Kac-Moody Root Systems and M-theory.

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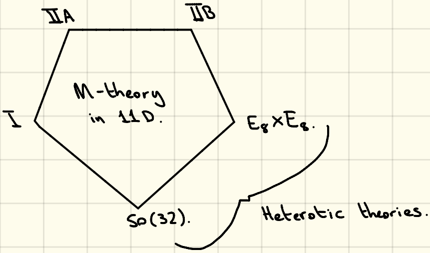
Some motivating comments.

What is M-theory?

String theories in 10 D are unified by U-duality.



[Witten '95].

The low energy description of M-theory is the maximal supergravity theory in 11D, whose bosonic part is:

$$\mathcal{L} = \underbrace{R * 1}_{\text{E.H. term.}} - \underbrace{\tfrac{1}{2} F_4 \wedge * F_4}_{\text{K.E. for a 3-form gauge field, } A_3.} + \underbrace{\tfrac{1}{6} F_4 \wedge F_4 \wedge A_3}_{\text{Self-interaction term. "Chern-Simons term".}} \qquad \text{where } F_4 = 4\, dA_3.$$

Comments:

1. it describes the degrees of freedom of an elf-bein, $e_\mu{}^a$, and a 3-form gauge field $A_{\mu_1\mu_2\mu_3}$.

   Comparison with Maxwell's electromagnetic field A_μ which sources a point charge will lead you to surmise correctly that $A_{\mu_1\mu_2\mu_3}$ sources a membrane (the M2-brane).

2. its equations of motion are

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R - \tfrac{1}{2}\tfrac{1}{3!} F_{\mu\mu_1\mu_2\mu_3} F_\nu{}^{\mu_1\mu_2\mu_3} + \tfrac{1}{2}\tfrac{1}{4!} g_{\mu\nu} F_{\mu_1\mu_2\mu_3\mu_4} F^{\mu_1\mu_2\mu_3\mu_4} = 0$$

$$\partial^\mu F_{\mu\mu_1\mu_2\mu_3} + \tfrac{1}{2} \varepsilon_{\mu_1\mu_2\mu_3\mu_4\cdots\mu_{11}} F^{\mu_4\mu_5\mu_6\mu_7} F^{\mu_8\mu_9\mu_{10}\mu_{11}} = 0.$$

3. It has simple solutions: the pp-wave, the M2-brane, the M5-brane and the KK6-brane.

4. M-theory must agree with SuGra at low energies but must also include all the string excitations in the 10D string theories.

Which Kac-Moody algebras are relevant to M-theory?

The two pioneering arguments:

1. Dimensional Reduction. (See the lectures on Kaluza-Klein theory by Chris Pope)

The rough idea: make one of the spatial dimensions compact, typically S^1, and let the cycle shrink to small volume. Excitations of the circle (for example) are standing waves satisfying $n\lambda = 2\pi R$ where R is the radius of the circle, λ the wavelength of the wave and $n \in \mathbb{Z}$. Such waves have energy

$$E = \hbar\nu = \hbar\left(\tfrac{1}{\lambda}\right) = \frac{n\hbar}{2\pi R} \qquad \text{hence} \quad R \to 0 \Rightarrow E \to \infty \quad \text{if} \quad n \neq 0.$$

So small radii imply very high-energy excitations (too high to have been reached in our colliders). Hence one may neglect the impact of the small compact coordinate in the low energy theory. In practise the effect on the field content of the theory is to neglect the compact index e.g. consider the metric being reduced in one-dimension (call it x^5) from 5D to 4D:

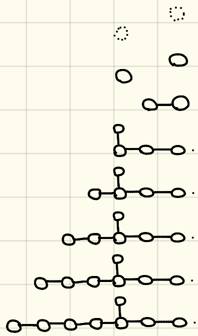
$$g_{\mu\nu} \longrightarrow g_{mn}\,, \quad g_{m5} \equiv A_m \quad \text{and} \quad g_{55} \equiv \phi.$$

From a theory of gravity in 5D emerges a theory of gravity, a vector (electromagnetism) and a scalar. This was the original observation of Kaluza in '21 and Klein in '26 made to propose a 5D unification of gravity and electromagnetism. The scalar was an unwanted extra, but it is the scalars appearing in the dimensional reduction of SuGra that give the first motivation for a Kac-Moody algebra in M-theory.

As one reduces the 11D SuGra the scalars that appear in D=10, 9, 8 ... have the symmetries in the Lagrangian of a coset $G/K(G)$ of greater complexity as the reduction descends to fewer dimensions:

D.	$G/K(G)$.
10.	$\mathbb{R}/1$.
9.	$(SL(2,\mathbb{R})\times\mathbb{R})/SO(2)$
8.	$SL(3,\mathbb{R})\times SL(2,\mathbb{R})/SO(3)\times SO(2)$
7.	$SL(5,\mathbb{R})/SO(5)$.
6.	$SO(5,5)/SO(5)\times SO(5)$.
5.	$E_6/USp(8)$.
4.	$E_7/SU(8)$.
3.	$E_8/SO(16)$.

Dynkin diagram for G.



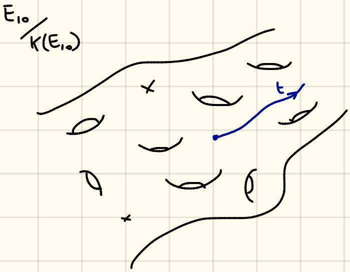
The early observation of Julia was that these hidden symmetries ought to continue and he argued for E_9 and E_{10} in $D=2$ and $D=1$. [Julia '78, '80, '85]

The extension would be that E_{11} should appear in $D=0$. Peter West argued that E_{11} was a symmetry of an extension of SuGra in 2001, N.B. E_{11} should be a symmetry of M-theory in 11D.

E_9, E_{10} and E_{11} are all Kac-Moody algebras.

2. Cosmological Billiards.

At the start of the millenium a different line of investigation led Damour, Henneaux and Nicolai to see the fingerprints of E_{10} within 11D SuGra. They were considering the physics in the vicinity of a cosmological singularity where they allowed the SuGra fields to depend on only time, t. The greatly simplified equations of motion had a solution which was identical to null-geodesic motion of a coset of E_{10}: $E_{10}/K(E_{10})$.



