# Calabi-Yau gauge theory with symmetries

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# Contents

Background:

- Monopoles, Instantons
- Co-homogeneity One, Asymptotically Conical

Results:

- Summary
- Sample Proof (Sketch)
- Application to $G_2$

# Local Differential Geometry

- $(M^6, g, \omega, \Omega)$ **Calabi-Yau 3-fold**, w. Riemannian metric $g$, Kähler form $\omega$, holomorphic vol. form $\Omega$.

On $U \subset M$, in local co-ordinates $(z_1, z_2, z_3) : U \to \mathbb{C}^3$:

- $g = \sum g_{ij} dz_i \otimes d\bar{z}_j$
- $\omega = -i \sum g_{ij} dz_i \wedge d\bar{z}_j$
- $\Omega = \det(g_{ij})\, dz_1 \wedge dz_2 \wedge dz_3$

where $g_{ij}$ positive, Hermitian.

# Gauge Theory

- Principal bundle $P \to M$ w. structure group $SU(2)$.
- Adjoint bundle $\mathrm{Ad}P = P \times_{SU(2)} \mathfrak{su}(2)$.

# Gauge Theory

- **Connection** form $A : TP \to \mathfrak{su}(2)$
- **Curvature** form $F_A \in \Omega^2\left(\mathrm{Ad}P\right)$
- **Covariant derivative** $\nabla_A : \Omega^0\left(\mathrm{Ad}P\right) \to \Omega^1\left(\mathrm{Ad}P\right)$
- **Section** $\Phi \in \Omega^0\left(\mathrm{Ad}P\right)$

# Local Differential Geometry

On $P|_U \cong U \times SU(2)$, with co-ordinates $(z, g)$:

- $A = g^{-1}dg + g^{-1}\, A|_U\, g$
- $\nabla_A \Phi = d\Phi + [\Phi, A|_U]$
- $F_A = d\, A|_U + [A|_U\,, A|_U]$

where $A|_U : TM|_U \to \mathfrak{su}(2)$.

### Definition

**Calabi-Yau monopole equations** for connection $A$ on $P$, $\Phi \in \Omega^0(\mathrm{Ad}P)$:

$$F_A \wedge \omega^2 = 0 \qquad\qquad F_A \wedge \mathrm{Re}\Omega = *\nabla_A \Phi$$

- $\Phi = 0$. $A$ a **Calabi-Yau instanton**.
- $\Phi \neq 0$. $(A, \Phi)$ a **Calabi-Yau monopole** w. **Higgs field** $\Phi$.

# Motivation

- Minimize **Yang-Mills-Higgs** energy $\int_M |F_A|^2 + |\nabla_A \Phi|^2$.
- Analogues of **anti-self-dual instantons** in dim. 4, **Bogomol'nyi monopoles** in dim. 3.
- Programme of **gauge-theoretic invariants** in higher dim. via moduli-space, Donaldson–Thomas–Joyce (1998).
- Analytic aspects difficult, e.g. compactness: Tian–Tao (2000) **bubbling phenomena** along calibrated sub-manifolds in co-dim. 3, 4 resp.
- Expected relationship to "count" of **complex curves**, **special Lagrangians** resp.

# Strategy

- Find sol. to CYM eqn. w. **continuous symmetries**.

$(M, g)$ Calabi-Yau $\Rightarrow \mathrm{Ric}(g) = 0$, so if $\mathrm{Hol}(g) = SU(3)$:

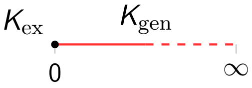
- $M$ **non-compact** w. a single end.
- $(M, g)$ cannot be homogeneous.

- $(M, g)$ **co-homogeneity one**, i.e. a (compact) Lie group $K \curvearrowright M$ with generic orbit co-dim. one.
- $P$, $(A, \Phi)$ also inv. $\Rightarrow$ PDE $\rightarrow$ ODE. Lotay–Oliveira (2018), Oliveira (2016).

