Torsion waves in metric–affine field theory

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Abstract. The approach of metric–affine field theory is to define spacetime as a real oriented 4-manifold equipped with a metric and an affine connection. The 10 independent components of the metric tensor and the 64 connection coefficients are the unknowns of the theory. We write the Yang–Mills action for the affine connection and vary it both with respect to the metric and the connection. We find a family of spacetimes which are stationary points. These spacetimes are waves of torsion in Minkowski space. We then find a special subfamily of spacetimes with zero Ricci curvature; the latter condition is the Einstein equation describing the absence of sources of gravitation. A detailed examination of this special subfamily suggests the possibility of using it to model the neutrino. Our model naturally contains only two distinct types of particles which may be identified with left-handed neutrinos and right-handed antineutrinos.

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1. Main results

We consider spacetime to be a real oriented 4-manifold M equipped with a non-degenerate symmetric metric g and an affine connection Γ. The 10 independent components of the metric tensor $g_{\mu\nu}$ and the 64 connection coefficients $\Gamma^\lambda{}_{\mu\nu}$ are the unknowns, as is the manifold M itself. This approach is known as metric–affine field theory. Its origins lie in the works of authors such as É Cartan, A S Eddington, A Einstein, T Levi-Civita, E Schrödinger and H Weyl; see, for example, Appendix II in [1], or [2]. Reviews of the more recent work in this area can be found in [3, 4, 5, 6].

The Yang–Mills action for the affine connection is

$$S_{\mathrm{YM}} := \int R^\kappa{}_{\lambda\mu\nu}\, R^\lambda{}_\kappa{}^{\mu\nu} \tag{1}$$

where R is the Riemann curvature tensor (14). Variation of (1) with respect to the metric g and the connection Γ produces Euler–Lagrange equations which we, for the time being, will write symbolically as

$$\partial S_{\mathrm{YM}}/\partial g = 0\,, \tag{2}$$

$$\partial S_{\mathrm{YM}}/\partial\Gamma = 0\,. \tag{3}$$

Equation (3) is the Yang–Mills equation for the affine connection. Equation (2) does not have an established name; we will call it the *complementary Yang–Mills equation*.

Our initial objective is the study of the combined system (2), (3). This is a system of 74 real non-linear partial differential equations with 74 real unknowns.

In order to state our results we will require the Maxwell equation

$$\delta \mathrm{d}u = 0 \tag{4}$$

as well as the *polarized Maxwell equation*

$$*\mathrm{d}u = \alpha \mathrm{i}\mathrm{d}u\,, \tag{5}$$

$\alpha = \pm 1$; here u is the unknown vector function. In calling (5) the polarized Maxwell equation we are motivated by the fact that any solution of (5) is a solution of (4). We call a solution u of the Maxwell equation (4) non-trivial if $\mathrm{d}u \not\equiv 0$.

If the metric is given and the connection is known to be metric compatible then the connection coefficients are uniquely determined by torsion (13) or contortion (16). The choice of torsion or contortion for the purpose of describing a metric compatible connection is purely a matter of convenience as the two are expressed one via the other in accordance with formulae (17).

We define Minkowski space $\mathbb{M}^4$ as a real 4-manifold with a global coordinate system (x^0, x^1, x^2, x^3) and metric $g_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$. Our definition of $\mathbb{M}^4$ specifies the manifold M and the metric g, but does not specify the connection Γ.

Our first result is

Theorem 1 *Let u be a complex-valued vector function on $\mathbb{M}^4$ which is a non-trivial plane wave solution of the polarized Maxwell equation (5), let $L \neq 0$ be a constant complex antisymmetric tensor satisfying*

$$*L = \tilde{\alpha}\mathrm{i}L\,, \tag{6}$$

$\tilde{\alpha} = \pm 1$, and let Γ be the metric compatible connection corresponding to contortion

$$K^{\lambda}{}_{\mu\nu} = \mathrm{Re}(u_\mu L^{\lambda}{}_{\nu})\,. \tag{7}$$

Then the spacetime $\{\mathbb{M}^4, \Gamma\}$ is a solution of the system of equations (2), (3).

Remark 1 *In abstract Yang–Mills theory it is not customary to consider the equation (2) because there is no guarantee that this would lead to physically meaningful results. As an illustration let us examine the Maxwell equation (4) for real-valued vector functions on a Lorentzian manifold, which is the simplest example of a Yang–Mills equation. Straightforward calculations show that it does not have non-trivial solutions which are stationary points of the Maxwell action with respect to the variation of the metric.*

It is easy to see that the connections from Theorem 1 are not flat, i.e., $R \not\equiv 0$.