$\longrightarrow$ Spacetime solution $g_{\mu\nu}(t)$, $A_{\mu_1\mu_2\mu_3}(t)$ near a cosmological singularity.

Later Englert and Houart extended this picture to restore space and time to an equal footing and their construction was called the brane σ-model and the coset symmetry was enlarged to $E_{11}/K(E_{11})$.

This is the setting we will work in for this talk and our aims are two-fold (time-permitting):

1. To construct the root system of E_{11}.

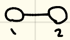
2. To use the brane σ-model to build solutions of M-theory.

Root Systems.

Given a Dynkin diagram (or equivalently a Cartan matrix) a set of generators in the algebra are singled out: those collections $\{H_i, E_i, F_i\}$ which form the $sl(2)$ algebras for each node.

Each generator E_i is associated with a simple positive root vector encoded in the definition of the Cartan matrix by $[H_i, E_j] = A_{ij} E_j = 2\frac{\langle\alpha_i, \alpha_j\rangle}{\langle\alpha_i, \alpha_i\rangle} E_j$. The Cartan matrix contains the inner products of the simple positive roots which allows one to geometrise the root system.

The simplest nontrivial root system belongs to $SL(3,\mathbb{R})$ whose Dynkin diagram is:



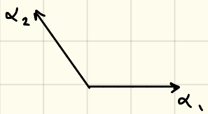
$$A = \begin{pmatrix} 2. & -1. \\ -1. & 2. \end{pmatrix}$$

The simple positive roots have inner product

$$\langle\alpha_1, \alpha_2\rangle = \tfrac{1}{2}(\alpha_1)^2 A_{12} = -\tfrac{1}{2}(\alpha_1)^2 \equiv -1$$

where in the last step we normalised $(\alpha_1)^2 = (\alpha_2)^2 = 2$ for simplicity.

Hence $\langle\alpha_1, \alpha_2\rangle = |\alpha_1||\alpha_2|\cos\theta_{12} = 2\cos\theta_{12} = -1 \implies \theta_{12} = {}^{2\pi}/_{3}$.



There exists a root β for every generator E_β in the algebra defined with respect to the Cartan sub-algebra and the simple positive roots (for the case where $(\alpha_i)^2 = 2 \ \forall i$)

$$[H_i, E_\beta] = \langle\alpha_i, \beta\rangle E_\beta \qquad \forall\, i.$$

So, if we know the algebra we can construct all the roots and likewise if we know all the roots we can construct the algebra.

It is frequently simpler to work with the root system:

- Root vectors add while in the algebra one must commute matrices:

Suppose that $[E_{\alpha_1}, E_{\alpha_2}] = E_{\alpha_3}$ in some algebra then

$$
\begin{aligned}
[H_i, E_{\alpha_3}] &= [H_i, [E_{\alpha_1}, E_{\alpha_2}]] \\
&= -[E_{\alpha_1}, [E_{\alpha_2}, H_i]] - [E_{\alpha_2}, [H_i, E_{\alpha_1}]] \qquad \text{(by the Jacobi identity).} \\
&= \langle \alpha_i, \alpha_2 \rangle [E_{\alpha_1}, E_{\alpha_2}] + \langle \alpha_i, \alpha_1 \rangle [E_{\alpha_1}, E_{\alpha_2}] \\
&= \langle \alpha_i, \alpha_1 + \alpha_2 \rangle E_{\alpha_3}.
\end{aligned}
$$

$$\Rightarrow \alpha_3 = \alpha_1 + \alpha_2.$$

- For Kac-Moody algebras the Matrix representations will (without a great inspiration) be of infinite rank, while the roots will (for a finite Dynkin diagram) reside in a finite-dimensional vector space.

Before completing the root system for $SL(3, \mathbb{R})$ let us introduce the defining relations for a Kac-Moody algebra where the simple roots all have $(\alpha_i)^2 = 2$:

A Very Brief Introduction to Kac-Moody Algebras.

Given an appropriate Cartan matrix A_{ij} a Kac-Moody algebra is formed of (Chevalley) generators E_i, F_i and H_i such that $\forall\ i,j$

$$[H_i, H_j] = 0, \quad [H_i, E_j] = \langle\alpha_i, \alpha_j\rangle E_j, \quad [H_i, F_j] = -\langle\alpha_i, \alpha_j\rangle F_j, \quad [E_i, F_j] = \delta_{ij} H_j$$

and the Serre relations:

$$\left.\begin{array}{l} [E_i,[E_i, \dots [E_i, E_j]\dots]] = 0 \\ [F_i,[F_i, \dots [F_i, F_j]\dots]] = 0. \end{array}\right\} \text{ where there are } (1 - A_{ij}) \text{ commutators.}$$

Comments.

1. If $\det(A_{ij}) > 0$ the above relations define a finite Lie algebra, if $\det(A_{ij}) \leqslant 0$ then the algebra is a Kac-Moody algebra.

2. The Serre relations guarantee that the adjoint representation is irreducible.

The Serre relations are worth exploring in detail as they will give a simple route to construct the root system of a Kac-Moody algebra.

The Serre Relations and Root Systems.

Recalling that we have limited our focus to root systems where simple roots all have the same length-squared (normalised to 2) [Such algebras are called simply-laced].

There are three distinct entries in the Cartan matrix:

	A_{ij}	$1-A_{ij}$.	Serre Relation.	
$(i=j)$.	2.	-1.	$[E_i, E_i] = 0$.	
	0.	1.	$[E_i, E_j] = 0$.	
	-1.	2.	$[E_i,[E_i,E_j]] = 0$.	$\Rightarrow [E_i,E_j] = E_{i+j}$

Starting from the simple roots α_i the Serre relations tell us that $\alpha_i + \alpha_j$ is a root if $\langle \alpha_i, \alpha_j \rangle = -1$. In this case we observe that

$$(\alpha_i+\alpha_j)^2 = (\alpha_i)^2 + 2\langle \alpha_i,\alpha_j\rangle + (\alpha_j)^2$$
$$= 2 - 2 + 2$$
$$= 2.$$

Consider now adding a third simple root to obtain $\alpha_i + \alpha_j + \alpha_k$. This is a root if the commutator

$[E_k,[E_i,E_j]] = E_{i+j+k}$ is not trivial.