# Co-homogeneity One

Co-homogeneity One:

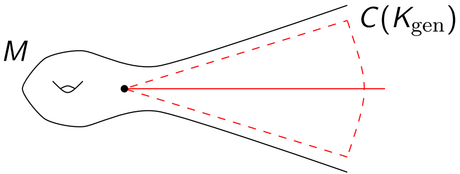
- $M/K \cong [0, \infty)$
- $M = \{\{0\} \times K_{\mathrm{ex}}\} \sqcup \{(0, \infty) \times K_{\mathrm{gen}}\}.$




- $g = dt^2 + g_t$ on $M \setminus K_{\mathrm{ex}}.$

# Asymptotically Conical (AC)

- $g_t \to t^2 g_\infty$ as $t \to \infty$, w. $g_\infty$ fixed metric on $K_{\mathrm{gen}}$




- i.e. $(M, g) \to \left(C(K_{\mathrm{gen}}), dt^2 + t^2 g_\infty\right)$ **Riemannian cone**

# Asymptotic Geometry

- Take $K = SU(2)^2$, $K_{\mathrm{gen}} = SU(2)^2/\triangle U(1) \cong S^2 \times S^3$.

## Conifold (Candelas–de la Ossa, 1990)

- Smooth mfld. $(0, \infty) \times S^2 \times S^3$.
- Complex mfld. $\{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 \mid \sum z_i^2 = 0\} \setminus \{\underline{0} \in \mathbb{C}^4\}$.
- Riemannian mfld. $C(S^2 \times S^3)$, Ricci-flat, Kähler.

# Complete Examples

- **Small resolution**: $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{CP}^1$ (Candelas–de la Ossa).
- **Smoothing**: $T^*S^3 \to S^3$ (Candelas–de la Ossa/Stenzel).
- **Canonical bundle**: $\mathcal{O}(-2,-2) \to \mathbb{CP}^1 \times \mathbb{CP}^1$, one-para. family (Pando-Zayas–Tseytlin–Calabi). Para. $\leftrightarrow$ relative vol. $\mathbb{CP}^1$.

- Asymptotic to conifold (up to cover).
- $SU(2)^2_{\mathrm{gen}} : S^2 \times S^3$.
- $SU(2)^2_{\mathrm{ex}} : \mathbb{CP}^1, S^3, \mathbb{CP}^1 \times \mathbb{CP}^1$ calibrated.

# Results: Summary

**Constr./classif.** for irr. $SU(2)^2$-inv. sol. to monopole eqn:

| | Monopoles | Instantons |
|---|---|---|
| $T^*S^3$ | 1-para. family | 1-para. family |
| $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ | none | two 1-para. families |
| $\mathcal{O}(-2,-2)$ | none | countably many 1-para. families |

- Monopoles on $T^*S^3$ (Oliveira, 2016).
- Sol. to ODE systems $\dot{\underline{x}}(t) = F(\underline{x}, t)$, $t \in [0, \infty)$.

# Asymptotic behaviour

Asymptotic models for inv. $(A, \Phi)$ as $t \to \infty$:

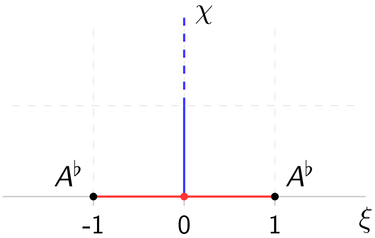
- $\Phi = 0$, $A \to A^\flat$, the **flat connection** .
- $\Phi = 0$, $A \to A^{\mathrm{can}}$ the **canonical connection**: unique non-flat inv. instanton on conifold, pulled back from $S^2 \times S^3$.
- $(A, \Phi) \to (A^{\mathrm{can}}, \Phi_\infty)$, $\Phi_\infty$ non-trivial, parallel.

