Non-Perturbative symplectic manifolds
and
Non-Commutative algebras

P. Boalch (CNRS & Orsay)

Some references: arXiv: 0806, 1307, 1501

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meromorphic Hitchin systems

Bottacin, Markman 1995

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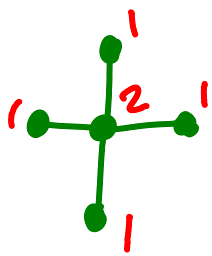
Eg



$\Rightarrow$ D_4 ALE space

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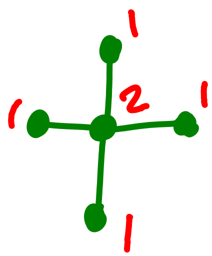
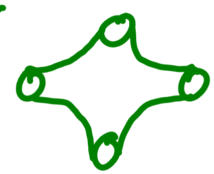
Eg 

$\Longrightarrow$ D_4 ALE space

$$\cap$$

$\mathcal{M}($  $, \ GL_2)$

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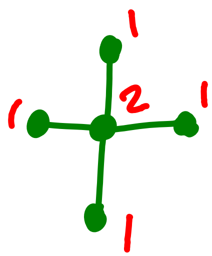
Eg 

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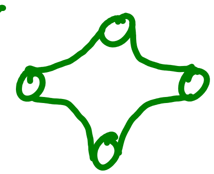
$\mathcal{M}($  $, GL_2)$

$$\cong$$

$$\mathcal{M}_{Betti} \cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \} \subset \mathbb{C}^3$$

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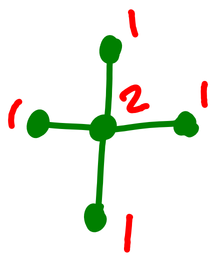
Eg 

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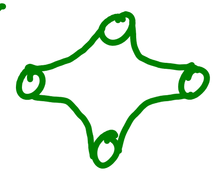
$\mathcal{M}($  $, GL_2)$

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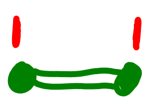
What about 

$\Rightarrow$ Eguchi-Hanson $\subset$?

$T^*\mathbb{P}^1$

Van den Bergh's spaces

$\Gamma =$ 

Van den Bergh's spaces

$$\Gamma = \circ\!\!-\!\!\!-\!\!\!-\!\!\!-\!\!\circ \qquad\qquad I = \{\text{nodes}(\Gamma)\}$$

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$V = V_1 \oplus V_2$ (I graded complex vector space)

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Thm (Van den Bergh '04) $\mathrm{Rep}^*(\Gamma, V)$ is a "multiplicative" (or "quasi") Hamiltonian H-space with group valued moment map $\mu(a,b) = (1+ab, (1+ba)^{-1}) \in H$

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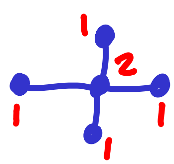
E.g. Mult. Quiver Var. () $\cong \{xyz + x^2 + y^2 + z^2 = ax + by + cz + d\}$

Qn Suppose $\Gamma =$  or  etc

then what is $\mathrm{Rep}^*(\Gamma, V)$?

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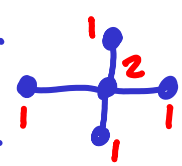
E.g. Mult. Quiver Var.() $\cong \{ xyz + x^2 + y^2 + z^2 = ax + by + cz + d \}$

SPECIMEN ALGORITHMI SINGVLARIS.

Auctore

L. EVLERO.

1.

Consideratio fractionum continuarum, quarum vsum vberrimum per totam Analysin iam aliquoties ostendi, deduxit me ad quantitates certo quodam modo ex indicibus formatas, quarum natura ita est comparata, vt singularem algorithmum requirat. Cum igitur summa Analyseos inuenta maximam partem algorithmo ad certas quasdam quantitates accommodato

6. Haec ergo teneatur definitio ſignorum ( ), inter quae indices ordine a ſiniſtra ad dextram ſcribere conſtitui; atque indices hoc modo clauſulis incluſi inpoſterum denotabunt numerum ex iſtis indicibus formatum. Ita a ſimpliciſſimis caſibus inchoando, habebimus:

$$(a) = a$$
$$(a,b) = ab + 1$$
$$(a,b,c) = abc + c + a$$
$$(a,b,c,d) = abcd + cd + ad + ab + 1$$
$$(a,b,c,d,e) = abcde + cde + ade + abe + abc + e + c + a$$

etc.

ex

"Euler's continuant polynomials"

G. G. Stokes 1857

VI. *On the Discontinuity of Arbitrary Constants which appear in Divergent Developments.* *By* G. G. STOKES, M.A., D.C.L., Sec. R.S., *Fellow of Pembroke College, and Lucasian Professor of Mathematics in the University of Cambridge.*

[Read *May* 11, 1857.]

IN a paper "On the Numerical Calculation of a class of Definite Integrals and Infinite Series," printed in the ninth volume of the *Transactions* of this Society, I succeeded in developing the integral $\int_0^\infty \cos\frac{\pi}{2}(w^3 - mw)\,dw$ in a form which admits of extremely easy numerical calculation when m is large, whether positive or negative, or even moderately large. The method there followed is of very general application to a class of functions which frequently occur in physical problems. Some other examples of its use are given in the same paper; and I was enabled by the application of it to solve the problem of the motion of the fluid surrounding a pendulum of the form of a long cylinder, when the internal friction of the fluid is taken into account *.

These functions admit of expansion, according to ascending powers of the variables, in series which are always convergent, and which may be regarded as defining the functions for all values of the variable real or imaginary, though the actual numerical calculation would involve a labour increasing indefinitely with the magnitude of the variable. They satisfy certain linear differential equations, which indeed frequently are what present themselves in the first instance, the series, multiplied by arbitrary constants, being merely their integrals. In my former paper, to which the present may be regarded as a supplement, I have employed these equations to obtain integrals in the form of descending series multiplied by exponentials. These integrals, when once the arbitrary constants are determined, are exceedingly convenient

How to define "multiplicative version"?

complex Lie group G $\Longrightarrow$ Lie algebra $\mathfrak{g} = T_e G$

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– can look at "monodromy" of many other connections

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$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$

monodromy of $X \dfrac{dz}{z}$

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$$\left(\frac{A}{z} + \frac{B}{z-1}\right)dz \Longrightarrow \text{all multizetas}$$

(generating series is perturbative expansion about trivial connection of connection matrix $0 \leftrightarrow 1$)

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$$\left(\frac{A}{z^2} + \frac{B}{z}\right)dz \implies \text{Poisson Lie group underlying } U_q\mathfrak{g}$$

$$\mathbb{C}[\mathfrak{g}^*] = \mathrm{Sym}\,\mathfrak{g} \xrightarrow[\text{quantize}]{} U\mathfrak{g}$$

(deform multiplication)

$$\begin{tikzcd}
 & U_q\mathfrak{g} \\
\mathbb{C}[\mathfrak{g}^*] = \operatorname{Sym}\mathfrak{g} \arrow[r, squiggly, "\substack{\text{quantize} \\ \text{(deform multiplication)}}"'] & U\mathfrak{g} \arrow[u, squiggly, "\substack{\text{``quantize''} \\ \text{(deform comultiplication)}}"']
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\end{tikzcd}$$

Thm (2001) G^* is the space of monodromy / Stokes data of connections $\left(\frac{A}{z^2} + \frac{B}{z}\right)dz \Big|_{\text{unit disc}}$ $A \in \mathfrak{t}_{reg}$ fixed, $B \in \mathfrak{g} \cong \mathfrak{g}^*$

and the desired nonlinear Poisson structure appears this way

Cartoon

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Hamiltonian geometry

$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

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$$\Big\downarrow \mu^{-1}(0)/G$$

Additive symplectic geometry

$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$

Cartoon

∞-d Ham^n geometry

e.g connections on C^∞ bundles / Riemann surfaces

$\cup$

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Multiplicative symplectic geometry

Betti spaces, character varieties

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$\mathcal{C} \subset G$, $D = G \times G$

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U

$/\!/ \mathcal{G}_1$

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Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$
$\mathcal{M}^*$

$\xRightarrow{RH}$

Multiplicative symplectic geometry
Betti spaces, character varieties
$\mathcal{M}_B$

Cartoon

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$\cup$ $\qquad\qquad$ $\Big\downarrow /\!/\mathcal{G}_1$

Hamiltonian geometry
$\mathcal{O} \subset \mathfrak{g}^*$, T^*G

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$\mathcal{C} \subset G$, $D = G \times G$

$\Big\downarrow \mu^{-1}(0)/G$

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Additive symplectic geometry
$\mathcal{O}_1 \times \cdots \times \mathcal{O}_m /\!/ G$
$\mathcal{M}^*$

$\xRightarrow{\text{RHB}}$

Multiplicative symplectic geometry
Betti spaces, wild character varieties
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Wild Character Varieties

