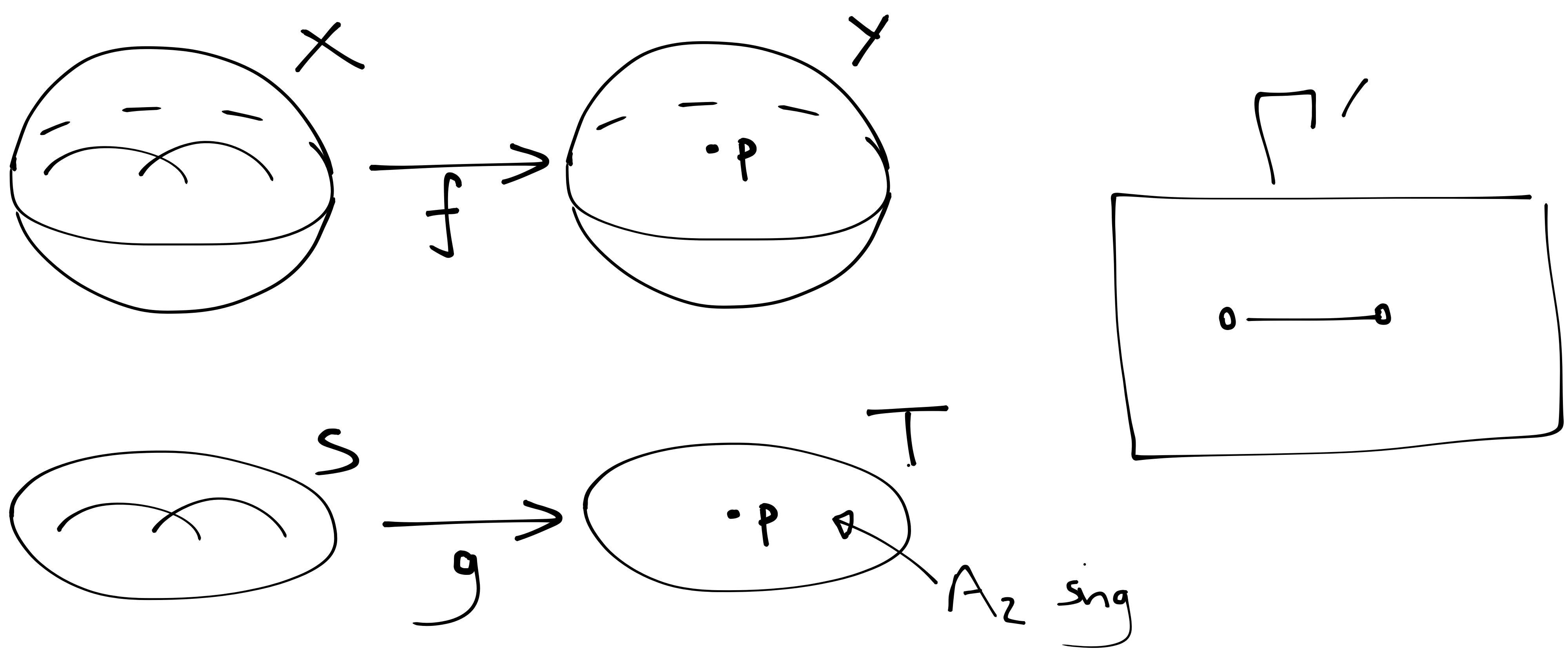
$$\begin{tikzcd} X \arrow[r, "f"] & Y \\ S \arrow[u, hook] \arrow[r, "g"] & T \arrow[u, hook] \end{tikzcd}$$