# Smoothing $T^*S^3$

- Fixed bundle w. irr. inv. conn.
- Inv. instantons in 1-para. family
- Inv. monopoles in 1-para. family

# Smoothing $T^*S^3$

- Nghd of $S^3 \subset T^*S^3$, inv. sol. in **2-para.** family $S_{\xi,\chi}$.
- $\chi = 0 \Rightarrow \Phi = 0$.




- $\chi = 0$, $\xi = \pm 1$ are $A^\flat$.
- $\chi = 0$, $\xi \in (-1, 1)$ asymp. to $A^{\mathrm{can}}$
- $\xi = 0$, $\chi \in (0, \infty)$ asymp. to $(A^{\mathrm{can}}, \Phi_\chi)$

# Monopoles

## Theorem (S. 2021)

*No irr., inv. monopoles on* $\mathcal{O}(-2,-2)$, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ *w. quadratic curvature decay.*

## Remark

- No (compact, inv.) special Lagrangians in $\mathcal{O}(-2,-2)$, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$.
- Also assume *intermediate energy* $\int_M |F_A \wedge \mathrm{Re}\Omega|^2 + |\nabla_A \Phi|^2$ finite, should hold generally using Fadel–Nagy–Oliveira (2021).

# Small Resolution $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

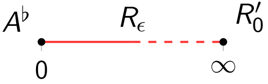
- Two bundles w. inv. irr. conn.
- Inv. instantons in one-para. family $R_\epsilon$, $\epsilon \in [0, \infty)$, resp. $R'_{\epsilon'}$, $\epsilon' \in [0, 1]$.


- $R_0$, $R'_1$ are $A^\flat$.
- $R'_0$ abelian (irr. else).
- $\epsilon \in (0, \infty)$, $\epsilon' \in [0, 1)$ asymp. to $A^{\mathrm{can}}$.

# Bubbling on $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$

As $\epsilon \to \infty$:

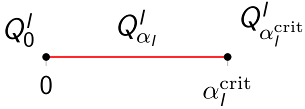
- Curvature $R_\epsilon$ concentrates at $\mathbb{CP}^1$.
- $R_\epsilon \to R_0'$ unif. on compact subsets of $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \setminus \mathbb{CP}^1$.




- $R_\epsilon$ (w. rescaling) bubbles off anti-self-dual conn. normal to $\mathbb{CP}^1 \subset \mathcal{O}(-1) \oplus \mathcal{O}(-1)$.

# Canonical Bundle $\mathcal{O}(-2,-2)$

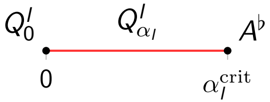
- Countably many ($l \in \mathbb{Z}$) bundles w. irr. inv. conn on $\mathcal{O}(-2,-2)$.
- For each CY str. on $\mathcal{O}(-2,-2)$: inv. instantons in one-para. families $Q^l_{\alpha_l}$, $\alpha_l \in \left[0, \alpha_l^{\mathrm{crit}}\right]$, some $0 < \alpha_l^{\mathrm{crit}} < \infty$.



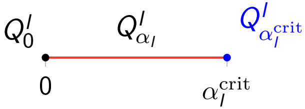
# Canonical Bundle $\mathcal{O}(-2,-2)$

- $l = 0, 1$, $\alpha_l = \alpha_l^{\mathrm{crit}}$ are $A^\flat$.
- $\alpha_l = 0$ abelian (irr. else).
- $\alpha_l \in \left[0, \alpha_l^{\mathrm{crit}}\right)$ asympt. to $A^{\mathrm{can}}$.
- $l \neq 0, 1$, $\alpha_l = \alpha_l^{\mathrm{crit}}$ asympt. to $A^\flat$ (finite YM energy).