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Fix G (e.g. $GL_n(\mathbb{C})$)

symplectic variety

Σ compact Riemann Surface $\Longrightarrow$ $\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface $\Longrightarrow$ $\overset{\text{symplectic variety}}{\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G}$

$$\wr\!\!\!\parallel \text{ RH}$$

$$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma\}/_{\text{isom}}$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$

symplectic variety

$$\mathcal{M}_B = \mathrm{Hom}(\pi_1(\Sigma), G)/G$$

$$\begin{array}{c} \| \wr \; RH \end{array}$$

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Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Σ compact Riemann Surface
with marked points
$\underset{\sim}{a} = (a_1, \dots, a_m)$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

Poisson variety

$\Longrightarrow \quad \mathcal{M}_B^{\text{tame}} = \operatorname{Hom}(\pi_1(\Sigma^\circ), G)/G$

$\wr\!\!\parallel$ RH

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \text{ with reg. sing.s}\}/\text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson scheme (∞-type)

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ $\Longrightarrow$ $\mathcal{M}_B$

$\cong$ RHB

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$\mathcal{M}_{DR} = \{\text{Alg. connections on } G\text{-bundles on } \Sigma^\circ\} / \text{isom}$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

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$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}$ $\}$ / isom

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$\mathcal{M}_{DR} = \{ \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \text{ with irreg. types } \underset{\sim}{Q} \} / \text{isom}$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$$Q_i \in \underbrace{\mathfrak{t}_i}_{\text{Cartan subalg.}} \subset \mathfrak{g}((z_i))$$

Wild Character Varieties

Fix G (e.g $GL_n(\mathbb{C})$)

Poisson variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$\| \wr$ RHB

$\mathcal{M}_{DR} = \{$ Alg. connections on G-bundles on Σ° with irreg. types $\underset{\sim}{Q}$ $\} / $ isom

$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} +$ holom.

e.g. $Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i))$

Cartan subalg.

$\mathfrak{t} \subset \mathfrak{g}$

Wild Character Varieties

Fix G (e.g. $GL_n(\mathbb{C})$)

Wild Riemann surface $(\Sigma, \underset{\sim}{a}, \underset{\sim}{Q}) \Longrightarrow$ wild character variety

Σ compact Riemann Surface with marked points $\underset{\sim}{a} = (a_1, \dots, a_m)$ and irregular types $\underset{\sim}{Q} = Q_1, \dots, Q_m$ $\Longrightarrow$ $\mathcal{M}_B$

$\Sigma^\circ = \Sigma \setminus \underset{\sim}{a}$

$\mathcal{M}_B \ \cong \ \text{RHB}$

$\mathcal{M}_{DR} = \{ \text{Alg. connections on } G\text{-bundles on } \Sigma^\circ \text{ with irreg. types } \underset{\sim}{Q} \} / \text{isom}$

$\nabla \cong dQ_i + \Lambda_i \frac{dz_i}{z_i} + \text{holom.}$

e.g. $Q_i \in \mathfrak{t}((z_i)) \subset \mathfrak{g}((z_i))$

Cartan subalg. $\mathfrak{t} \subset \mathfrak{g}$

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Fix G (e.g $GL_n(\mathbb{C})$)

E.g. $(\text{Disc}, 0, Q)$ $\quad G = GL_2(\mathbb{C})$

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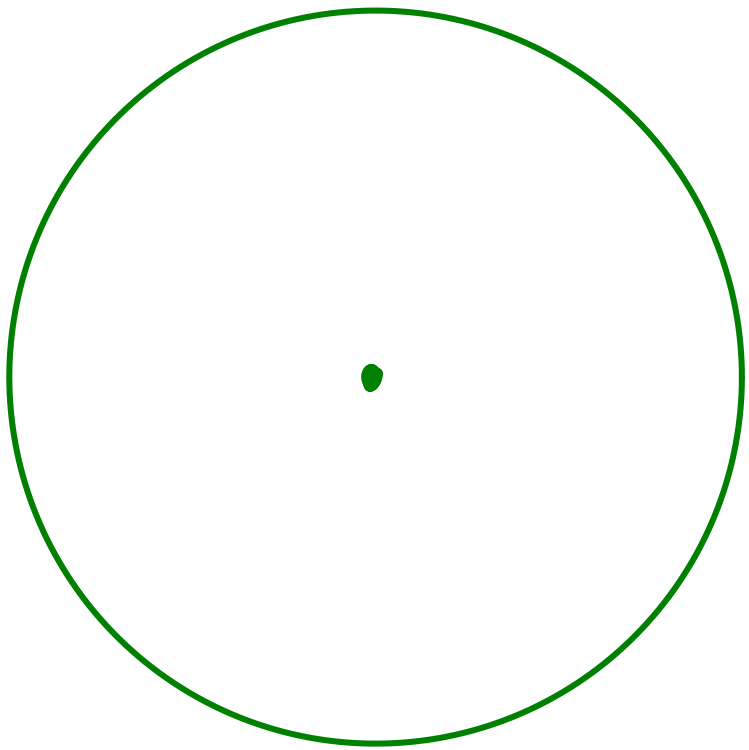

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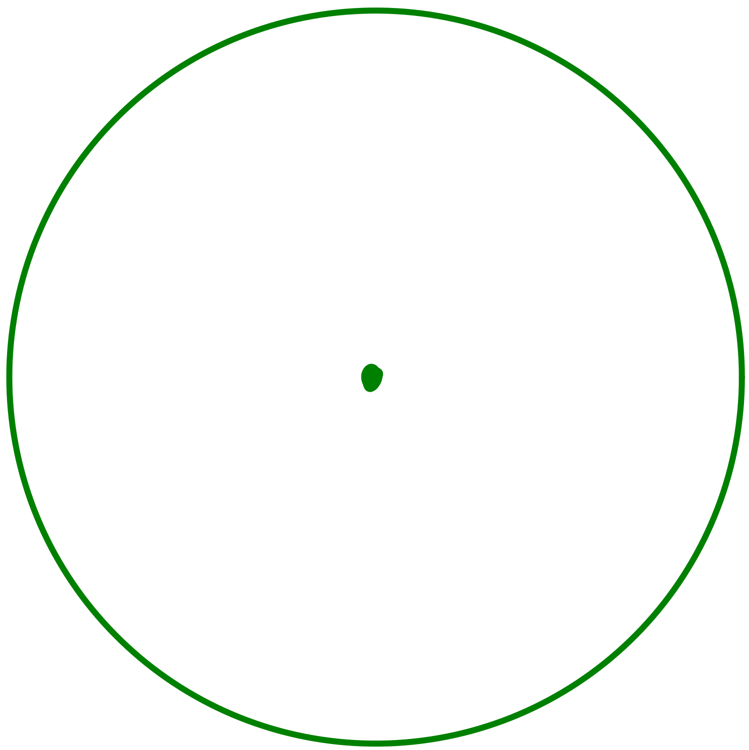

$Q \Rightarrow$

- centraliser group $H = T = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \subset G$
  $C_G(Q)$

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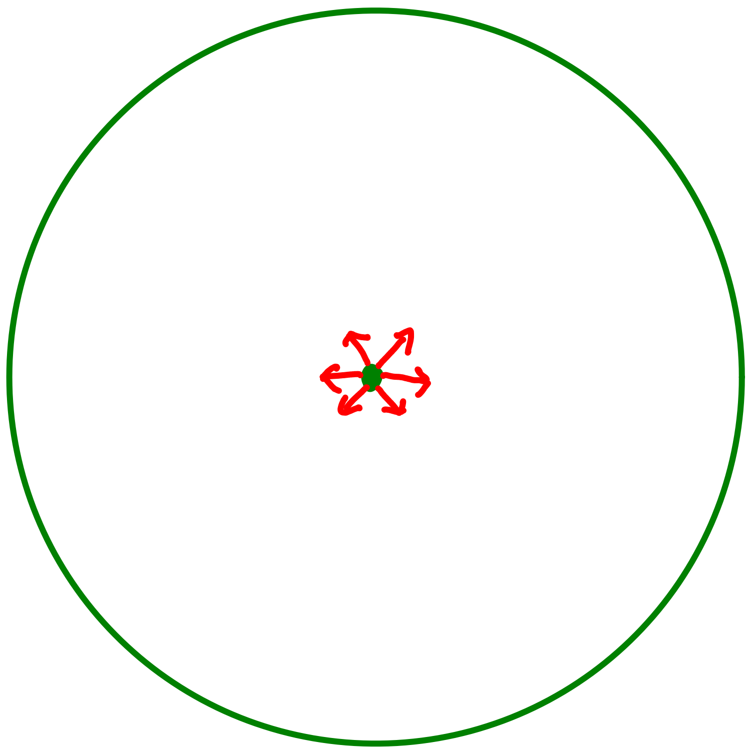