The non-trivial plane wave solutions of (5) can, of course, be written down explicitly: up to a proper Lorentz transformation they are

$$u(x) \;=\; w\,\mathrm{e}^{-\mathrm{i}k\cdot x} \tag{8}$$

where

$$w_\mu = C(0, 1, -\alpha\mathrm{i}, 0)\,, \qquad k_\mu = \beta(1, 0, 0, 1)\,, \tag{9}$$

$\beta = \pm 1$, and C is an arbitrary positive constant (amplitude).

Let us now introduce an additional equation into our model:

$$Ric = 0 \tag{10}$$

where Ric is the Ricci curvature tensor. This is the Einstein equation describing the absence of sources of gravitation.

Remark 2 *If the connection is that of Levi-Civita then (10) implies (3). In the general case equations (3) and (10) are independent.*

The question we are about to address is whether there are any spacetimes which simultaneously satisfy the Yang–Mills equation (3), the complementary Yang–Mills equation (2), and the Einstein equation (10). More specifically, we are interested in spacetimes whose connections are not flat and not Levi-Civita connections.

The following theorem provides an affirmative answer to the above question.

Theorem 2 *A spacetime from Theorem 1 satisfies equation (10) if and only if L is proportional to $(\mathrm{d}u)|_{x=0}$, in which case torsion equals contortion up to a natural reordering of indices:*

$$T_{\lambda\mu\nu} = K_{\mu\lambda\nu}. \tag{11}$$

When describing the spacetimes from Theorem 2 it is convenient to take $L = \mathrm{d}u$ rather than $L = (\mathrm{d}u)|_{x=0}$. This leads to a rescaling of the wave vector k which can, of course, be incorporated into a Lorentz transformation. Thus, the torsion of spacetimes from Theorem 2 can be written as

$$T_{\lambda\mu\nu} = \mathrm{Re}(u_\lambda(\mathrm{d}u)_{\mu\nu})\,. \tag{12}$$

The paper has the following structure.

Sections 3 and 4 contain the proof of Theorem 1, whereas Section 5 contains the proof of Theorem 2. The central elements of our construction are the linearization ansatz (Lemma 2) and the double duality ansatz (Lemma 3).

The rest of the paper is a detailed examination of the spacetimes from Theorem 2.

In Section 6 we establish general invariant properties of the spacetimes from Theorem 2. In particular, it turns out (Lemma 6) that their Riemann curvature tensors possess *all* the symmetry properties of the "usual" curvature tensors generated by Levi-Civita connections. This means that in observing such connections we might be led to believe (mistakenly) that we live in a Levi-Civita universe.

In Section 7 we show that the Riemann curvature tensors corresponding to spacetimes from Theorem 2 have an algebraic structure which makes them equivalent to bispinors. It turns out (Lemma 8) that these bispinors satisfy the Weyl equation, which suggests the possibility of interpreting such spacetimes as a model for the neutrino. Our model naturally contains only two distinct types of particles which may be identified with left-handed neutrinos and right-handed antineutrinos.

Finally, in Section 8 we compare our results with those of Einstein [7] who performed a double duality analysis of Riemann curvatures with the aim of modelling elementary particles. We show that the spacetimes from Theorem 2 are in agreement with the results of Einstein's analysis, in that we get the sign predicted by Einstein.

2. Notation

We denote $\partial_\mu = \partial/\partial x^\mu$ and define the covariant derivative of a vector function as $\nabla_\mu v^\lambda := \partial_\mu v^\lambda + \Gamma^\lambda{}_{\mu\nu} v^\nu$. We define the torsion tensor as

$$T^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \Gamma^\lambda{}_{\nu\mu}\,, \tag{13}$$

the Riemann curvature tensor as

$$R^\kappa{}_{\lambda\mu\nu} := \partial_\mu \Gamma^\kappa{}_{\nu\lambda} - \partial_\nu \Gamma^\kappa{}_{\mu\lambda} + \Gamma^\kappa{}_{\mu\eta}\Gamma^\eta{}_{\nu\lambda} - \Gamma^\kappa{}_{\nu\eta}\Gamma^\eta{}_{\mu\lambda}\,, \tag{14}$$

and the Ricci curvature tensor as $Ric_{\lambda\nu} := R^\kappa{}_{\lambda\kappa\nu}$.

We employ the usual convention of raising or lowering tensor indices by contraction with the contravariant or covariant metric tensor. Some care is, however, required when performing covariant differentiation: the operations of raising or lowering of indices do not commute with the operation of covariant differentiation unless the connection is metric compatible.

By d we denote the exterior derivative and by δ its adjoint. Of course, these operators do not depend on the connection.