base change $g: S \to T$ generic surface $\ni p$

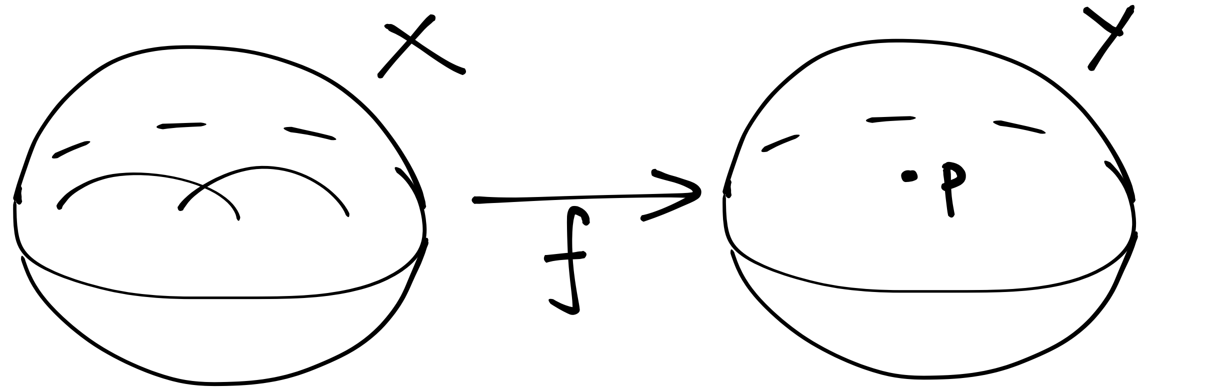
is partial resolution of ADE sing at p

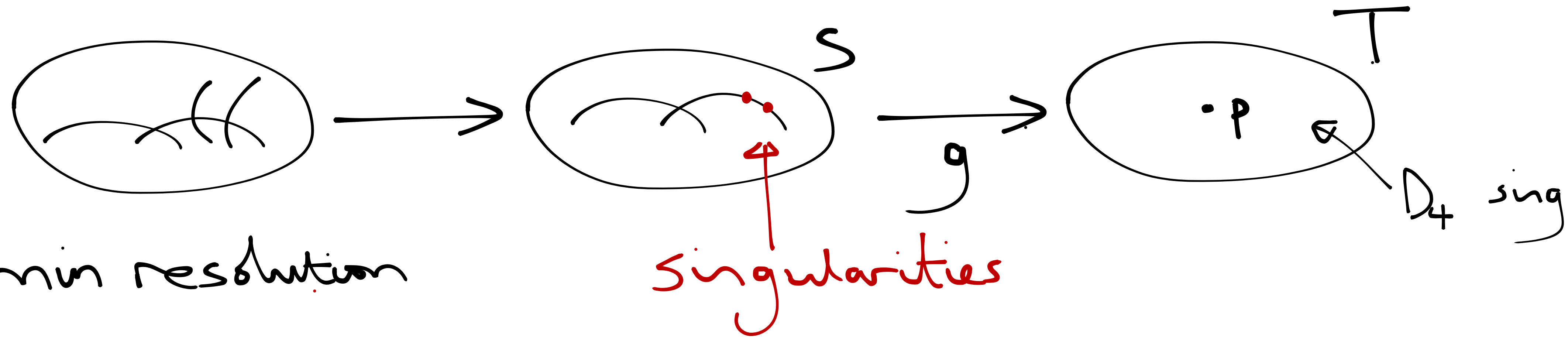
$\xleftrightarrow[1:1]{}$ marked Dynkin diagram Γ'