By the Jacobi identity we have:

$$[E_k,[E_i,E_j]] = -[E_i,[E_j,E_k]] - [E_j,[E_k,E_i]]$$

The right-hand-side is non-trivial if $\langle \alpha_j, \alpha_k \rangle = -1$ or $\langle \alpha_k, \alpha_i \rangle = -1$ or even both are true.

This means that if $\alpha_i + \alpha_j + \alpha_k$ is a root then

$$(\alpha_i+\alpha_j+\alpha_k)^2 = (\alpha_i)^2 + (\alpha_j)^2 + (\alpha_k)^2 + 2\langle\alpha_i,\alpha_j\rangle + 2\langle\alpha_i,\alpha_k\rangle + 2\langle\alpha_j,\alpha_k\rangle$$

$$= \begin{cases} 2 & \text{if only one of } \langle\alpha_i,\alpha_k\rangle \text{ or } \langle\alpha_j,\alpha_k\rangle \text{ equals } -1. \\ 0 & \text{if } \langle\alpha_i,\alpha_k\rangle = -1 = \langle\alpha_j,\alpha_k\rangle = -1. \end{cases}$$

This line of argument can be generalised to non-simple roots so that we can say that the roots of any simply-laced Dynkin diagram satisfy $\beta^2 = 2, 0, -2, -4, -6 \ldots$

This is almost, but not quite, a sufficient algebraic condition to find all roots of E_{11}.

Let us return to $SL(3,\mathbb{R})$ and complete the construction of its root system.

Recall we have that $(\alpha_1)^2 = (\alpha_2)^2 = 2$ and $\langle\alpha_1,\alpha_2\rangle = -1$ and now we wish to find all $\beta = n\alpha_1 + m\alpha_2$ where $n, m \in \mathbb{Z}$ such that $\beta^2 = 2, 0, -2, \ldots$

So $\beta^2 = 2n^2 + 2m^2 - 2nm = 2(n+m)^2 - 6nm$

For $n=0$ we have $2m^2 \leq 2 \Rightarrow m = \pm 1$.

$n=1$ we have $2 + 2m^2 - 2m \leq 2$

$\Rightarrow m^2 - m \leq 0 \Rightarrow m=0$ or $m=1$.

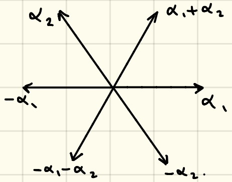
$n=2$ we have $8 + 2m^2 - 8m \leq 2$.

$\Rightarrow m^2 - 4m + 6 \leq 0$.

$(m-2)^2 + 2 \leq 0. \Rightarrow$ no solutions.

We also have that $(-\beta)^2 \leq 2$.

Hence the root system of $SL(3,\mathbb{R})$ is



At this point we may realise that we might have employed the Weyl reflections (reflections in the planes perpendicular to the roots) starting from just the simple positive roots to construct the root system. The reason being that reflections preserve inner products and the inner products contain all the information in the root system.

For semisimple Lie algebras the root systems are finite, i.e. there are finite solutions to $(\beta)^2 \leq 2$, this is because there are only roots of positive length-squared.

For affine Kac-Moody algebras there exists a root of length-squared zero (a null root) whose inner product with the simple positive roots α_i is zero i.e.

$$\langle \delta, \delta \rangle = 0 \quad \text{and} \quad \langle \alpha_i, \delta \rangle = 0.$$

So that one can construct an infinite set of roots of the form $\alpha_i + n\delta$ $n \in \mathbb{Z}$ as

$$(\alpha_i + n\delta)^2 = (\alpha_i)^2 + 2n\langle \delta, \alpha_i \rangle + n^2 \langle \delta, \delta \rangle = 2.$$

For general Kac-Moody algebras there are roots of negative length-squared too (imaginary roots) and the root systems are also infinite, and of faster growth than the affine case.

In passing through the example of $SL(3,\mathbb{R})$ it is useful to highlight that a canonical matrix representation of the generators exists which has a simple extension to all $SL(N,\mathbb{R})$:

$$H_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv K^1{}_2 \qquad F_1 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv K^2{}_1$$

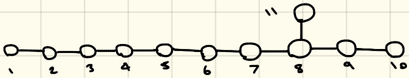
$$H_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \qquad E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \equiv K^2{}_3 \qquad F_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \equiv K^3{}_2$$

$$E_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \equiv K^1{}_3 \qquad F_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \equiv K^3{}_1 .$$

For $SL(N,\mathbb{R})$ algebras the matrices $K^i{}_j$, which are $N \times N$ matrices with a 1 at row i, column j, but is otherwise filled with zeroes, give a representation of the $\frac{N}{2}(N-1)$ positive generators and their transposes represent the negative generators.

Let us now graduate to the main example of this talk: E_{11}.

The Roots of E_{11}.

Dynkin diagram: 

The problem: find all coefficients $m_i \in \mathbb{Z}$ such that for $\beta \equiv \sum_{i=1}^{11} m_i \alpha_i$, $\beta^2 \leq 2$.

We will solve the problem in a way that allows us to grade the roots and collect them into highest weight representations of the $SL(11,\mathbb{R})$ sub-algebra formed from nodes 1 to 10 above.