- $l = 0, 1$




- $l \neq 0, 1$



# Bubbling on $\mathcal{O}(-2,-2)$

## Adiabatic limit

- Fix vol. copy of $\mathbb{CP}^1$.
- Rescale metric normal to fixed vol. $\mathbb{CP}^1$, shrink vol. of $\mathbb{CP}^1$ in normal direction.
- vol. $\to 0$, $\mathcal{O}(-2,-2) \to$ **Eguchi-Hanson** on $T^*\mathbb{CP}^1$ fibred over $\mathbb{CP}^1$.

- Anti-self-dual instantons on $T^*\mathbb{CP}^1$ (Nakajima, 1990).
- Rescale $Q^I_{\alpha_I} \to$ 1-para. families ASD instantons on $T^*\mathbb{CP}^1$.
- $\alpha_I \to \alpha_I^{\mathrm{crit}}$ corresp. to curv. conc. at $\infty$, bubbling off ASD instanton on $\mathbb{C}^2/\mathbb{Z}_2$.

# Sample Proof: Instantons on $\mathcal{O}(-2,-2)$.

- On $\mathcal{O}(-2,-2) \setminus \mathbb{CP}^1 \times \mathbb{CP}^1$, write CYI eq.:

$$\dot{x} = -\frac{4\lambda}{\mu^2}\left(y^2(u_1 - u_0) + xu_1 + u_0\right) \quad \dot{y} = -\frac{3}{2\lambda} y\,(x+1)$$

- As $t \to \infty$ i.e. on the conifold, CYI $\to$:

$$\dot{x} = -\frac{4}{t}\left(y^2 + x\right) \qquad \dot{y} = -\frac{3}{2t} y\,(x+1)$$

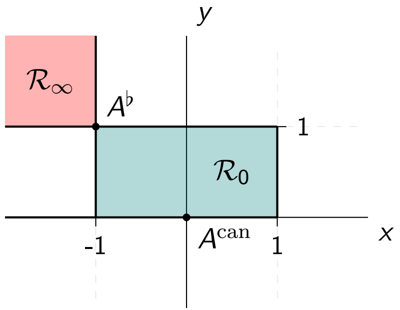
- $A^{\mathrm{can}} : (x,y) = (0,0)$, $A^{\flat} : (x,y) = (-1,1)$

# Sample Proof: Instantons on $\mathcal{O}(-2,-2)$.

- Eschenburg–Wang (2000): extend sol. $(x(t), y(t))$ to $\mathbb{CP}^1 \times \mathbb{CP}^1$ at $t = 0$.
- Power-series sol. $Q^{I}_{\alpha_I}$ to CYI near $t = 0$.
- Inv. sets for CYI (i.e. sol. inside at initial $t^*$ inside $\forall t \geq t^*$), determine sol. behaviour in inv. sets.

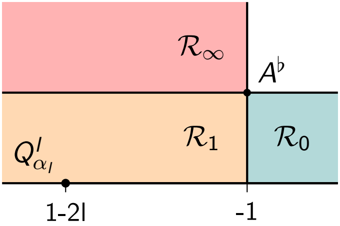
# Sample Proof: Instantons on $\mathcal{O}(-2,-2)$.

- $(x(t), y(t)) \in \mathcal{R}_\infty \Rightarrow (x(t), y(t))$ unbbd
- $(x(t), y(t)) \in \mathcal{R}_0 \Rightarrow (x(t), y(t)) \to A^{\mathrm{can}}$ as $t \to \infty$



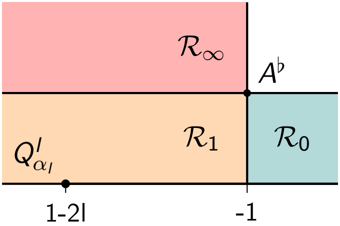
# $Q^l_{\alpha_l}$, when $l > 1$

- $Q^l_{\alpha_l}(0) = (1-2l, 0)$, para. $\alpha_l$ at $O(t^l)$ in power-series at $t = 0$.
- $Q^l_{\alpha_l}(t^*) \in \mathcal{R}_1$ at $1 \gg t^* > 0$, $\forall \alpha_l \in [0, \infty)$.
- **But** $\mathcal{R}_1$ not inv., cannot apply power-series directly.