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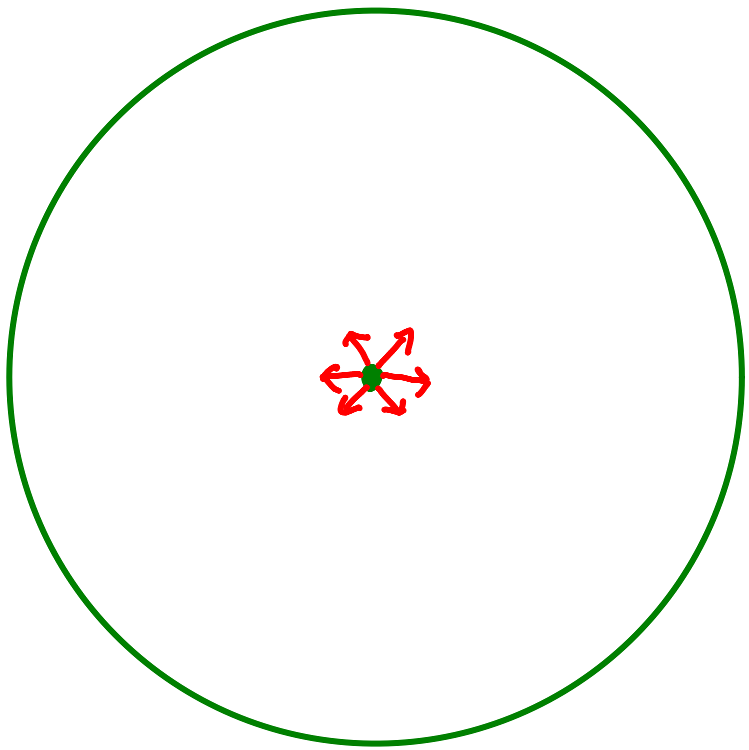

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  $\cong U_+$ or U_- here
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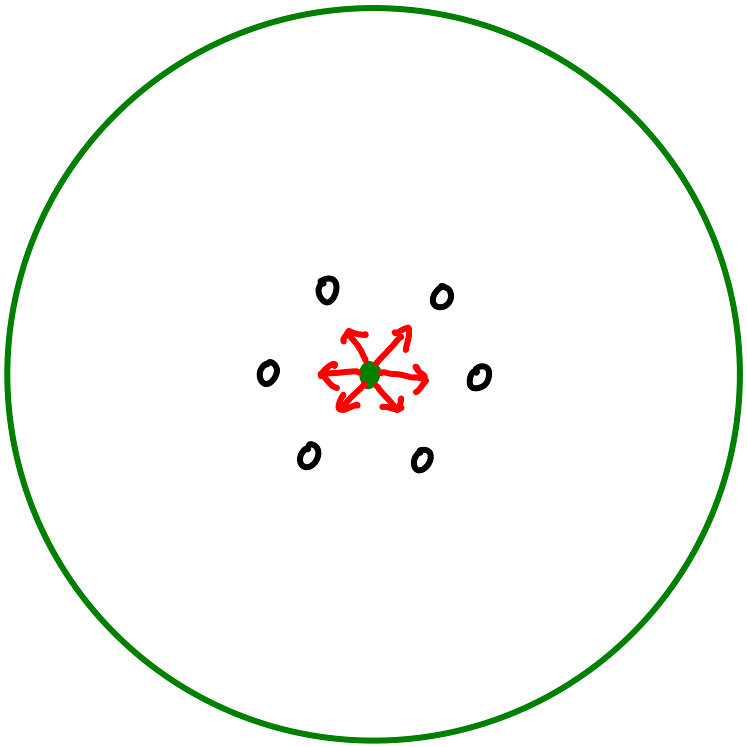

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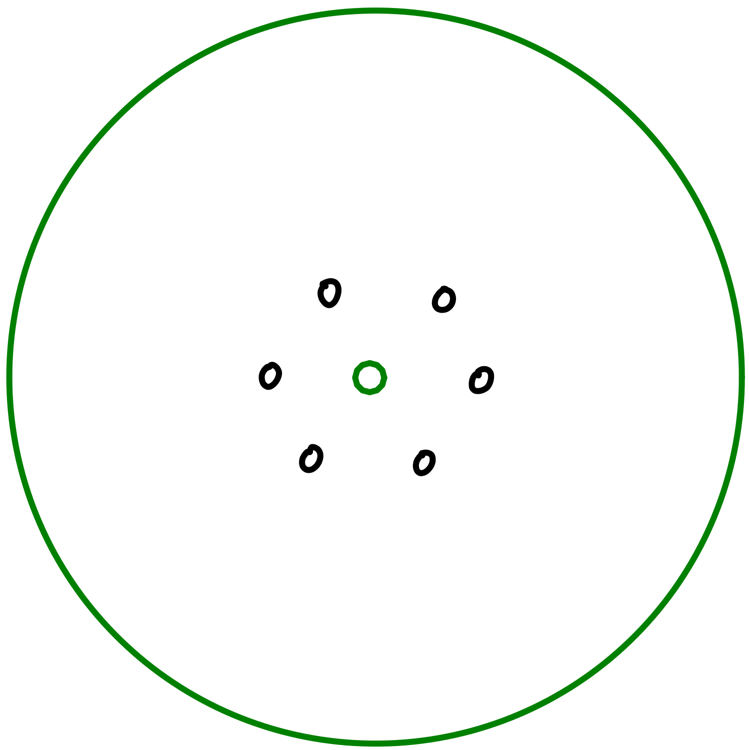

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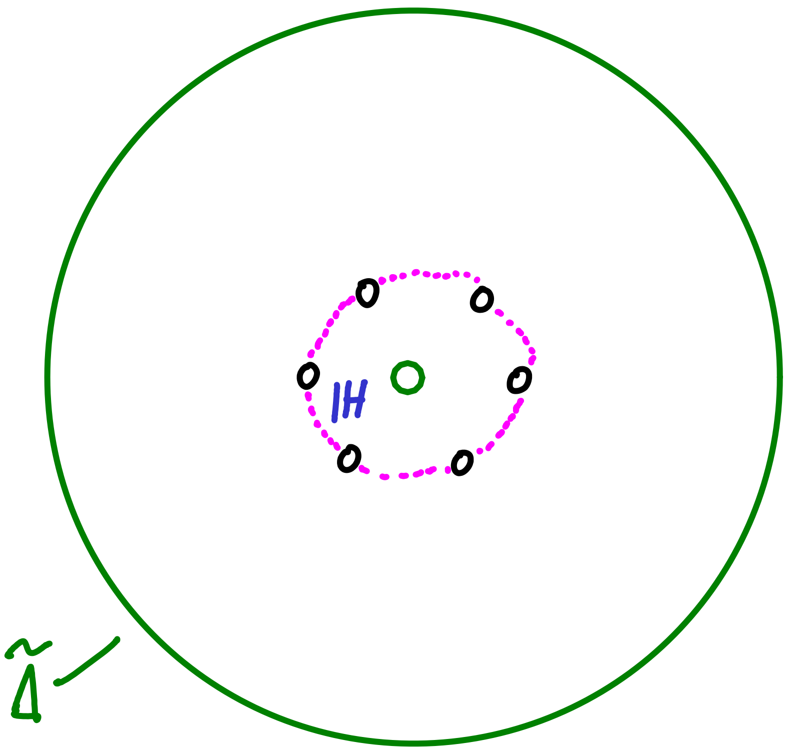

$\circ$ $e(d)$ extra punctures

$\mathbb{H}$ halo/annulus

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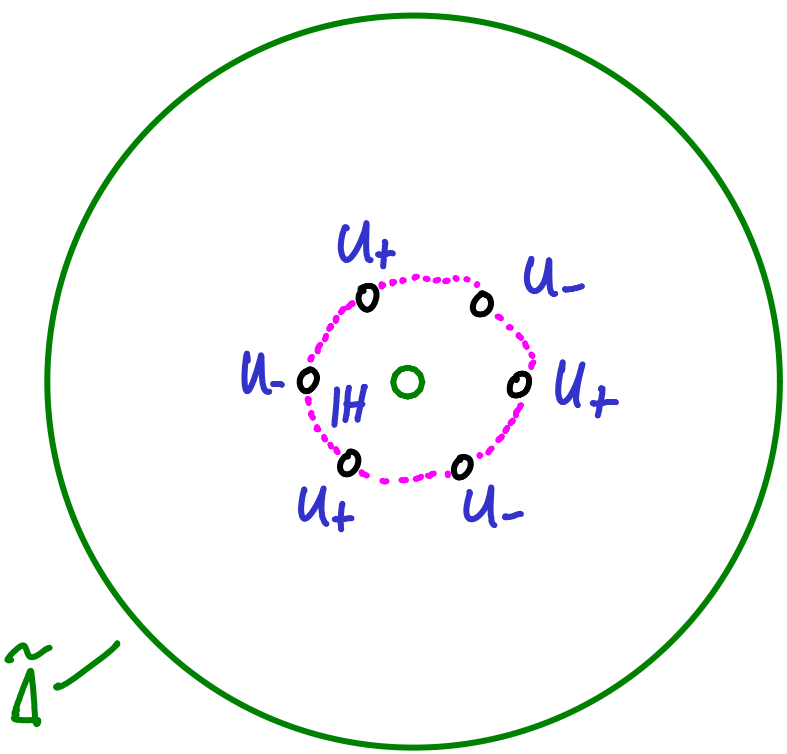