To do this we note that it is straightforward to split β into a root in $SL(11,\mathbb{R})$ and a multiple of α_{11}:

$$\beta = \sum_{i=1}^{10} m_i \alpha_i + m_{11}\alpha_{11} = \sum_{i=1}^{10} m_i \alpha_i + m_{11}(\gamma - \lambda_8) = m_{11}\gamma - m_{11}\lambda_8 + \sum_{i=1}^{10} m_i \alpha_i = m_{11}\gamma - \Lambda_\beta$$

Where $\alpha_{11} = \gamma - \lambda_8$ where λ_i are fundamental weights of $SL(11,\mathbb{R})$ satisfying $\langle \lambda_i, \alpha_j \rangle \equiv \delta_{ij}$ and γ is orthogonal to the roots of $SL(11,\mathbb{R})$. Hence $\langle \alpha_{11}, \alpha_8 \rangle = \langle \gamma - \lambda_8, \alpha_8 \rangle = -1$ as required.

Λ_β is a weight of $SL(11,\mathbb{R})$ associated to β labelled by the unique highest weight.

The root $\alpha_{11} = \gamma - \lambda_8$ is associated with the highest weight of the $SL(11,\mathbb{R})$ tensor representation with three antisymmetric indices, i.e. $E_{\alpha_{11}} = R^{91011}$. The addition of $SL(11,\mathbb{R})$ roots lowers the index labels, e.g. $E_{\alpha_{11}+\alpha_8} = R^{81011}$ (recall that $E_{\alpha_8} = K^8{}_9$ and $[\underset{\alpha_8}{K^8{}_9}, \underset{\alpha_{11}}{R^{91011}}] = \underset{\alpha_8+\alpha_{11}}{R^{81011}}$)

Fortunately it is not necessary to already have an intimate knowledge of the algebra to identify the $SL(11,\mathbb{R})$ tensor irreps that appear in the decomposition of E_{11} into highest weights of $SL(11,\mathbb{R})$ and a level m_{11}. Instead by making a choice of basis for α_i we can have a quick way to read off the highest weight of $SL(11,\mathbb{R})$. We choose:

$\alpha_i = e_i - e_{i+1}$ for $i = 1$ to 10 (compare with $K^i{}_{i+1}$ the corresponding generator of $SL(11,\mathbb{R})$)

$\alpha_{11} = e_9 + e_{10} + e_{11}$ (compare with $E_{\alpha_{11}} = R^{91011}$).

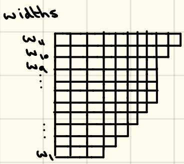
Then we have $\beta = \sum_{i=1}^{11} m_i \alpha_i = \sum_{i=1}^{11} w_i e_i$

where for roots corresponding to the highest weight of Λ_β w_i are the widths of the $SL(11,\mathbb{R})$ Young tableau, the sign of the e_i coefficient indicates where the corresponding index is covariant (-ve) or contravariant (+ve),

e.g. given a simple root $\alpha_1 = e_1 - e_2$ we may read of the tensor $K^1{}_2$

or $\alpha_{11} = e_9 + e_{10} + e_{11}$ we read off R^{91011}.

and if β is a highest weight under the $SL(11,\mathbb{R})$ action then $\beta \equiv \sum_{i=1}^{11} w_i e_i$ the Young tableau is



$R^{a_1 \ldots a_{11} | b_1 \ldots b_{11} | c_1 \ldots c_{11} | d_1 \cdots d_{10} | e_1 \ldots e_9 | \ldots | i_1 i_2 | j_1}$.

We are able to pick such an embedding into $\mathbb{R}^{11}$ so long as we can define an inner product on $\mathbb{R}^{11}$ such that inner products of E_{11}'s simple roots are reproduced.

This is achieved by

$$\langle \beta_1, \beta_2 \rangle = \sum_{i=1}^{11} w_i^{(1)} w_i^{(2)} - \frac{1}{9} \sum_{i=1}^{11} w_i^{(1)} \sum_{j=1}^{11} w_j^{(2)}.$$

where $\beta_1 \equiv \sum_{i=1}^{11} w_i^{(1)} e_i$ and $\beta_2 \equiv \sum_{i=1}^{11} w_i^{(2)} e_i$. The $-\frac{1}{9}$ comes from $(\alpha_{11})^2 = 2$.

Note that $\sum_{i=1}^{11} w_i$ = the number of boxes on the corresponding Young tableau $\equiv \#_\beta$.

Furthermore at level $L \equiv m_{11}$ each generator is formed from L commutators of $R^{M_1 M_2 M_3}$ so

$\#_\beta = 3L$, hence,

$$\langle \beta_1, \beta_2 \rangle = \sum_{i=1}^{11} w_i^{(1)} w_i^{(2)} - L^{(1)} L^{(2)}.$$

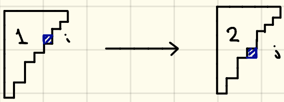
and importantly for our construction of the root space:

$$\beta^2 = \sum_{i=1}^{11} (w_i)^2 - L^2$$

where L is the level of β.

To restate our problem in the context of Young tableaux: at level L we aim to find Young tableaux formed of $3L$ boxes satisfying $\beta^2 \leq 2$.

We will need a helpful trick: moving a box on a Young tableau one column to the left reduces β^2 by 2.



$w_i \rightarrow w_i - 1$

$w_j \rightarrow w_j + 1$

such that $w_i = w_j + 2$.

$$\begin{aligned} \therefore \beta_2^2 &= (w_1)^2 + \dots (w_i - 1)^2 + \dots + (w_j + 1)^2 + \dots + (w_{11})^2 - L^2 \\ &= \sum_{i=1}^{11} (w_i)^2 - L^2 - 2w_i + 2w_j + 2 \\ &= \beta_1^2 - 2. \end{aligned}$$

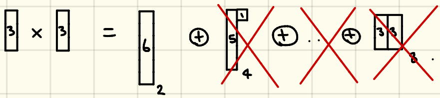
Let us construct the roots of E_{11} at each level L as Young Tableau:

Level

0 $\qquad \alpha_i, \ i=1,\dots 10 \qquad K^i{}_j$

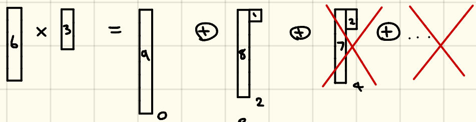
1  $\qquad R^{\mu_1\mu_2\mu_3}$.

β_3 ↖ Length-squared.

2. 

β_6

× ruled out by root length (Serre relations).