eg 1

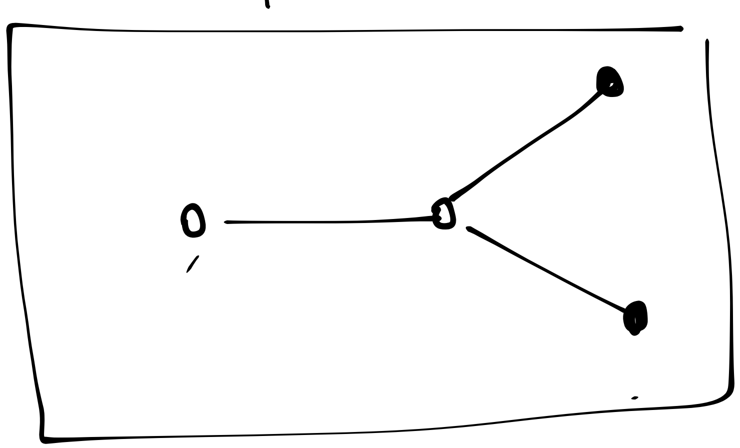


eg 2 compound D_4





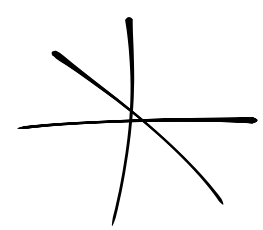
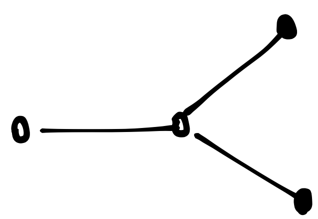
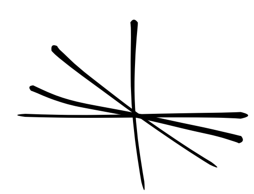
Γ'



Def $V = \langle \text{marked roots} \rangle_{\mathbb{C}} \subset \text{Cartan } \mathfrak{h}_{\Gamma}$ ← dim = # cpts E

Def $V^\circ = V - \text{root vector h'planes}$

Thm [80s] $\pi_0(V^\circ_{\mathbb{R}}) = \{\text{min. models } X_\tau \longleftrightarrow X\}$

eg	Γ'	V°	$V^\circ_{\mathbb{R}}$	# min models
1		$\mathbb{C}^2 - 3L$		6
2		$\mathbb{C}^2 - 4L$		8

$X \xrightarrow{f} Y$ flopping contraction

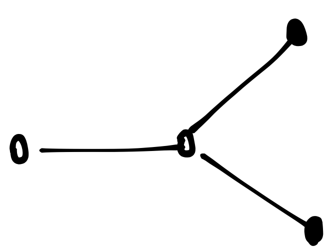
$E \longmapsto p$ normal, Gor term, $\mathbb{R}f_*\mathcal{O} = \mathcal{O}$

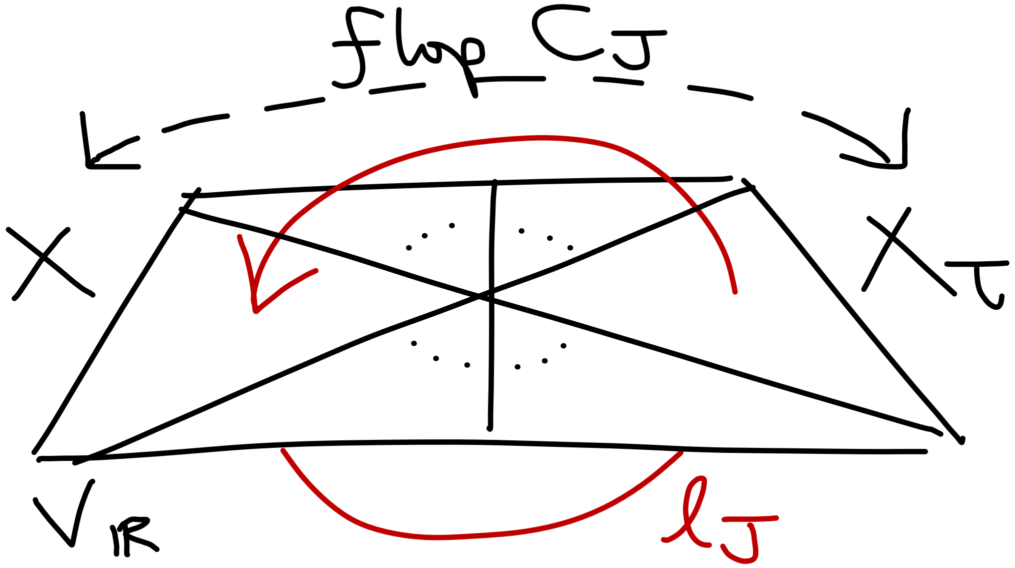
<u>Def</u> Write $E = \bigcup_{1\ldots n} C_j$, $C_j \cong \mathbb{P}^1$