Given a scalar function f we write for brevity

$$\int f := \int_M f\,\sqrt{|\det g|}\,dx^0dx^1dx^2dx^3\,, \quad \det g := \det(g_{\mu\nu})\,. \tag{15}$$

Throughout the paper we work only in coordinate systems with positive orientation. Moreover, when we restrict our consideration to Minkowski space we assume that our coordinate frame is obtained from a given reference frame by a proper Lorentz transformation. We use these conventions when defining the notions of left-handedness and right-handedness, as well as those of the forward and backward light cone.

We define the Hodge star as $(*Q)_{\mu_{q+1}\ldots\mu_4} := (q!)^{-1}\sqrt{|\det g|}\,Q^{\mu_1\ldots\mu_q}\varepsilon_{\mu_1\ldots\mu_4}$ where ε is the totally antisymmetric quantity, $\varepsilon_{0123} := +1$.

When dealing with a connection which is compatible with a given metric it is convenient to introduce the *contortion* tensor

$$K^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \left\{ {\lambda \atop \mu\nu} \right\} \tag{16}$$

where $\left\{ {\lambda \atop \mu\nu} \right\} := \frac{1}{2} g^{\lambda\kappa}(\partial_\mu g_{\nu\kappa} + \partial_\nu g_{\mu\kappa} - \partial_\kappa g_{\mu\nu})$ is the Christoffel symbol. Contortion has the antisymmetry property $K_{\lambda\mu\nu} = -K_{\nu\mu\lambda}$. A metric and contortion uniquely determine the metric compatible connection. Torsion and contortion are related as

$$T^\lambda{}_{\mu\nu} = K^\lambda{}_{\mu\nu} - K^\lambda{}_{\nu\mu}\,, \qquad K^\lambda{}_{\mu\nu} = \tfrac{1}{2}(T^\lambda{}_{\mu\nu} + T_\mu{}^\lambda{}_\nu + T_\nu{}^\lambda{}_\mu)\,, \tag{17}$$

see formula (7.35) in [8].

The remainder of this section is devoted to the special case of Minkowski space.

Lorentz transformations are assumed to be "passive" in the sense that we transform the coordinate system and not the tensors or spinors themselves.

Consider a complex-valued tensor or spinor function of the form const $\times\ \mathrm{e}^{-\mathrm{i}k\cdot x}$ where $k \ne 0$ is a constant real vector and $k \cdot x := k_\mu x^\mu$. We call such a function a plane wave and the vector k a wave vector. In defining a plane wave as $\sim \mathrm{e}^{-\mathrm{i}k\cdot x}$ rather than $\sim \mathrm{e}^{\mathrm{i}k\cdot x}$ we follow the convention of [9, 10, 11]. We say that a lightlike wave vector k lies on the forward (respectively, backward) light cone if $k_0 > 0$ (respectively, $k_0 < 0$).

A bispinor is a column of four complex numbers $(\ \xi^1 \quad \xi^2 \quad \eta_{\dot 1} \quad \eta_{\dot 2}\ )^T$ which change under Lorentz transformations in a particular way, see Section 18 in [10] for details. The Pauli and Dirac matrices are

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma^0 = \begin{pmatrix} 0 & -\sigma^0 \\ -\sigma^0 & 0 \end{pmatrix}, \qquad \gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix}, \quad j = 1, 2, 3,$$

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} \sigma^0 & 0 \\ 0 & -\sigma^0 \end{pmatrix}.$$

We chose the sign of γ^5 as in [11] (in [10] it is opposite).

3. Solving the Yang–Mills equation

When dealing with the Yang–Mills equation it is convenient to use matrix notation to hide two indices: $R_{\mu\nu} = R^\kappa{}_{\lambda\mu\nu}$, $\Gamma_\rho = \Gamma^\kappa{}_{\rho\lambda}$, with κ enumerating the rows and λ the columns. Formulae (1), (14) can be rewritten in this notation as

$$S_{\mathrm{YM}} := \int \mathrm{tr}(R_{\mu\nu}\, R^{\mu\nu})\,, \tag{18}$$

$$R_{\mu\nu} = \partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + [\Gamma_\mu, \Gamma_\nu]\,, \tag{19}$$

where $\mathrm{tr}\, L := L^\kappa{}_\kappa$ (trace of a matrix) and $[L, N]^\tau{}_\lambda := L^\tau{}_\kappa N^\kappa{}_\lambda - N^\tau{}_\kappa L^\kappa{}_\lambda$ (commutator of matrices). Straightforward analysis of formulae (18), (19), (15) shows that the Yang–Mills equation which we initially wrote down symbolically as (3) is actually

$$(\partial_\mu + [\Gamma_\mu,\ \cdot\ ])(\sqrt{|\det g|}\, R^{\mu\nu}) = 0\,. \tag{20}$$