3. 

$\beta_{8,1}$

We can complete the prescription for finding all the roots by noting that $\delta = e_3 + \dots + e_{11}$

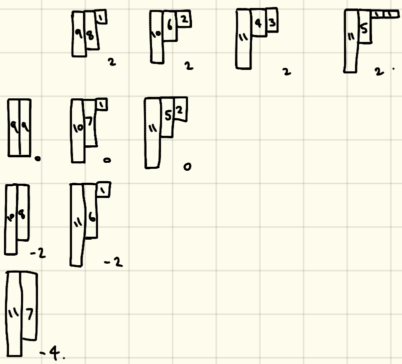
$$(\beta_3 + n\delta)^2 = 2, \quad (\beta_6 + n\delta)^2 = 2 \quad \text{and} \quad (\beta_{8,1}+n\delta)^2 = 2.$$

This gives a real root at any level of length-squared 2 as $\beta_3 + n\delta$ has level $n+1$

$\beta_6 + n\delta$ has level $n+2$

$\beta_{8,1} + n\delta$ has level $n+3$.

From these real roots we may construct all roots at any level by hand, e.g. at level 6 we have:



There is a caveat in our replacement of Serre relations with a condition on the root length we have discarded properties of the algebra coming from the symmetries of the Lie bracket. In particular the Jacobi identity which projects out some generators has been lost, the first example is $\boxed{9}$ as

$$[R^{123},[R^{456},R^{789}]] + [R^{456},[R^{789},R^{123}]] + [R^{789},[R^{123},R^{456}]] = 0 .$$

$$\text{i.e.} \quad R^{123456789} + R^{456789123} + R^{789123456} = 3R^{123456789} = 0 .$$

The null root appears as a weight within the $R^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6\mu_7\mu_8|\nu}$, when $\nu \notin \{\mu_1,\dots,\mu_8\}$.

Part II: M-theory and E_{11}.

Recall the low energy bosonic description of M-theory is 11D bosonic supergravity:

$$\mathcal{L} = R*1 - \tfrac{1}{2}F_4\wedge *F_4 + \tfrac{1}{6}F_4\wedge F_4\wedge A_3.$$

This is a Lagrangian describing the elf-bein $e_\mu{}^a$ (gravity) and $A_{\mu_1\mu_2\mu_3}$ (gauge theory)

The Hodge dual $*F_4 \equiv G_7 = dA_6 + \ldots$ where A_6 is a six-form $A_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}$.

If one were to extend this theory so that the dual of the gravity degrees of freedom are also included this would require the addition of a field $A_{\mu_1\ldots\mu_8 1}{}^a$ as $*\partial_{\mu_1}e_{\mu_2}{}^a = \partial_{\mu_3}A_{\mu_4\ldots\mu_{11}}{}^a + \ldots$

The fields are

$e_\mu{}^a$ (gravity).

$A_{\mu_1\mu_2\mu_3}$ (membrane)

$*$ $*$

$A_{\mu_1\ldots\mu_6}$ (fivebrane)

$A_{\mu_1\ldots\mu_8 1}{}^a$ (dual gravity).

Supergravity.

E_{11} extension of Supergravity.

These fields are the algebra coefficients of the level 0, 1, 2 and 3 generators of E_{11}.

This is more than a coincidence of the tensor index structure: the corresponding roots of E_{11} can be used to reconstruct the solutions of supergravity precisely.

The Brane σ-model.

The Lagrangian should be invariant under the coset $E_{11}/K(E_{11})$ where $K(E_{11})$ is a real-form of E_{11}: a mathematical object which deserves more investigation. $K(E_{11})$ is the extension of $SO(1,10)$ a real-form of $SO(11)$ relevant to supergravity and E_{11}.

How does one construct a Lagrangian for scalars taking values in $E_{11}/K(E_{11})$?

The ingredients are:

- a coset representative in Borel (upper triangular gauge):

$$g = \underbrace{\exp(\phi \cdot H)}_{\text{gravity}} \underbrace{\exp(C \cdot E)}_{\text{gauge.}} \qquad \text{where } \phi \equiv \phi(\xi),\ C \equiv C(\xi)$$

- the Maurer-Cartan form:

$$\nu \equiv dg \cdot g^{-1} = P + Q$$

where $Q \in K(E_{11})$ and $P \in E_{11} \setminus K(E_{11})$.

The Lagrangian:

$$\mathcal{L} = \eta^{-1} (P|P) \qquad \text{where } (M|N) = \mathrm{Tr}(MN) \text{ and } \eta \text{ is the lapse function.}$$

η is included to guarantee that the Action $\int d\xi \, \mathcal{L}$ is invariant under the reparameterisation of ξ.

$\mathcal{L}$ is invariant under a global transformation g_0 (i.e. g_0 does not depend on ξ):

$$g \to g \cdot g_0$$

as this leaves the Maurer-Cartan form unchanged:

$$\nu \longrightarrow d(g \cdot g_0)(g \cdot g_0)^{-1} = (dg \cdot) g_0 g_0^{-1} g = dg \cdot g^{-1} = \nu.$$

The local transformation under $K(E_{11})$ given by:

$$g \longrightarrow kg \qquad \text{where } k \equiv k(\xi) \in K(E_{11})$$

transform ν as:

$$\nu = d(k \cdot g)(k \cdot g)^{-1} = dk \cdot k^{-1} + k \nu k^{-1}$$

hence $P \longrightarrow kPk^{-1}$ and $Q \longrightarrow kQk^{-1} + dk k^{-1}$, leaving $\mathcal{L}$ unchanged.

Abstractly the equations of motion are:

$$(P, P) = 0 \qquad \text{(from varying } \eta\text{).}$$
$$dP - [Q, P] = 0$$

These define a null geodesic on the coset.

We are unable to carry this procedure out for $E_{11}/K(E_{11})$ instead we may use it for cosets $G/K(G)$ where $G \subset E_{11}$. We will find that:

(i.) brane solutions are given by null geodesics on $SL(2,\mathbb{R})/SO(1,1)$

(ii.) bound states of branes are given by null geodesics on $G/K(G)$ where $\mathrm{Rank}(G) > 1$.