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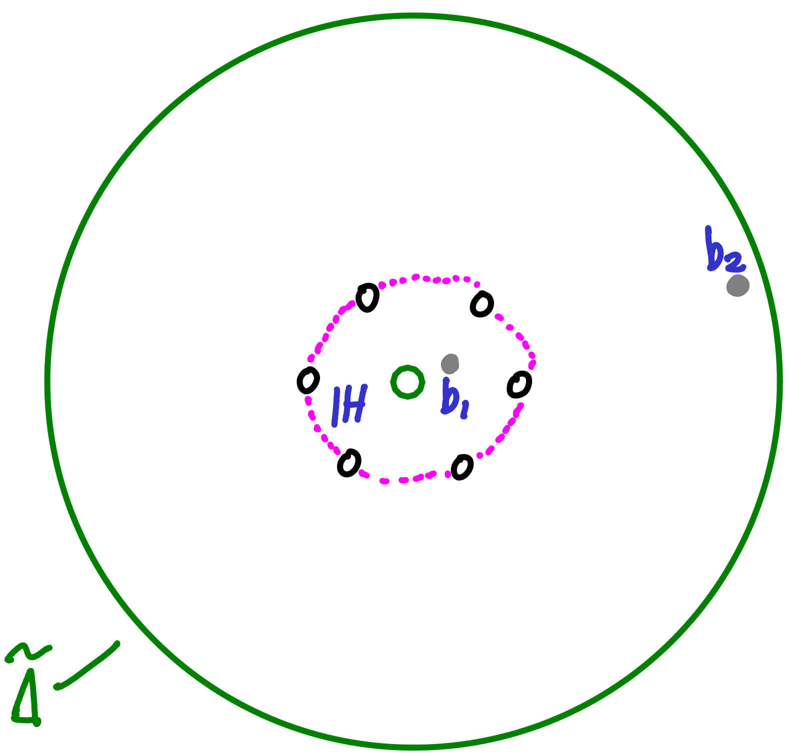

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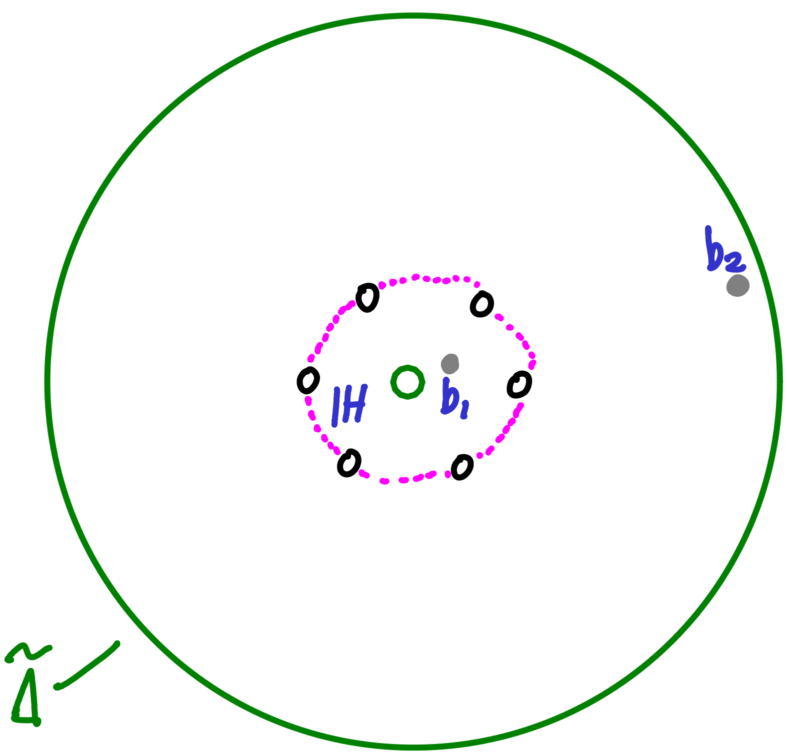

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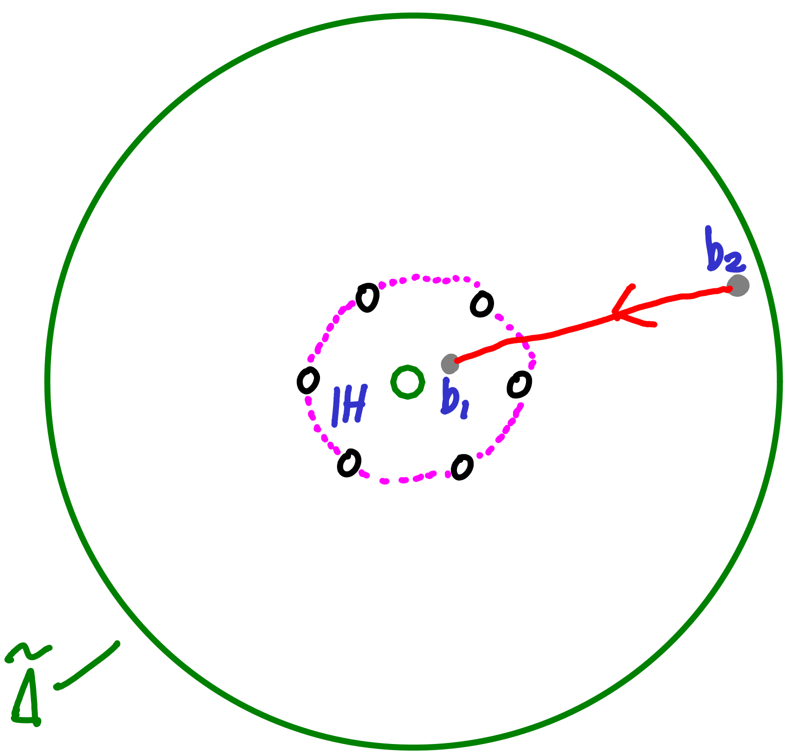

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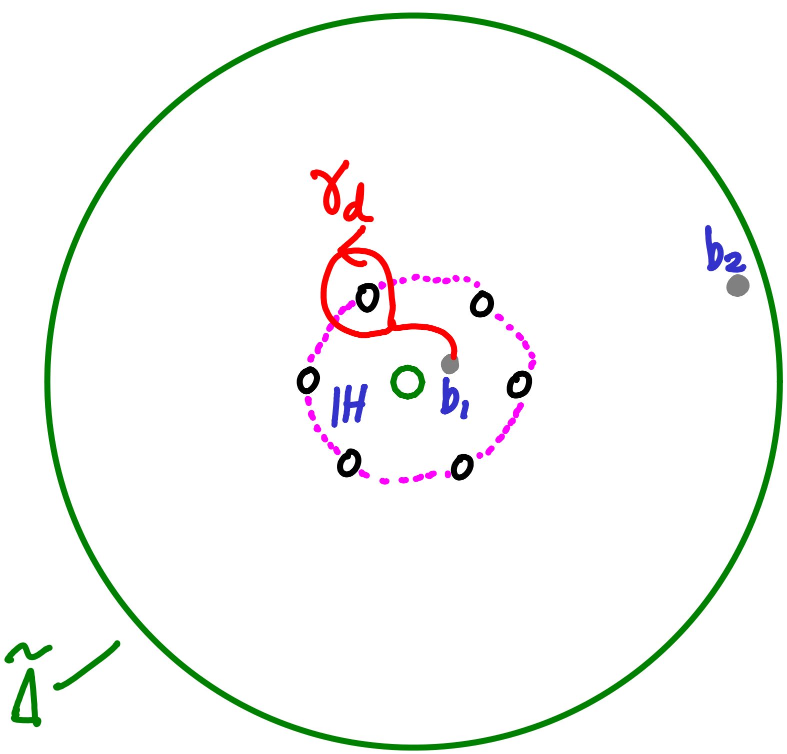

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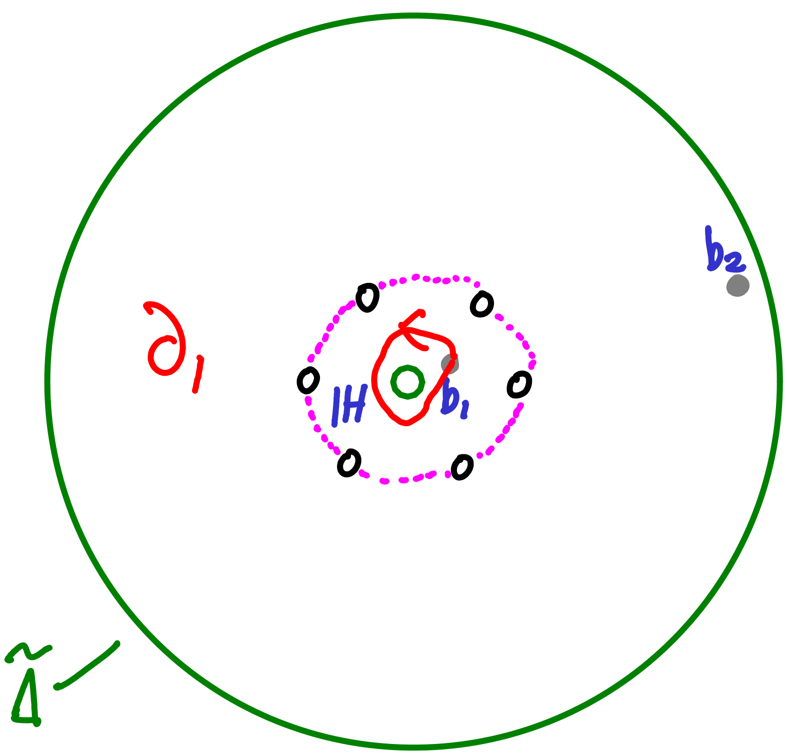