From now on we work only in Minkowski space and only with metric compatible connections. This leads to a number of simplifications. Connection coefficients now coincide with contortion, for which we continue using matrix notation $K_\rho = K^\kappa{}_{\rho\lambda}$. Formula (19) becomes

$$R_{\mu\nu} = \partial_\mu K_\nu - \partial_\nu K_\mu + [K_\mu, K_\nu]\,, \tag{21}$$

and the Yang–Mills equation (20) becomes

$$(\partial_\nu + [K_\nu,\,\cdot\,])R^{\mu\nu} = 0\,. \tag{22}$$

The Yang–Mills equation (22) appears to be overdetermined as it is a system of 64 equations with only 24 unknowns (24 is the number of independent components of the contortion tensor). However 40 of the 64 equations are automatically fulfilled. This is a consequence of the fact that the 6-dimensional Lie algebra of real antisymmetric rank 2 tensors is a subalgebra of the 16-dimensional general Lie algebra of real rank 2 tensors.

The fundamental difficulty with the Yang–Mills equation is that it is nonlinear with respect to the unknown contortion K. The following lemma plays a crucial role in our construction by allowing us to get rid of the nonlinearities.

Lemma 1 *If L is an eigenvector of the Hodge star then* $[\mathrm{Re}L, \mathrm{Im}L] = 0$.

Proof of Lemma 1 The result follows from the general formula $[*L, N] = *[L, N]$. □

Lemma 1 can be rephrased in the following way: the 6-dimensional Lie algebra of real antisymmetric rank 2 tensors has 2-dimensional abelian subalgebras which can be explicitly described in terms of the eigenvectors of the Hodge star.

Lemma 1 immediately implies the following *linearization ansatz.*

Lemma 2 *Suppose contortion is of the form (7) where u is a complex-valued vector function and $L \neq 0$ is a constant complex antisymmetric tensor satisfying (6). Then the nonlinear terms in the formula for Riemann curvature (21) and in the Yang–Mills equation (22) vanish.*

Substituting (7) into (21) and the latter into (22) we see that the Yang–Mills equation reduces to the Maxwell equation (4) for the complex-valued vector function u.

4. Solving the complementary Yang–Mills equation

Straightforward analysis of formulae (1), (15) shows that the complementary Yang–Mills equation which we initially wrote down symbolically as (2) is actually

$$H - \tfrac{1}{4}\,(\mathrm{tr}\,H)\,\delta = 0 \tag{23}$$

where $H = H_\nu{}^\rho := R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_\kappa{}^{\mu\rho}$ and $\delta = \delta_\nu{}^\rho$ is the identity tensor. Note the important difference between the Yang–Mills equation (20) and the complementary Yang–Mills equation (23): equation (20) is linear in curvature, whereas (23) is quadratic.

Equation (23) was written down without any assumptions on the connection. We, however, will be interested in solving (23) in the class of spacetimes with metric compatible connections, in which case the Riemann curvature tensor has the symmetries

$$R_{\kappa\lambda\mu\nu} = -R_{\lambda\kappa\mu\nu} = -R_{\kappa\lambda\nu\mu}\,. \tag{24}$$

Let $\mathcal{R}$ be the 36-dimensional linear space of real rank 4 tensors satisfying (24). We define in $\mathcal{R}$ the following two commuting endomorphisms

$$R \to {}^*R\,, \qquad ({}^*R)_{\kappa\lambda\mu\nu} := \tfrac{1}{2}\sqrt{|\det g|}\;\varepsilon^{\kappa'\lambda'}{}_{\kappa\lambda}\,R_{\kappa'\lambda'\mu\nu}\,,$$

$$R \to R^* , \qquad (R^*)_{\kappa\lambda\mu\nu} := \tfrac{1}{2}\sqrt{|\det g|}\; R_{\kappa\lambda\mu'\nu'}\,\varepsilon^{\mu'\nu'}{}_{\mu\nu} ,$$

and we also consider their composition

$$R \to {}^*R^* . \tag{25}$$

Clearly, the endomorphism (25) has eigenvalues ± 1.

Remark 3 *It is easy to see that the endomorphism (25) is well defined even if the manifold is not orientable. This observation is related to a much deeper fact established in [12]: the rank 8 tensor* $(\det g)\,\varepsilon_{\kappa'\lambda'\kappa\lambda}\,\varepsilon_{\mu'\nu'\mu\nu}$ *is a purely metrical quantity, i.e., it is expressed via the metric tensor.*

The following lemma is the *double duality ansatz* which reduces the complementary Yang–Mills equation to an equation linear in curvature.