Along the way we will highlight ambiguities in this construction and investigate its meaning for spacetime.

The $SL(2,\mathbb{R})/SO(1,1)$ brane σ-model.

$$g = \exp(\phi(\xi)H)\exp(C(\xi)E)$$

where $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ hence

$$g = \begin{pmatrix} e^{\phi} & Ce^{\phi} \\ 0 & e^{-\phi} \end{pmatrix}$$

$$\nu = \partial g . g^{-1} \qquad \text{where } \partial = \frac{\partial}{\partial \xi}.$$

$$= \partial\phi \cdot H + e^{2\phi}\partial C\, E.$$

Now as $Q = SO(1,1)$ then $k = E - F$ hence

$$\nu = \underbrace{\partial\phi \cdot H + \tfrac{1}{2}e^{2\phi}\partial C\,(E+F)}_{P} + \underbrace{\tfrac{1}{2}e^{2\phi}\partial C\,(E-F)}_{Q}.$$

$$\therefore \quad P = \begin{pmatrix} \partial\phi & \tfrac{1}{2}e^{2\phi}\partial C \\ -\tfrac{1}{2}e^{2\phi}\partial C & -\partial\phi \end{pmatrix}$$

$$\Rightarrow \quad \mathcal{L} = \eta^{-1}\left(2(\partial\phi)^2 - \tfrac{1}{2}e^{4\phi}(\partial C)^2\right).$$

The equations of motion for ϕ, C and η respectively are:

$$\partial^2\phi + \tfrac{1}{2}(\partial C)^2 e^{4\phi} = 0 \qquad \text{-[I]}$$

$$\partial(\partial C . e^{4\phi}) = 0 \qquad \text{-[II]}$$

$$(\partial\phi)^2 - \tfrac{1}{4}(\partial C)^2 e^{4\phi} = 0. \qquad \text{-[III]}$$

As $\partial(\partial C e^{4\phi}) = 0$ then $\partial C e^{4\phi} = A$ a constant. Substitution into [III] gives:

$$(\partial\phi)^2 = \tfrac{1}{4}(Ae^{-4\phi})^2 e^{4\phi} = \tfrac{1}{4}A^2 e^{-4\phi} \qquad \Rightarrow \quad \partial\phi = \pm\tfrac{1}{2}Ae^{-2\phi}.$$

$$\int e^{2\phi}d\phi = \int \pm\tfrac{1}{2}A\,d\xi$$

$$e^{2\phi} = \pm A\xi + B \qquad \therefore \quad \phi = \tfrac{1}{2}\ln(\pm A\xi + B).$$

Let $N = a\xi + b$ then trivially N is a harmonic function in ξ and $\phi = \frac{1}{2}\ln(N)$

$$\partial C e^{4\phi} = A \quad \Rightarrow \quad \partial C = A/N^2 = \frac{\partial N}{N^2} \quad \Rightarrow \quad C = -N^{-1} + D.$$

Comment: If you have solved the SuGra equations this all sounds familiar, in that case a brane solution is characterised by a harmonic function (no longer a trivial one) with field strength given by $F = e^{2\phi}\partial C$.

In this simple model are all the necessary parts for a brane solution all that remains is to embed the coset in $E_{11}/K(E_{11})$ and identify ξ with a spacetime parameter.

<u>Example: Level 1 - the M2 brane.</u>

Let the embedding of $SL(2,\mathbb{R}) \subset E_{11}$ be:

$$E \equiv E_{\alpha_{11}} = R^{91011}, \quad F \equiv E_{-\alpha_{11}} = R_{91011} \quad \text{and} \quad H \equiv H_{\alpha_{11}} = -\tfrac{1}{3}(K^1{}_1 + \ldots + K^8{}_8) + \tfrac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11}).$$

So $g = \exp(\phi \cdot H_{\alpha_{11}}).\exp(C \cdot R^{91011}) = \exp(\frac{1}{2}\ln N \cdot H)\exp((-N^{-1}+D)R^{91011})$

Let $h_i{}^i$ denote coefficient of $K^i{}_i$ then we read off:

$$h_1{}^1 = h_2{}^2 = \ldots = h_8{}^8 = -\tfrac{1}{6}\ln N, \quad h_9{}^9 = h_{10}{}^{10} = h_{11}{}^{11} = \tfrac{2}{6}\ln N.$$

Noting that under $\hat{g} P_\mu \hat{g}^{-1} = \exp(h_a{}^b K^a{}_b) P_\mu \exp(-h_c{}^d K^c{}_d)$ where $[P_\mu, K^a{}_b] = \delta^a_\mu P_b$

$$= \exp(-h)_\mu{}^a P_a + \ldots$$

then $\exp(-h)_\mu{}^a = e_\mu{}^a$ the elf-bein, so $g_{\mu\nu} = e_\mu{}^a e_\nu{}^b \eta_{ab}$ with x^{11} time-like gives:

$$ds^2 = N^{1/3}\left((dx^1)^2 + (dx^2)^2 + \ldots + (dx^8)^2\right) + N^{-2/3}\left((dx^9)^2 + (dx^{10})^2 - (dx^{11})^2\right)$$

Supergravity dictionary:

$$F_{\xi\, 9\, 10\, 11} = e^{2\phi}\, \partial_\xi C = N\, \partial_\xi N^{-1}$$

Embed in spacetime (using elf-bein)

curvilinear coordinate with hat

$$\xi \equiv \hat{i}$$

$$F_{\hat{i}\hat{9}\hat{10}\hat{11}} = \partial_{\hat{i}} N^{-1}$$

Make the solution spherically symmetric $\Rightarrow$ $N = b + \frac{a}{r^6}$ (keep $\partial^2 N = 0$)

This is the full M2 brane solution of SuGra.

N.B. this means it solves 66 Einstein equations and 165 gauge field equations, although in practise only the same 3 equations we solved are non-trivial and distinct.

It describes a membrane from the root $\begin{array}{|c|}\hline 11\\\hline 10\\\hline 9\\\hline\end{array}$ whose worldvolume directions are x^9, x^{10} and t^{11}.