# $Q^l_{\alpha_l}$, when $l > 1$.

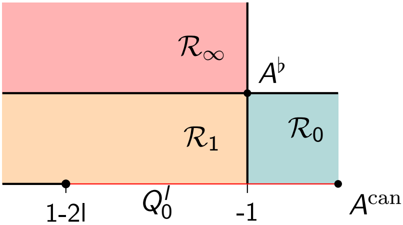
- Sol. in $\mathcal{R}_1$ must either leave in finite time via $\mathcal{R}_0$, $\mathcal{R}_\infty$, or stay in $\mathcal{R}_1$, asymp. to $A^\flat$.




- $I_0 := \{\alpha_l \in [0,\infty) \mid Q^l_{\alpha_l} \text{ enters } \mathcal{R}_0\}$,
  $I_\infty := \{\alpha_l \in [0,\infty) \mid Q^l_{\alpha_l} \text{ enters } \mathcal{R}_\infty\}$,
  $I_1 := \{\alpha_l \in [0,\infty) \mid Q^l_{\alpha_l} \text{ stays in } \mathcal{R}_1\}$

# $Q^l_{\alpha_l}$, when $l > 1$

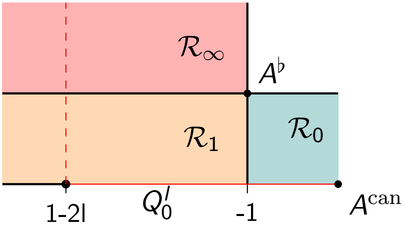
- $I_0$ non-empty, explicit (abelian) sol. $Q^l_0$ to CYI w. $y(t) = 0$.



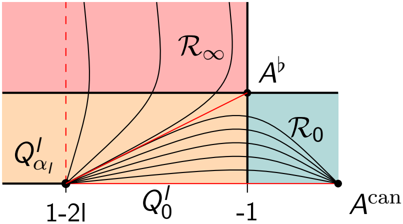
# $Q^l_{\alpha_l}$, when $l > 1$

$I_\infty$ also non-empty:

- Assume $I_\infty$ empty $\Rightarrow Q^l_{\alpha_l}(t)$ exists $\forall t \in [0, \infty)$.
- Rescale $t$, fibre normal to $\mathbb{CP}^1 \times \mathbb{CP}^1 \subset \mathcal{O}(-2,-2)$. Take $\alpha_l \to \infty$, rescaled sol. converges to vertical. Contradiction.



- Continuity $\Rightarrow I_0,\ I_\infty$ open.
- Comparison argument $\Rightarrow I_0$ , $I_\infty$ connected. Also $I_1 = \{\alpha_l^{\text{crit}}\}, \emptyset$.
- $\Rightarrow \exists \alpha_l^{\text{crit}}$ s.t. $Q^l_{\alpha_l^{\text{crit}}}(t) \to A^\flat$ as $t \to \infty$. QED.



# Further Work (ongoing, joint w. Matt Turner, Bath)

- $\exists$ 1-para. families $SU(2)^2$-inv. co-ho. 1 $G_2$-metrics, Foscolo–Haskins–Nordström (2021).
- Geometry at end: $S^1$ fixed length $\ell > 0$, fibred over conifold.
- As $\ell \to 0$, $G_2$-metrics $\to \mathcal{O}(-2,-2)$, $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$, $T^*S^3$.

- View CYM eq. as dim. red. of $G_2$-instanton eq.
- CY gauge th. $\to$ circle-inv. $G_2$-instantons near $\ell = 0$.
- (?) Interpolate between AC $G_2$ limit as $\ell \to \infty$.