Lemma 3 *If* $R \in \mathcal{R}$ *is an eigenvector of (25) then it satisfies (23).*

Proof of Lemma 3 We have

$$H_\nu{}^\rho = R^\kappa{}_{\lambda\mu\nu} R^\lambda{}_\kappa{}^{\mu\rho} = R_{\kappa\lambda\nu\mu} R^{\kappa\lambda\mu\rho} = \tfrac{1}{2}(R_{\kappa\lambda\nu\mu} R^{\kappa\lambda\mu\rho} + ({}^*R^*)_{\kappa\lambda\nu\mu}({}^*R^*)^{\kappa\lambda\mu\rho}) . \tag{26}$$

For antisymmetric rank 2 tensors we have the identities

$$(*L)_{\kappa\lambda}(*N)^{\kappa\lambda} = -L_{\kappa\lambda}N^{\kappa\lambda} ,$$

$$(*L)_{\nu\mu}(*N)^{\mu\rho} + (*N)_{\nu\mu}(*L)^{\mu\rho} = L_{\nu\mu}N^{\mu\rho} + N_{\nu\mu}L^{\mu\rho} + L_{\mu\tau}N^{\mu\tau}\delta_\nu{}^\rho ,$$

so formula (26) can be continued as

$$\begin{aligned} H_\nu{}^\rho &= \tfrac{1}{2}(R_{\kappa\lambda\nu\mu} R^{\kappa\lambda\mu\rho} - (R^*)_{\kappa\lambda\nu\mu}(R^*)^{\kappa\lambda\mu\rho}) \\ &= \tfrac{1}{4}((R_{\kappa\lambda\nu\mu} R^{\kappa\lambda\mu\rho} + R^{\kappa\lambda}{}_{\nu\mu} R_{\kappa\lambda}{}^{\mu\rho}) - ((R^*)_{\kappa\lambda\nu\mu}(R^*)^{\kappa\lambda\mu\rho} + (R^*)^{\kappa\lambda}{}_{\nu\mu}(R^*)_{\kappa\lambda}{}^{\mu\rho})) \\ &= -\tfrac{1}{4} R_{\kappa\lambda\mu\tau} R^{\kappa\lambda\mu\tau} \delta_\nu{}^\rho . \end{aligned}$$

The tensor $H_\nu{}^\rho$ is proportional to the identity tensor $\delta_\nu{}^\rho$, therefore it satisfies (23). $\square$

Let us now apply Lemma 3 to the spacetimes constructed in the previous section. In view of Lemma 2 the Riemann curvature in this case is

$$R_{\kappa\lambda\mu\nu} = \mathrm{Re}(L_{\kappa\lambda}(\mathrm{d}u)_{\mu\nu}) \tag{27}$$

where L is an eigenvector of the Hodge star and u is a non-trivial complex-valued solution of the Maxwell equation (4). Clearly, (27) is an eigenvector of the endomorphism (25) if and only if $\mathrm{d}u$ is an eigenvector of the Hodge star. The latter means that u is a solution of the polarized Maxwell equation (5). The proof of Theorem 1 is complete.

5. Solving the Einstein equation

Substituting (27) into the Einstein equation (10) we get $\mathrm{Re}(L^\kappa{}_\lambda(\mathrm{d}u)_{\kappa\nu})=0$. As the expression under the Re sign is a plane wave, the latter is equivalent to

$$L^\kappa{}_\lambda((\mathrm{d}u)|_{x=0})_{\kappa\nu}=0\,. \tag{28}$$

It is convenient to perform further calculations in the coordinate system in which u has the canonical form (8), (9). Then

$$(\mathrm{d}u)_{\kappa\nu}=C\,\mathrm{i}\begin{pmatrix}0&-1&\alpha\mathrm{i}&0\\1&0&0&1\\-\alpha\mathrm{i}&0&0&-\alpha\mathrm{i}\\0&-1&\alpha\mathrm{i}&0\end{pmatrix}\mathrm{e}^{-\mathrm{i}\beta(x^0+x^3)} \tag{29}$$

and (28) becomes an explicit system of linear algebraic equations with respect to the unknown components of the tensor L; namely, it is a system of 16 equations with 3 unknowns (recall that L has to be an eigenvector of the Hodge star). Elementary analysis shows that equation (28) is satisfied if and only if L is proportional to $(\mathrm{d}u)|_{x=0}$. Finally, formula (11) is established by straightforward calculations (see also Lemma 5 in the next section). The proof of Theorem 2 is complete.

6. Invariant properties of our solutions

It is known [4, 5, 6] that the 24-dimensional space of real torsions decomposes into the following 3 irreducible subspaces: tensor torsions, trace torsions, and axial torsions. The dimensions of these subspaces are 16, 4, and 4, respectively.