Higher Level Real Roots.

The level 2 root $\begin{array}{|c|}\hline 11\\\hline 10\\\hline 9\\\hline 8\\\hline 7\\\hline 6\\\hline\end{array}$ gives the spacetime solution of the fivebrane (using the $SL(2,\mathbb{R})/SO(1,1)$ coset & supergravity dictionary construction).

Other real roots all correspond to gauge fields of mixed symmetry.

The level 3 root gives a pure gravity solution: The KK 6 monopole.

Problems with mixed symmetry fields

The mixed symmetry fields present immediate ambiguities for the supergravity dictionary.

Consider the level 4 real root, corresponding to the highest weight of the .

The gauge-field $A_{34567891011|91011}$ is treated as a scalar in the $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ coset model,

but at the point of embedding the model in space-time the gauge field is imbued with the

structure of a mixed symmetry tensor i.e.

$$F_{\xi} = e^{2\phi}\partial_{\xi} C \longrightarrow \partial_{\xi} A_{34567891011|91011} \in \begin{array}{c} F_{\xi 34\ldots 11|91011} \equiv F_{10|3} \\ \text{OR} \\ F_{34\ldots 11|\xi 91011} \equiv F_{9|4}. \end{array}$$

Both choices work i.e. the null geodesic on $SL(2,\mathbb{R})/SO(1,1)$ gives a solution to the

equations of motion of both:

$$S_1 = \int R*1 - \tfrac{1}{2} F_{10|3} \wedge *_1 F_{10|}{}^{3}$$

and

dualise on the set of indices including the derivative index.

$$S_2 = \int R*1 - \tfrac{1}{2} F_{9|4} \wedge *_2 F^{9|}{}_{4}.$$

But it is no longer evident how to relate these field strengths back to F_4 of SuGra

as:

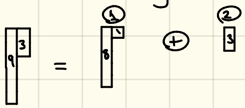
$$*_1 F_{10|3} = F_{1|3} = \partial_{\nu} A_{\mu_1\mu_2\mu_3} \longrightarrow A_3. \quad \text{level } 1.$$

$$*_2 F_{9|4} = F_{9|7} = \partial_{\nu_1} A_{\mu_1\ldots\mu_9|\nu_2\ldots\nu_7} \longrightarrow A_{9|6}. \quad \text{level } 5.$$

An extension of Hodge duality is required to write an E_{11} invariant action.
First steps: embed the Hodge duality within an affine duality [See E_9 Multiplet of BPS States by Englert, Houart, Kleinschmidt, Nicolai & Tabti].

Interpretations of Mixed-Symmetry Fields.

Fundamental idea: Break up the high level root into its low level parts, i.e. play with the Young tableau as if they were Lego bricks.



KK6. ⊕ M2.

$$\underline{\beta}_1 = \underline{e}_4 + \dots + \underline{e}_{10} + 2\underline{e}_{11} \qquad \underline{\beta}_2 = \underline{e}_3 + \underline{e}_9 + \underline{e}_{10}$$

Now $\underline{\beta}_1 \cdot \underline{\beta}_2 = -1$ and $\underline{\beta}_1^2 = \underline{\beta}_2^2 = 2$.

These form the root system of $SL(3|\mathbb{R}) \longrightarrow$ we may construct the level 4 solution by solving the brane σ-model for $\frac{SL(3|\mathbb{R})}{SO(1,2)}$.

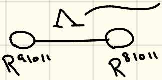
"Exotic E_{11} branes as..." [PPC '09 & Kleinschmidt, Houart & Hörnlund-Lindmann] '09 "Some algebraic Aspects..."]

There are simpler examples than this to commence with!

For example consider $R^{81011} = [\underset{\substack{\uparrow \\ \text{Lorentz boost}}}{K^8{}_9}, \underset{\text{M2}}{R^{91011}}]$ as an $SL(3|\mathbb{R})$ sigma model.

boost parameter.

The solution describes two M2 branes



$$H_1 = -\tfrac{1}{3}(K^1{}_1 + \dots + K^8{}_8) + \tfrac{2}{3}(K^9{}_9 + K^{10}{}_{10} + K^{11}{}_{11})$$
$$H_2 = K^8{}_8 - K^9{}_9$$
$$E_1 = R^{91011}$$
$$E_2 = K^8{}_9$$
$$E_{12} = R^{81011}$$

$$g = \exp(\phi_1 H_1 + \phi_2 H_2)\exp(C_1 E_1 + C_2 E_2 + C_{12} E_{12}).$$

Null geodesic equations are solved by:

$$\phi_1 = \tfrac{1}{2}\ln N_1 \quad , \quad \phi_2 = \tfrac{1}{2}\ln N_2$$

$$C_1 = \frac{\tan\theta}{N_1} \quad , \quad C_2 = \frac{\sin\theta}{N_2} \quad \text{and} \quad C_{12} = \frac{1}{2\cos\theta}\left(\frac{\cos^2\theta}{N_1} + \frac{1}{N_2}\right)$$

where $N_1 = 1 + Q\xi$ and $N_2 = 1 + Q\cos^2\theta \cdot \xi$.

Note the solution is described by 2 harmonic functions: N_1 and N_2 and an interpolating parameter θ. When $\theta = \pi/2$, $N_2 = 1$ is trivial and the solution reduces to that of the $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ model. When $\theta = 0$, $N_1 = N_2$ and the solution becomes a solution of another $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ coset model inside E_{11}.