Take $C_J \subseteq E$ for $J \subseteq \{1 \ldots n\}$

<u>Thm</u> If C_J flop algebraically $\forall J$ then

[D-Wemyss] $\exists\, \Omega : \pi_1(\mathcal{V}^0) \longrightarrow \operatorname{Aut} D(X)$

eg	Γ'	$\mathcal{V}^0$	$\pi_1(\mathcal{V}^0)$
1		$\mathbb{C}^2 - 3L$	$PBr_3 = \ker(Br_3 \to S_3)$
2		$\mathbb{C}^2 - 4L$	braid-type group



$$\begin{tikzcd} D(X) \arrow[r, bend left, "\Psi"] & D(X_\tau) \arrow[l, bend left, "\Psi'"] \end{tikzcd}$$

$$\Omega : \ell_J \longmapsto \Psi' \circ \Psi \cong T = \text{twist by } {}_A\mathcal{F}$$

$$A = \operatorname{Def}(\mathcal{O}_{C_j}(-1))$$ augmented $\mathbb{C}^{|J|}$-alg of deformations [Landal, Eriksen ...]

$$\mathcal{F} \in \operatorname{mod-} A \otimes \mathcal{O}_X$$ univ family

$\mathcal{F} \in \operatorname{mod-} A \otimes \mathcal{O}_X$ univ family

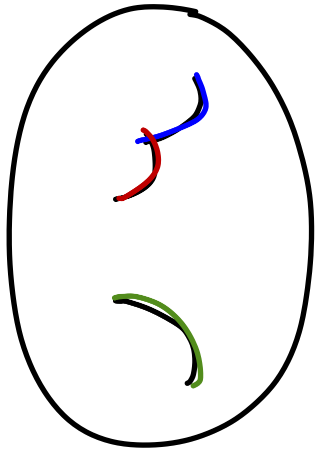
T = twist by ${}_A\mathcal{F}$, determined by

$$\mathbb{R}\mathrm{Hom}(\mathcal{F}, E) \overset{\mathbb{L}}{\underset{A}{\otimes}} \mathcal{F} \longrightarrow E \longrightarrow T(E) \xrightarrow{[1]}$$

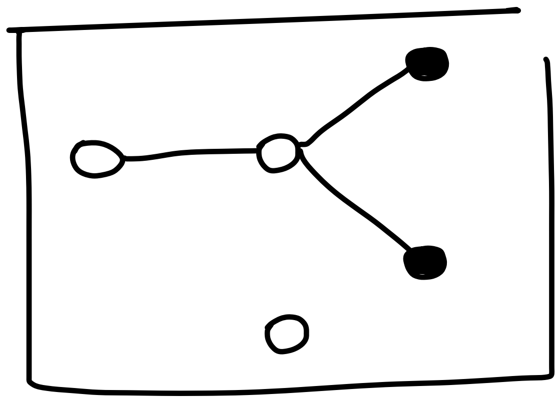
<u>Thm</u> [Hirano–Wemyss] Action is faithful