Lemma 4 *The torsions of spacetimes from Theorem 2 are purely tensor.*

Proof of Lemma 4 The trace component of a torsion tensor $T_{\lambda\mu\nu}$ is zero if $T^\lambda{}_{\lambda\nu}=0$, and the axial component is zero if $T_{\lambda\mu\nu}\,\varepsilon^{\lambda\mu\nu\kappa}=0$. These identities are established by direct examination of the explicit formulae (12), (8), (9). □

It has been suggested [13] to interpret the axial component of torsion as the Hodge dual of the electromagnetic vector potential. If one takes this point of view then Lemma 4 implies that in spacetimes from Theorem 2 the electromagnetic field is zero.

Let us mention (without proof) the following useful general result.

Lemma 5 *Equation (11) is satisfied if and only if the axial component of torsion is zero.*

Lemmas 4 and 5 imply that when working with spacetimes from Theorem 2 one can switch from contortion to torsion and back without acquiring cumbersome expressions.

Lemma 6 *The Riemann curvatures of spacetimes from Theorem 2 have all the symmetry properties of Riemann curvatures in the Levi-Civita setting, that is,*

$$R_{\kappa\lambda\mu\nu}=-R_{\lambda\kappa\mu\nu}=-R_{\kappa\lambda\nu\mu}=R_{\mu\nu\kappa\lambda}\,, \tag{30}$$

$$R_{\kappa\lambda\mu\nu}\,\varepsilon^{\kappa\lambda\mu\nu}=0\,. \tag{31}$$

Proof of Lemma 6 Let us define the complex Riemann curvature tensor

$$\mathbb{C}R_{\kappa\lambda\mu\nu} := F_{\kappa\lambda}F_{\mu\nu} \tag{32}$$

where

$$F := \mathrm{d}u \tag{33}$$

and u is a plane wave solution of (5). Then the Riemann curvature generated by torsion (12) can be written as

$$R = \mathrm{Re}\,\mathbb{C}R \tag{34}$$

(cf. (27)). Direct examination of formulae (32)–(34), (29) establishes the identities (30), (31). $\square$

7. Weyl's equation

The torsions (and, therefore, spacetimes) from Theorem 2 are described, up to a proper Lorentz transformation and a scaling factor $C > 0$, by a pair of indices $\alpha, \beta = \pm 1$; see (12), (8), (9). It may seem that this gives us 4 essentially different spacetimes. However, formula (12) contains the operation of taking the real part and, as a result, the transformation $\{\alpha, \beta\} \to \{-\alpha, -\beta\}$ does not change our torsion. Thus, Theorem 2 provides us with only two essentially different spacetimes labelled by the product $\alpha\beta = \pm 1$. The purpose of this section is to show that it is natural to interpret these two spacetimes as the neutrino and antineutrino.

We base our interpretation on the analysis of the Riemann curvature tensor. We choose to deal with curvature rather than with torsion because curvature is an accepted physical observable.

We will work with the complex curvature (32) rather than the real curvature (34) because the complex one has a simpler structure. Indeed, according to formula (32) the complex Riemann curvature tensor $\mathbb{C}R$ factorizes as the square of a rank 2 tensor F and is, therefore, completely determined by it.

Working with the rank 2 tensor F is much easier than with the original rank 4 tensor $\mathbb{C}R$, but one would like to simplify the analysis even further by factorizing F itself. It is impossible to factorize F as the square of a vector but it is possible to factorize F as the square of a bispinor.

Lemma 7 *A complex rank 2 antisymmetric tensor F satisfying the conditions*

$$F_{\mu\nu}F^{\mu\nu} = 0, \qquad (*F)_{\mu\nu}F^{\mu\nu} = 0 \tag{35}$$

is equivalent to a bispinor ψ, the relationship between the two being

$$F^{\mu\nu} = -\tfrac{\mathrm{i}}{4}\,\psi^T\gamma^0\gamma^2\gamma^\mu\gamma^\nu\psi\,. \tag{36}$$

Proof of Lemma 7 Formula (36) is a special case of the general equivalence relation between rank 2 antisymmetric tensors and rank 2 symmetric bispinors, see end of Section 19 in [10]. Conditions (35) are necessary and sufficient for the factorization of the symmetric rank 2 spinors as squares of rank 1 spinors. $\square$

Remark 4 *The corresponding text in the end of Section 19 in [10] contains mistakes. These can be corrected by replacing everywhere* i *by* −i.

Remark 5 *For a given* F *formula (36) defines the individual spinors* $\xi = (\ \xi^1 \quad \xi^2\ )^T$ *and* $\eta = (\ \eta_{\dot 1} \quad \eta_{\dot 2}\ )^T$ *uniquely up to choice of sign. This is in agreement with the general fact that a spinor does not have a specific sign, see the beginning of Section 19 in [10].*

Remark 6 *Conditions (35) are equivalent to* $\det F = 0$, $\det *F = 0$.