Example: The Dyonic Membrane.

$$\underset{\beta_1+\beta_2}{\boxed{6}} = \underset{\beta_1}{\boxed{3}} \oplus \underset{\beta_2}{\boxed{3}}.$$

$$M5 = S2 + M2. \qquad (\text{where } x^6 \text{ is timelike}).$$

$$ds^2_{M2,S2} = (N_1N_2)^{\frac{1}{3}}\left((dx^1)^2 + \dots + (dx^5)^2 + N_1^{-1}\left(-(dx^6)^2 + (dx^7) + (dx^8)^2\right) + N_2^{-1}\left((dx^9)^2 + (dx^{10})^2 + (dx^{11})^2\right)\right).$$

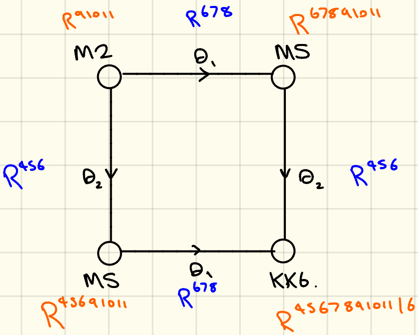
When $\theta = \pi/2$, $N_2 = 1$ the solution is the M2 brane along (x^6, x^7, x^8)

$\theta = 0$, $N_2 = N_1$ the solution is the M5 brane along $(x^6, x^7, x^8, x^9, x^{10}, x^{11})$.

This is a bound state of an M2 brane and an M5 brane first found in N=8 SuGra by Izquierdo, Lambert, Papadopoulos & Townsend in '95.

Other solutions of bound states of branes have been found [PPC '11] for larger groups G in the context of $D=10$ string theory (types IIA and IIB).

e.g. an $\frac{SL(4,\mathbb{R})}{SO(1,3)}$ null geodesic:



The $SO(1,3)$ orbit of the M2 brane gives the full solution.

Comments:

- All real roots of E_{11} can be interpreted as bound states of M2 branes.
  - Many branes are space-filling which presents a problem embedding ξ in space-time transverse to the brane world-volume, as the SuGra dictionary suggests.
  - General $G/K(G)$ cosets do not have G semisimple but G is Kac-Moody itself, or worse G may not be recognisable as a Dynkin diagram.
- All boosts under $K(E_{11})$ are extensions of the Lorentz group $SO(1,3)$ and on the same footing $\Rightarrow$ that spacetime should be extended to an infinite-dimensional manifold, constructed from the 1st fundamental representation of E_{11}: L_1 i.e. the full theory has symmetries $L_1 \ltimes E_{11}/K(E_{11})$ (cf. $P_a \ltimes \frac{GL(4,\mathbb{R})}{SO(1,3)}$).

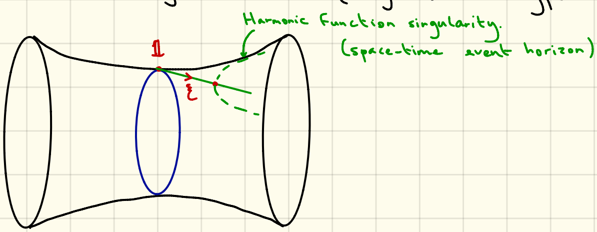
Where are the cosets?

Recall that the scalar cosets appeared in the dimensional reduction of SuGra and that the SuGra dictionary advocates embedding the null geodesic's parameter on the coset with a spacetime coordinate e.g. $\xi \longrightarrow x^{\hat{i}}$. However ther coset space of $SL(2,\mathbb{R})/SO(1,1)$ is two-dimensional. There is the possibility of considering a 2-parameter solution (the world volume of a string moving on $\frac{SL(2,\mathbb{R})}{SO(1,1)}$) and embedding both parameters in spacetime [Work in progress with Sarban Sarkar.] The topology of $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ is $S^1 \times \mathbb{R}^1$. It is simple to see $SL(2,\mathbb{R})/SO(1,1)$ as a single-sheeted hyperboloid:

Let $M \in SL(2,\mathbb{R}) \setminus SO(1,1)$ be $M = \exp\left(aH + b(E-F)\right) = \begin{pmatrix} \cosh(r) + \frac{a}{r}\sinh(r). & \frac{b}{r}\sinh(r). \\ -\frac{b}{r}\sinh(r). & \cosh(r) - \frac{a}{r}\sinh(r). \end{pmatrix}$

where $r^2 = a^2 - b^2$. Now writing $x = \frac{b}{r}\sinh(r)$, $y = \frac{a}{r}\sinh(r)$ and $z = \cosh(r)$:

$$\det M = 1 \Rightarrow x^2 - y^2 + z^2 = 1. \quad \text{(Single-sheeted hyperboloid)}$$



The single-parameter solution is given by the path from $\mathbb{1}$ to the point where N becomes singular.

The solution has no knowledge of the compact cycle on $\frac{SL(2,\mathbb{R})}{SO(1,1)}$: this is due to fixing the Borel gauge in the set up of the brane σ-model. If two-parameter solutions on $\frac{SL(2,\mathbb{R})}{SO(1,1)}$ exist then

spacetime needs to be enlarged, as $\dim\left(\frac{G}{K}\right) = \dim(G) - \dim(K) > 11$ when $G = SL(5,\mathbb{R})$
$K = SO(1,4)$.

N.B. $\dim(SL(5,\mathbb{R})) = 24$, $\dim(SO(1,4)) = \frac{5}{2}(4) = 10$.

This offers another motivation for enlarging spacetime to have coordinates sitting in the fundamental representation of E_{11}. [Kleinschmidt & West '03].

In this setting the cosets would be geometrised in an enlarged space-time.

Concluding Remarks.

- Much mathematical work is needed on the representation theory of E_{11} and $K(E_{11})$
  - see the attempts to construct spinor representations of $K(E_{10})$ by Kleinschmidt & Nicolai.

- Spacetime generalised in the manner described by E_{11} has produced many results recently under the title of Double-Field Theory [See Siegel, Hull, Hohm, Zweibach, Samtlebean and others] and more recently exceptional Field theory [Hohm, Samtleben]
Both of these directions involve investigation of truncated versions of the l_1 coordinates :

$$P_a, \quad Z^{ab}, \quad Z^{a_1 \ldots a_5}, \quad Z^{a_1 \ldots a_7, b}, \quad Z^{a_1 \ldots a_8} \ldots$$
$$11 \qquad 55 \qquad 385 \qquad 3465 \qquad 165 \quad \ldots$$

- Recent progress has been made by Tumanov & West by finding the equation of motion for the dual graviton.