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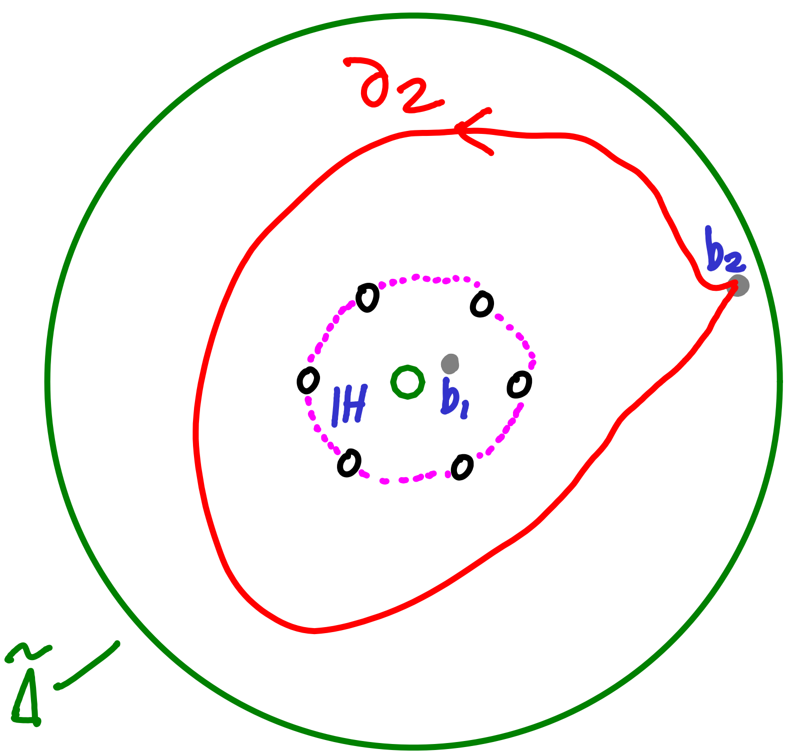

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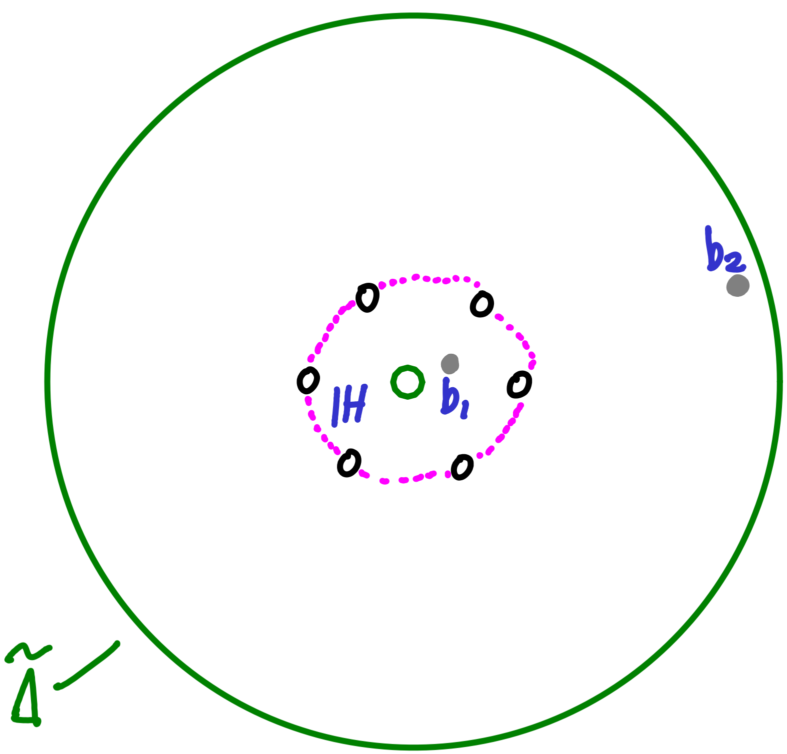

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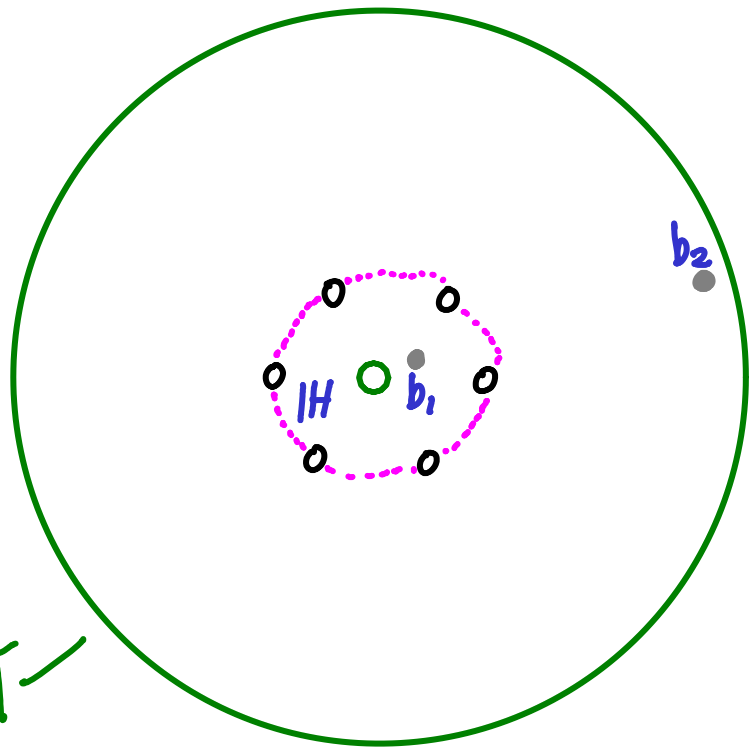

$\tilde{\Delta}$

basepoints b_1, b_2

$$\Pi = \Pi_1(\tilde{\Delta}, \{b_1, b_2\})$$

$$\tilde{\mathcal{M}}_B = \mathrm{Hom}_{\mathbb{S}}(\Pi, G)$$

$$= \left\{ \rho : \Pi \to G \;\middle|\; \begin{array}{l} \rho(\partial_1) \in H \\ \rho(\gamma_d) \in \mathrm{Sto}_d \quad \forall d \in \mathbb{A} \end{array} \right\}$$

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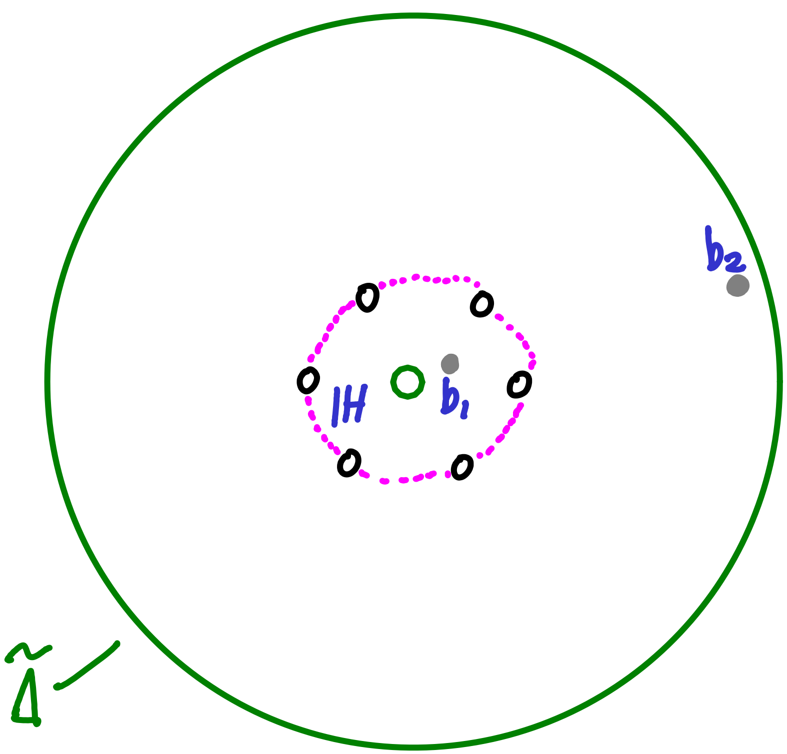