Our particular tensor F defined by formula (33) satisfies conditions (35). Indeed, $F_{\mu\nu}F^{\mu\nu} = 0$ is the statement that the complex scalar curvature is zero (consequence of the complex Ricci curvature being zero), whereas $(*F)_{\mu\nu}F^{\mu\nu} = 0$ is the statement that the complex Riemann curvature satisfies the cyclic sum identity, cf. (31). Thus, the complex Riemann curvature tensor (32) has an algebraic structure which makes it equivalent to a bispinor. We will now establish which equations this bispinor satisfies.

We say that two solutions u and u' of the Maxwell equation (4) belong to the same equivalence class if $\mathrm{d}u = \mathrm{d}u'$. We say that two bispinor functions ψ and ψ' belong to the same equivalence class if $\psi = \pm\psi'$.

Lemma 8 *Formula (36) establishes a one–to–one correspondence between the equivalence classes of non-trivial plane wave solutions of the polarized Maxwell equation (5) and of the system*

$$\gamma^\mu \partial_\mu \psi = 0\,, \tag{37}$$

$$\gamma^5 \psi = \alpha\psi\,. \tag{38}$$

Equation (37) is, of course, the Weyl equation (Dirac equation for massless particle).

Proof of Lemma 8 If u is a non-trivial plane wave solution of the polarized Maxwell equation (5) then, up to a proper Lorentz transformation, our tensor F is given by formula (29), where C is a positive constant. If ψ is a non-trivial plane wave solution of the system (37), (38) then, up to a proper Lorentz transformation,

$$\psi = \pm\sqrt{C}\,\mathrm{i}\begin{pmatrix} 0 \\ 1+\alpha \\ \mathrm{i}-\alpha\mathrm{i} \\ 0 \end{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}}{2}\beta(x^0+x^3)} \tag{39}$$

where C is a positive constant. Straightforward calculations show that the tensor function (29) and the bispinor function (39) are related in accordance with formula (36). $\square$

The parameter $\beta = \pm1$ in formula (39) determines whether the wave vector lies on the forward ($\beta = +1$) or backward ($\beta = -1$) light cone. Non-trivial plane wave solutions of (37), (38) whose wave vector lies on the forward light cone are called neutrinos whereas those whose wave vector lies on the backward light cone are called antineutrinos.

The parameter $\alpha = \pm1$ in formula (39) determines whether the solution is left- or right-handed. A neutrino is said to be left-handed if $\alpha = -1$ and right-handed if $\alpha = +1$. An antineutrino is said to be left-handed if $\alpha = +1$ and right-handed if $\alpha = -1$.

Remark 7 *The above definitions of left- and right-handedness are given in terms of helicity. See Section 2-4-3 in [11] for a detailed explanation of why one should use helicity rather than chirality for these purposes.*

As explained in the beginning of this section, the transformation $\{\alpha,\beta\}\to\{-\alpha,-\beta\}$ does not change the resulting spacetime. This means that the torsion wave which models the left-handed neutrino is identical to that for the left-handed antineutrino, and the torsion wave which models the right-handed neutrino is identical to that for the right-handed antineutrino. Thus, our model contains as many distinct types of neutrinos as are currently observed experimentally.

8. Einstein's double duality analysis

Let us examine in more detail the linear space of Riemann curvatures $\mathcal{R}$ introduced in Section 4. For $R\in\mathcal{R}$ we define its transpose R^T as $(R^T)_{\kappa\lambda\mu\nu}:=R_{\mu\nu\kappa\lambda}$. We consider the following two commuting endomorphisms in $\mathcal{R}$:

$$R\to R^T \tag{40}$$

and (25). The endomorphisms (40) and (25) have no associated eigenvectors and their eigenvalues are ±1. Therefore, $\mathcal{R}$ decomposes into a direct sum of 4 invariant subspaces

$$\mathcal{R}=\oplus_{a,b=\pm}\mathcal{R}_{ab}\,,\qquad \mathcal{R}_{ab}:=\{R\in\mathcal{R}\mid R^T=aR,\ {}^*R^*=bR\}\,. \tag{41}$$

The decomposition (41) was suggested in [14] and analyzed in [7, 12]. Actually, [14, 7, 12] dealt only with the case of a Levi-Civita connection, but the generalization to an arbitrary metric compatible connection is straightforward. Lanczos called tensors $R\in\mathcal{R}$ self-dual (respectively, antidual) if ${}^*R^*=-R$ (respectively, ${}^*R^*=R$). Such a choice of terminology is due to the fact that Einstein and Lanczos defined their double duality endomorphism as

$$R\to(\operatorname{sgn}\det g)\ {}^*R^* \tag{42}$$

rather than as (25). The advantage of (42) is that this linear operator is expressed via the metric tensor as a rational function. The endomorphism (42) is, in a sense, even more invariant than (25) because it does not "feel" the signature of the metric.