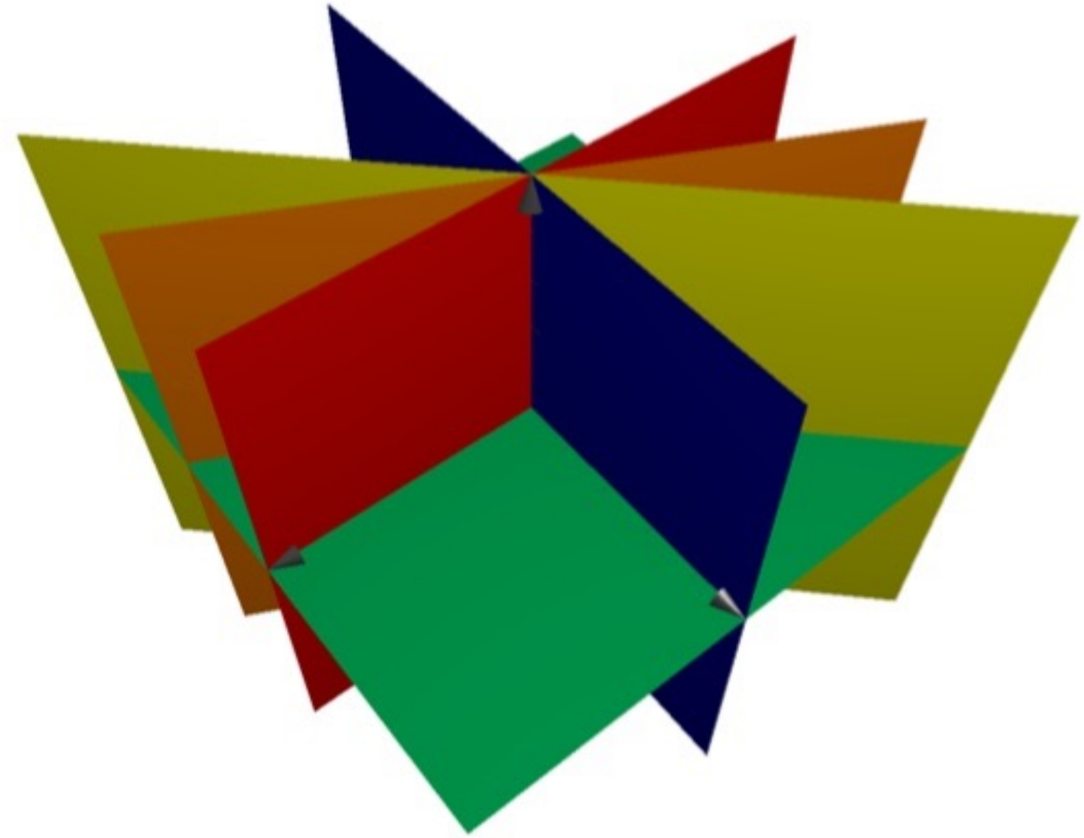
C_J	T	A
$(-1,-1)$ curve	Seidel–Thomas	$\mathbb{C}$
$(-2,0)$ curve	Toda	$\mathbb{C}[\varepsilon]/\varepsilon^k$
otherwise		noncomm

eg $V^\circ = \mathbb{C}^3 - 5H$

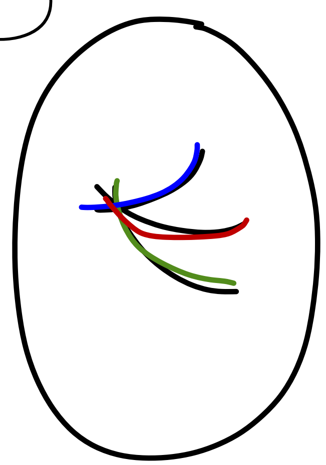


braid rel^{n} order 4.