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Thm (arXiv 0203.*****)

$\tilde{\mathcal{M}}_B$ is a quasi-Hamiltonian $G \times H$ space

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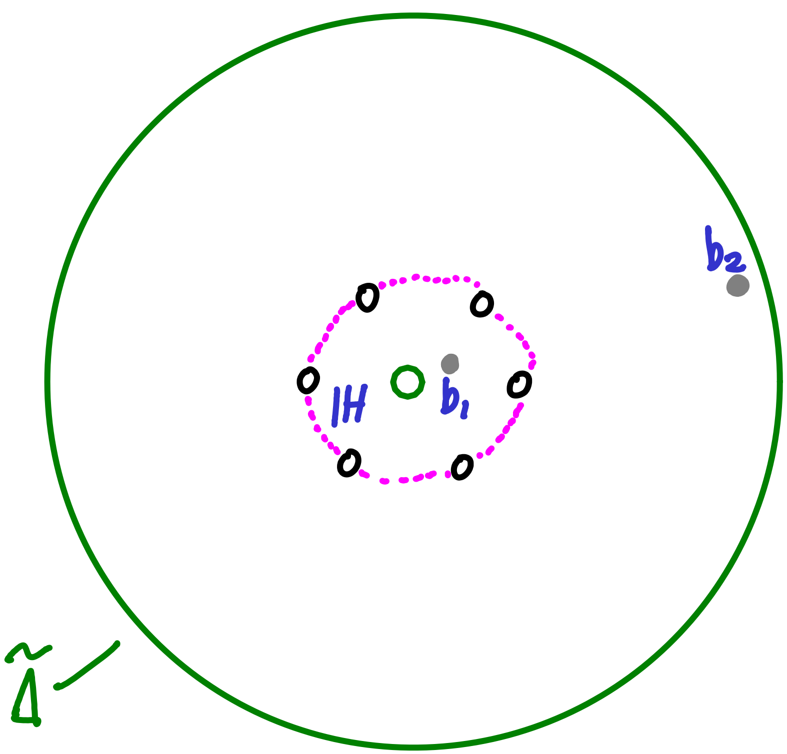

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$\mathcal{A}(Q) = G \times (U_+ \times U_-)^k \times H$ is a quasi-Hamiltonian $G \times H$ space ("fission space")

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Moment map $\quad \mu(C, \underset{\sim}{S}, h) = (C^{-1} h S_{2k} \cdots S_2 S_1 C, \ h^{-1}) \in G \times H$

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Cor. $\mathcal{B}(Q) := \mathcal{A}(Q) /\!/ G$ is a quasi-Hamiltonian H-space

$$= \mu_G^{-1}(1)/G \qquad = \widetilde{\mathcal{M}}_B((\mathbb{P}^1, 0, Q))$$

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$$\cong \{ (S_2, \dots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \} \quad (\text{Gauss})$$

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E.g. $k=2$ $\qquad \left(\begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\right)_{11} = 1 + ab$

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Lemma

$$\left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_r & 1 \end{pmatrix} \right)_{11} = (a_1, b_1, \ldots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

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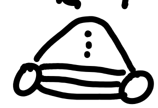
$\cong \{ (S_2, \ldots, S_{2k-1}) \mid (S_{2k-1} \cdots S_3 S_2)_{11} \neq 0 \}$ (Gauss)

$\cong \{ \underset{\sim}{a}, \underset{\sim}{b} \in \mathrm{Rep}(\Gamma, V) \mid (a_1, b_1, \ldots, a_{k-1}, b_{k-1}) \neq 0 \}$

$\Gamma =$  , $V = \mathbb{C} \oplus \mathbb{C}$

Lemma

$$\left(\begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ b_r & 1 \end{pmatrix} \right)_{11} = (a_1, b_1, \ldots, a_r, b_r)$$

— Euler's continuants are group valued moment maps

Wild Character Varieties

Cor.

$$\{(\underset{\sim}{S}, h) \in (U_+ \times U_-)^k \times H \mid h S_{2k} \cdots S_2 S_1 = 1\}$$ is a quasi-Hamiltonian H-space

$$\cong \{(S_2, \ldots, S_{2k-1}) \mid S_{2k-1} \cdots S_3 S_2 \in G^0 = U_- H U_+ \subset G\}$$

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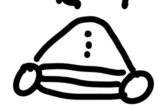
$k-1$

$\Gamma =$  , $V = \mathbb{C} \oplus \mathbb{C}$

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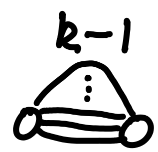
$\Gamma =$ 

, $V = \mathbb{C} \oplus \mathbb{C}$

[Similarly for $V = V_1 \oplus V_2$ any dimension
(2009-2015)
Γ any "fission graph"]

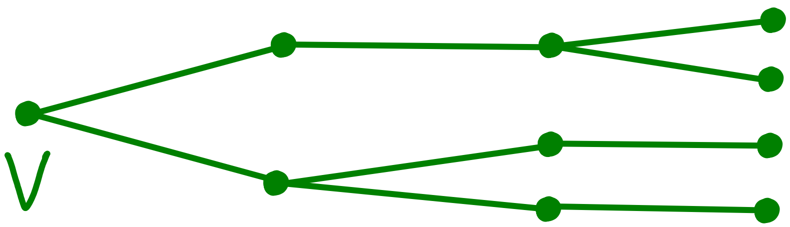
$$\mu(a_1, \ldots, b_{k-1}) = \left((a_1, b_1, \ldots, a_{k-1}, b_{k-1}), (b_{k-1}, \ldots, b_1, a_1)^{-1} \right)$$

Fission graphs (arxiv 0806, apx C) $\quad G = GL(V)$

$$Q = A_r/z^r + \cdots + A_1/z \qquad (A_i \in \mathfrak{t})$$

$$= A_r w^r + \cdots + A_1 w \qquad w = 1/z$$

$r = 3$:



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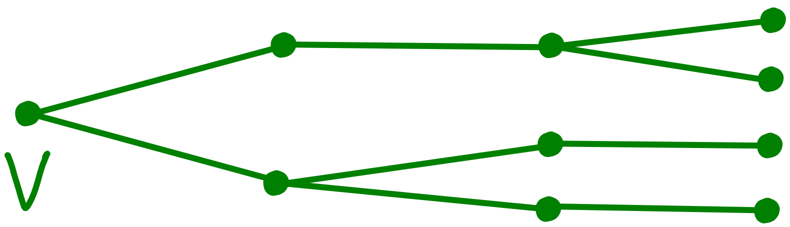

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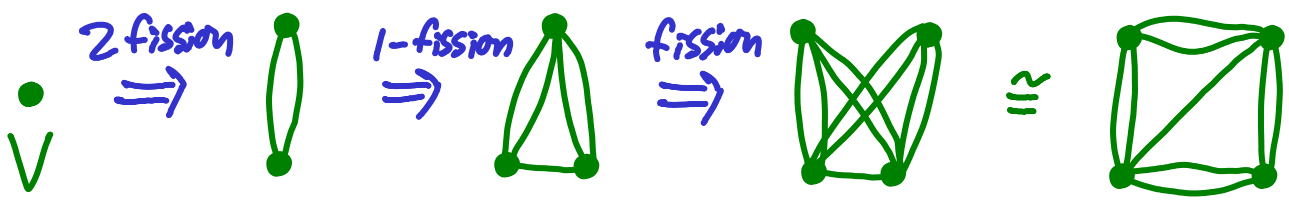
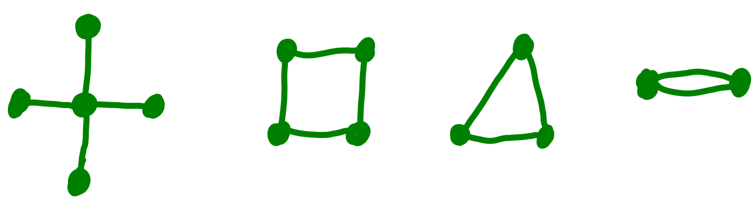
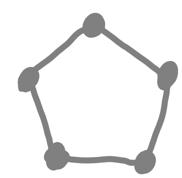


"fission graph" $\Gamma(Q)$

- $r=2$ get all complete k-partite graphs
- e.g.



but not



$Q = \mathrm{diag}(q_1, \dots, q_n) \Rightarrow$ nodes $= \{1, \dots, n\}$, # edges $i \leftrightarrow j$ $= \deg_w(q_i - q_j) - 1$

Wild Character Varieties

In this example $\left((\mathbb{P}^1, 0, Q) \quad Q = A/z^k, \quad GL_2(\mathbb{C}) \right)$

$$\mathcal{M}_B = \widetilde{\mathcal{M}}_B \, /\!/_{(q_1,q_2)} \, H$$

$$= \mathrm{Rep}^*(\Gamma, V) \, /\!/_{(q_1,q_2)} \, H$$

$\Gamma =$ 

$, \quad V = \mathbb{C} \oplus \mathbb{C}$

"multiplicative quiver variety"