Lemma 9 (Rainich [14]) *The subspaces $\mathcal{R}_{++}$ and $\mathcal{R}_{+-}$ have dimensions 9 and 12, respectively.*

Remark 8 *In Rainich's article the dimensions are actually given as 9 and 11. The reason behind this is that Rainich imposed on curvatures the cyclic sum condition (31). This excludes from $\mathcal{R}_{+-}$ curvatures of the type $R_{\kappa\lambda\mu\nu}=\mathrm{const}\times\varepsilon_{\kappa\lambda\mu\nu}$ and, therefore, reduces the dimension by 1.*

Lemma 10 (Einstein [7]) *Let $R\in\mathcal{R}_{++}$. Then the corresponding Ricci tensor is symmetric and trace free. Moreover, R is uniquely determined by its Ricci tensor and the metric tensor according to the formula*

$$R_{\kappa\lambda\mu\nu}=\tfrac12(g_{\kappa\mu}Ric_{\lambda\nu}+g_{\lambda\nu}Ric_{\kappa\mu}-g_{\kappa\nu}Ric_{\lambda\mu}-g_{\lambda\mu}Ric_{\kappa\nu})\,. \tag{43}$$

Einstein's goal in [7] was to construct a relativistic model for the electron; note that this paper was published in 1927, a year before Dirac published his equation. (For a basic exposition of [7] in English see the Introduction in [12].) Einstein based his search for a mathematical model on the decomposition (41). As in this particular paper Einstein restricted his analysis to the case of a Levi-Civita connection he had to make the choice between the invariant subspaces $\mathcal{R}_{++}$ and $\mathcal{R}_{+-}$. The difference between these two invariant subspaces is fundamental: it has nothing to do with the choice between forward and backward light cones or the choice of orientation of the coordinate system, and, as a consequence, it has nothing to do with the notions of "particle" and "antiparticle" or the notions of "left-handedness" and "right-handedness".

Lemmas 9 and 10 led Einstein to the conclusion that curvatures from $\mathcal{R}_{++}$ are too trivial and the dimension of the subspace too low to associate it with the electron. Namely, the main argument against $\mathcal{R}_{++}$ is that dimension 9 is not enough to account for the 10 independent components of the energy–momentum tensor. This suggests that if the electron were to be modelled in terms of General Relativity then one would expect its Riemann curvature to lie in the invariant subspace $\mathcal{R}_{+-}$, that is, satisfy the equation

$$ {}^*R^* = -R\,. \tag{44} $$

Formulae (32)–(34), (5) imply that the spacetimes from Theorem 2 satisfy equation (44). Our paper falls short of constructing a metric–affine field model for the electron, but, nevertheless, we find it encouraging that our metric–affine field model for the neutrino agrees with the results of Einstein's analysis.

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References

[1] Einstein A 1960 *The meaning of relativity* 6th edition (London: Methuen & Co)
[2] Schrödinger E 1985 *Space-time structure* (Cambridge: CUP)
[3] Mielke E W 1987 *Geometrodynamics of gauge fields* (Berlin: Akademie-Verlag)
[4] Hehl F W, McCrea J D, Mielke E W and Ne'eman Y *Phys. Rep.* **258** 1–171
[5] Gronwald F *Int. J. Mod. Phys.* D **6** 263–303
[6] Hehl F W and Macias A *Int. J. Mod. Phys.* D **8** 399–416
[7] Einstein A *Mathematische Annalen* **97** 99–103
[8] Nakahara M 1998 *Geometry, Topology and Physics* (Bristol: IOP)
[9] Landau L D and Lifshitz E M 1975 *The Classical Theory of Fields*, Course of Theoretical Physics vol. **2**, 4th edition (Oxford: Pergamon Press)
[10] Berestetskii V B, Lifshitz E M and Pitaevskii L P 1982 *Quantum Electrodynamics*, Course of Theoretical Physics vol. **4**, 2nd edition (Oxford: Pergamon Press)
[11] Itzykson C and Zuber J-B 1980 *Quantum field theory* (New York: McGraw-Hill)
[12] Lanczos C *Rev. Mod. Phys.* **34** 379–389
[13] Vassiliev D 2000 A tensor interpretation of the 2D Dirac equation *Preprint* math-ph/0006019
[14] Rainich G Y *Nature* **115** 498