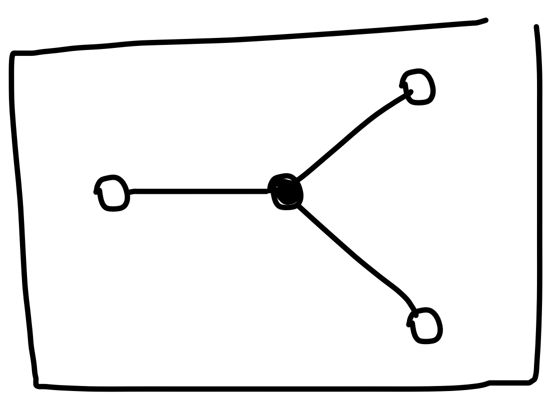


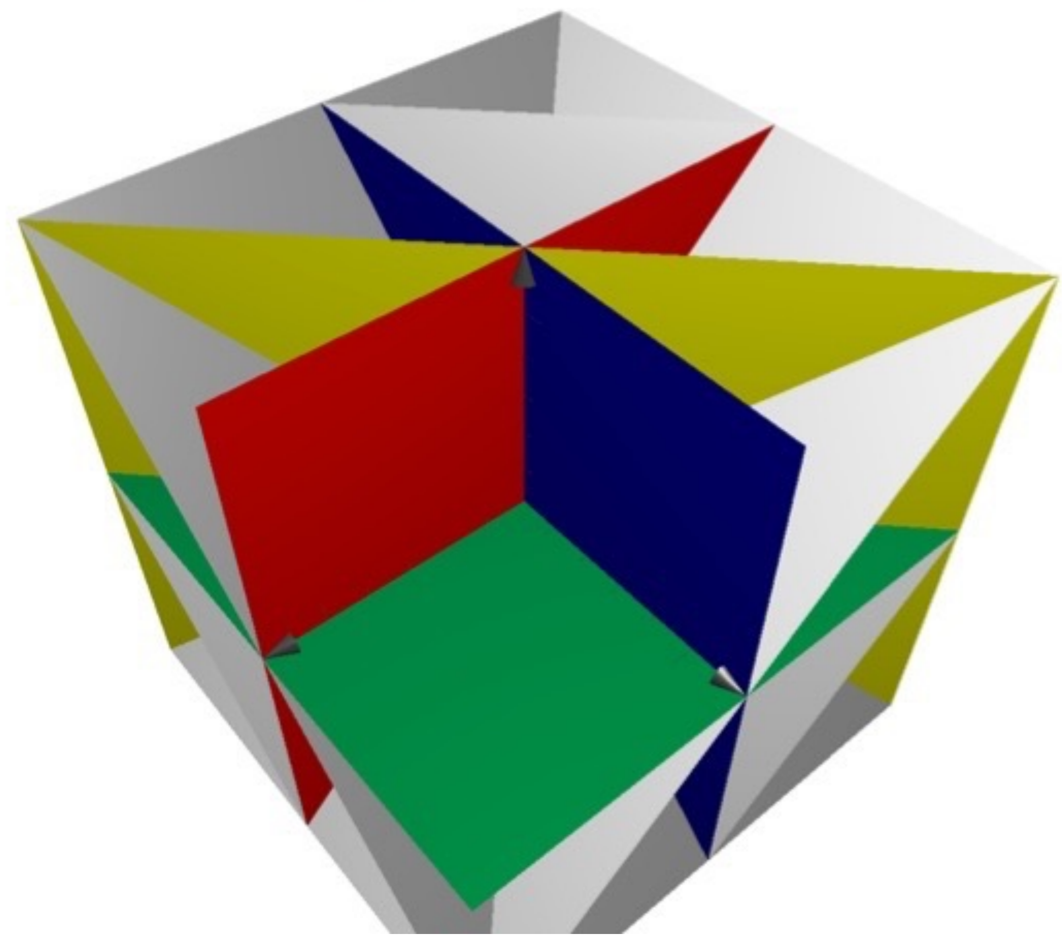


eg $V^\circ = \mathbb{C}^3 - 7H$



pairwise braid rel^{n}s order 3.





<u>Rem</u> simplicial h'plane arr. $\Longrightarrow V^\circ$ is $K(\pi, 1)$

THANK YOU