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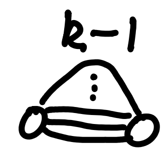
$\Gamma =$ 

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E.g. $k=3$ (Painlevé 2 Betti space)

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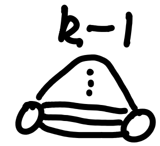
$\Gamma =$  , $V = \mathbb{C} \oplus \mathbb{C}$

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$$\boxed{\begin{array}{ccc} \mathcal{M}^* & \xRightarrow{\text{RHB}} & \mathcal{M}_B \\ \cong & & \cong \\ \mathrm{Rep}(\Gamma, V) /\!\!/_{\lambda} H & & \mathrm{Rep}^*(\Gamma, V) /\!\!/_{q} H \end{array}}$$

Conjectural classification (of $\mathcal{M}$'s) in $\dim_{\mathbb{C}} = 2$:

(Nonabelian Hodge surfaces) (1203.6607) "K2 surfaces"

E_8	E_7	E_6	D_4	$A_3 = D_3$	D_2	D_1	D_0	A_2	A_1	A_0
6	4	3	2	2	2	2	2	2	2	2
1+1+1	1+1+1	1+1+1	1+1+1+1	2+1+1	2+2	2+2	2+2	3+1	4	4

affine Weyl group

minimal rank of bundles

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Tame ← | → Wild

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D_4: P_6	$A_3 = D_3$: P_5	A_2: P_4	A_1: P_2	A_0: P_1

Phase spaces for Painlevé differential equations

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D_4 $A_3 = D_3$

D_2 D_1 D_0 — Atiyah-Hitchin

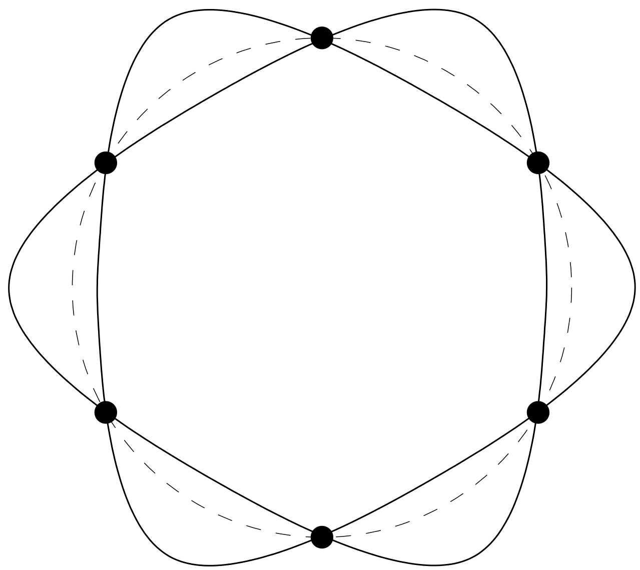
A_2 A_1 A_0

A_1: $T^*\mathbb{P}^1$, A_0: $\mathbb{C}^2$

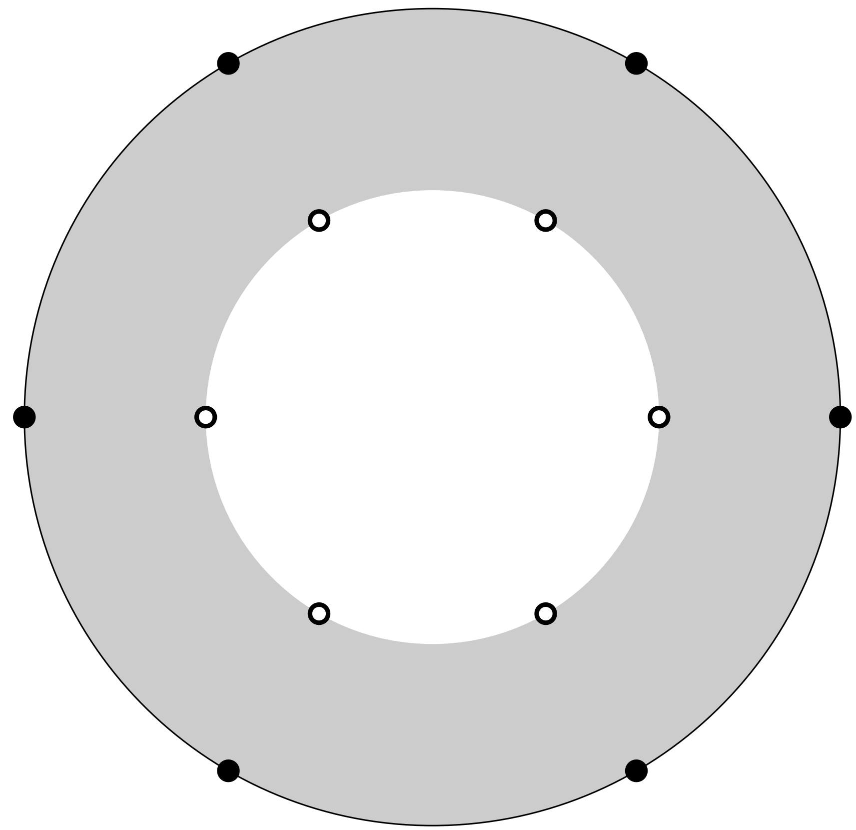
[$\mathcal{M}^* \subset \mathcal{M}$ open piece where bundle holom. trivial]

Stokes structures

(Sibuya 1975, Deligne 1978, Malgrange 1980 ...)



Stokes diagram with Stokes directions



Halo at ∞ with singular directions

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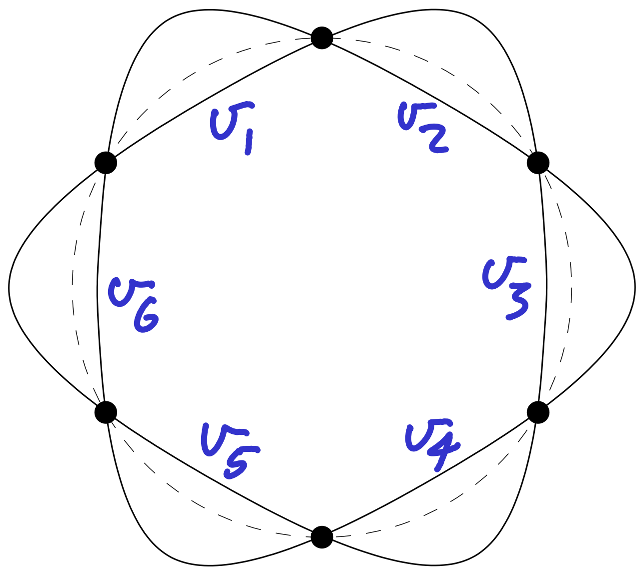

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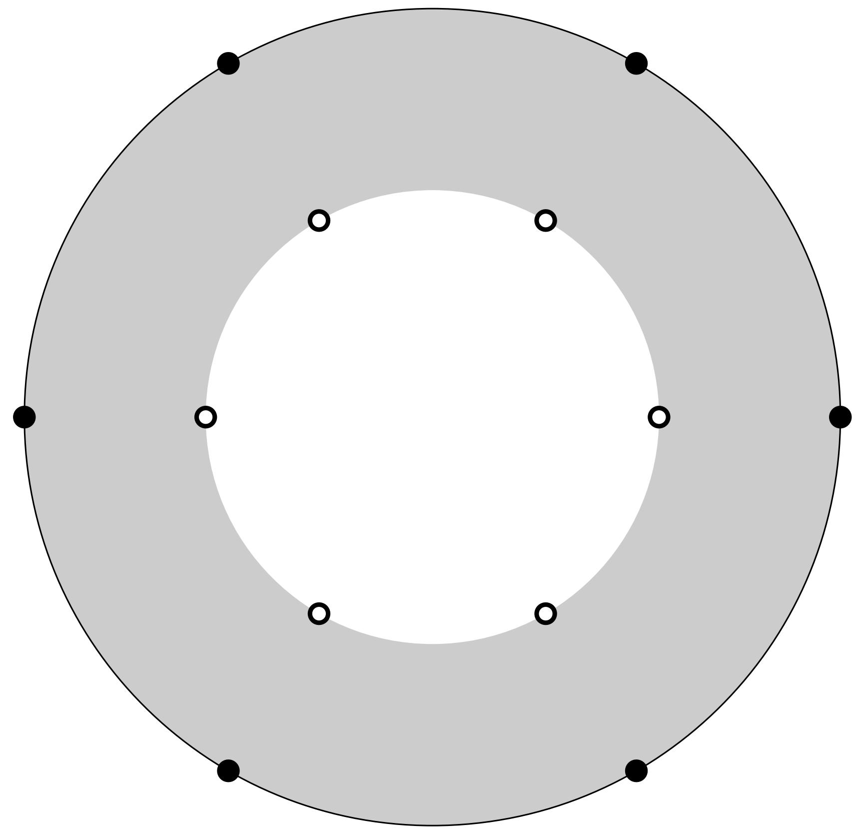

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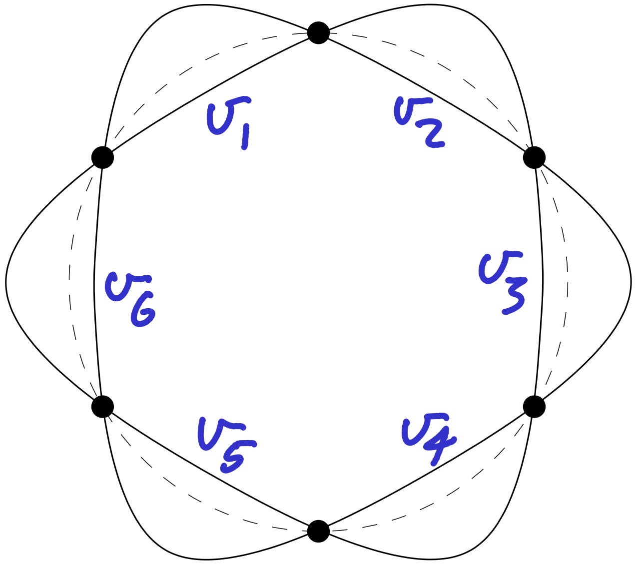

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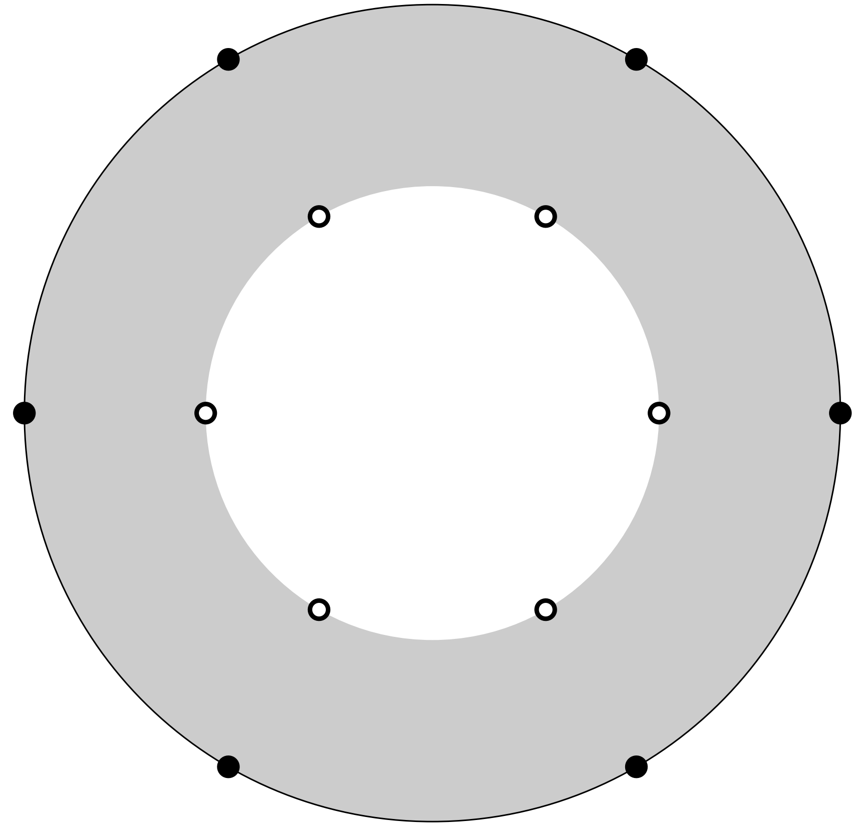

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$$\mathcal{M}_B \cong \{ xyz + x + y + z = b - b^{-1} \}$$

$$\cong \left\{ (p_1, \dots, p_6) \in (\mathbb{P}^1)^6 \;\middle|\; \begin{array}{c} p_i \neq p_{i+1} \bmod 6 \\ \dfrac{(p_1-p_2)(p_3-p_4)(p_5-p_6)}{(p_2-p_3)(p_4-p_5)(p_6-p_1)} = b^2 \end{array} \right\} \Big/ PSL_2(\mathbb{C})$$

§2 Algebras

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(Replace linear maps by symbols)

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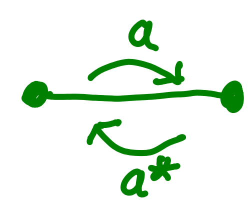

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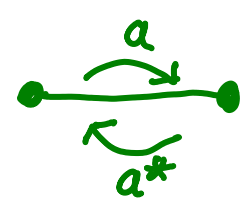

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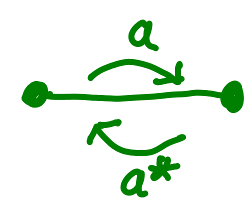

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Crawley-Boevey — Holland

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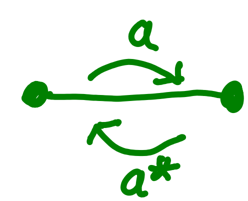

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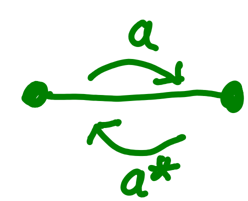

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We can now replace Van den Bergh edges $\mathrm{Rep}^*(\bullet\!\!-\!\!\bullet, V)$ by $\mathrm{Rep}^*(\Gamma, V)$ for arbitrary fission graph Γ (e.g. $\circ\overset{k}{\equiv}\circ$)

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"Fission algebras" $\mathcal{F}^q(\Gamma)$

E.g. $\Gamma = \circ \overset{k}{\equiv} \circ$ $\qquad q = (q_1, q_2) \in (\mathbb{C}^*)^I$

$$\mathcal{F}^q(\Gamma) \cong \mathbb{C}\overline{\Gamma} \Big/ \begin{pmatrix} (a_1, b_1, \dots, a_k, b_k) e_1 = q_1 e_1, \\ (b_k, a_k, \dots, b_1, a_1) e_2 = q_2^{-1} e_2 \end{pmatrix}$$

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(more examples in arXiv:1307.****)

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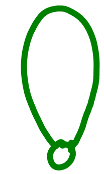
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$

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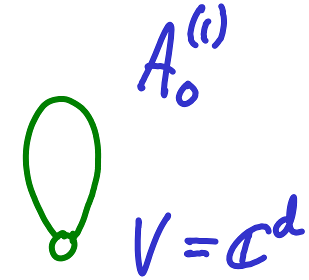
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$ 

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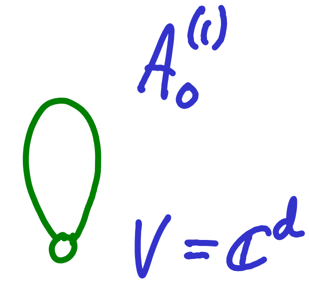
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$ 

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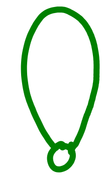
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$ $V = \mathbb{C}^d$

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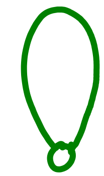
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$  $A_0^{(1)}$ $V = \mathbb{C}^d$

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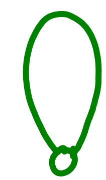
- From Okamoto expect Painlevé 1 Betti space $\sim \Gamma =$ 

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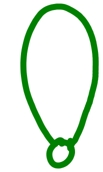
§3 Odd continuants (work with D. Yamakawa) $\Gamma =$  $V = \mathbb{C}^d$

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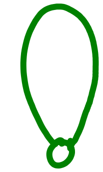
$\Gamma =$  $V = \mathbb{C}^d$

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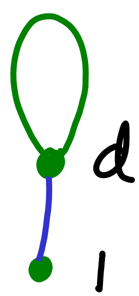
$\mathrm{Rep}^*\Big($ 

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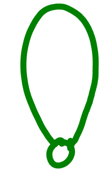
§3 Odd continuants (work with D. Yamakawa) $\Gamma =$  $V = \mathbb{C}^d$

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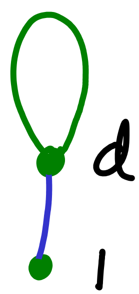
$\mathrm{Rep}^*\Big($  $, \mathbb{C}^d \oplus \mathbb{C} \Big) /\!\!/_q H \cong \mathcal{M}_B\left(h P_1^{(d)}\right)$ dim $2d$

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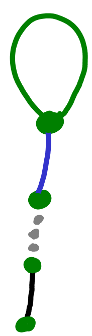
$\mathrm{Rep}^*\Big($  $, \oplus V_i \Big) /\!\!/_q H \cong \mathcal{M}_B(\text{matrix } P_1)$

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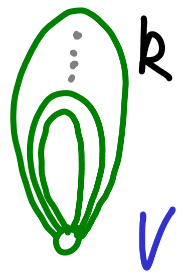
More generally if $\Gamma =$ 

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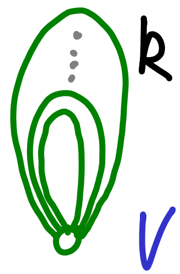
More generally if $\Gamma =$ 

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– and similarly for any twisted irregular type Q (any G)