On the homology of free 2-step nilpotent Lie algebras

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We find an explicit formula for the total dimension of the homology of a free 2-step nilpotent Lie algebra. We analyse the asymptotics of this formula and use it to find an improved lower bound on the total dimension of the homology of any 2-step nilpotent Lie algebra.

1. INTRODUCTION

The free 2-step nilpotent Lie algebra of rank r is $\mathcal{N}_r = V \oplus \Lambda^2 V$, where V is an r dimensional vector space over $\mathbb{C}$. The only non-zero Lie brackets are for $v, w \in V$, when $[v, w] = v \wedge w \in \Lambda^2 V$. The centre of $\mathcal{N}_r$ is $\Lambda^2 V$.

In [5], Sigg describes how to decompose the Lie algebra homology $H_*(\mathcal{N}_r)$ into its irreducible components as a representation of $GL(V)$.

$$H_*(\mathcal{N}_r) = \bigoplus_{I \subset \{1,\ldots,r\}} H_I(\mathcal{N}_r), \tag{1}$$

where the summand $H_I(\mathcal{N}_r)$ is isomorphic to the irreducible tensor representation $R_\lambda(V)$ corresponding to the self-conjugate partition $\lambda = (I; I)$ in Frobenius notation. (This notation will be explained in Section 2.) The homology grading of $H_I(\mathcal{N}_r)$ is $\Sigma(I) = \sum_{i \in I} i$.

Despite claims to the contrary in [5], this decomposition is a special case of Kostant's decomposition [4] of the Lie algebra cohomology (or homology) of the nilradical of a parabolic subalgebra of a semisimple Lie algebra as a representation of the Levi factor. In this case, the parabolic is the Lie algebra of the stabiliser in $SO(V \oplus \mathbb{C} \oplus V^*)$ of the maximal isotropic subspace V. The Levi factor is $GL(V)$ and the indexing set for the indecomposable summands is a transversal to the smaller Weyl group S_r in the large one $S_r \ltimes \mathbb{Z}_2^r$. Such a transversal is naturally in one-one correspondence with the subsets of $\{1, \ldots, r\}$.

The decomposition yields a formula for the Poincaré polynomial

$$P(\mathcal{N}_r; t) = \sum_{n \in \mathbb{N}} \dim H_n(\mathcal{N}_r) t^n = \sum_{I \subset \{1,\ldots,r\}} \dim R_{(I;I)}(V) t^{\Sigma(I)} \tag{2}$$

and thus for the total homology

$$T(r) = \dim H_*(\mathcal{N}_r) = P(\mathcal{N}_r; 1).$$

The sum may be computed using one of the standard formulae for the dimension of an irreducible representation of $GL(r)$ (e.g. from [2]). For example, the first nine values of $T(r)$ are as follows.

r	$T(r)$
1	2
2	6
3	36
4	420
5	9800
6	452760
7	41835024
8	7691667984
9	2828336198688

Since we are taking a sum of 2^r positive terms, the length of the computation and the size of the answer grow exponentially.

A well-known lower bound for the total homology of any 2-step nilpotent Lie algebra is 2^z, where z is the dimension of the centre, i.e., the so-called Toral Rank Conjecture is true in this case. Recently, in [6], the bound has been improved to $2^{z+\lceil r/2 \rceil}$, where r is the codimension of the centre. For the free 2-step nilpotent Lie algebra, $z = r(r-1)/2$ and hence a lower bound for $T(r)$ is $2^{r^2/2}$.

In this paper we find the following explicit formula for $T(r)$.

Theorem 1.1. *For* $n \geq 0$

$$T(2n+1) = 2^{n+1}\beta(n)^2 \tag{3}$$
$$T(2n+2) = 2^{n+1}\beta(n)\beta(n+1) \tag{4}$$

where

$$\beta(n) = \prod_{1\leq i\leq j\leq n} \frac{2(i+j)-1}{2i-1} \tag{5}$$
$$= \prod_{1\leq k\leq n} \frac{(4k)!k!^2}{(2k)!^3} \tag{6}$$

Note from (5) that $\beta(n)$ is always odd. For example, the first five values are as follows.

n	0	1	2	3	4
$\beta(n)$	1	3	35	1617	297297

Hence the power of 2 dividing $T(r)$ is precisely $2^{\lceil r/2\rceil}$.

Theorem 1.1 is proved in Section 2 using Giambelli's determinant formula for the representation dimensions and observing that several simplifications can be made for self-conjugate partitions leading to an expression for $T(r)$ as a single determinant. This determinant can be further simplified by elementary row and column operations. A remarkable fact, that appears as the finishing step in the proof, is that $\beta(n)$ is the dimension of an irreducible $SO(2n+1)$ representation. Indeed, (3) is valid at the level of characters of $SO(2n+1)$ and not just dimensions, but we have no deeper understanding of why this is true.

One consequence of Theorem 1.1 is that the asymptotic behaviour of $T(r)$ can be analysed more closely and we discover that the lower bound $2^{r^2/2}$ is in fact the dominant term in the asymptotics. We do this in Section 3 by analysing the asymptotic behaviour of $\beta(n)$ using (6) and find the following.

Theorem 1.2. *There is a constant* $\kappa \simeq 1.3814$ *such that*

$$T(r) \sim 2^{r^2/2} r^{1/8} \kappa \tag{7}$$

In fact, we obtain a stronger result (Theorem 3.1) by finding actual upper and lower bounds on $T(r)$. In Section 4, we then apply Theorem 3.1 to further improve the lower bound on the total homology of any 2-step nilpotent Lie algebra.

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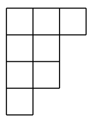
2. FROBENIUS' NOTATION AND GIAMBELLI'S FORMULA

A partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_k \geq 0)$ is often represented by its Young diagram $Y(\lambda)$, a graphical arrangement of λ_i boxes in the i-th row starting in the first column. The conjugate partition λ' of λ has Young diagram $Y(\lambda')$ obtained by reflecting $Y(\lambda)$ in the diagonal.

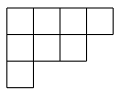
Another way to denote a partition λ is due to Frobenius. Let $d = d_\lambda$ be the number of diagonal boxes of $Y(\lambda)$. For $i = 1, \ldots, d$, let α_i to be the number of boxes in the i-th row to the right of and including the diagonal. Let β_i to be the number of boxes in the i-th column below and including the diagonal. Then one writes $\lambda = (I; J)$ where $I = \{\alpha_1, \ldots, \alpha_d\}$ and $J = \{\beta_1, \ldots, \beta_d\}$. Note that $\alpha_1 > \cdots > \alpha_d \geq 1$ and $\beta_1 > \cdots > \beta_d \geq 1$, so the sets I and J determine the sequences α_i and β_i.

An example is given below, showing a partition λ and its conjugate λ' in standard notation and Frobenius notation, together with their Young diagrams.

$$\lambda = (3,2,2,1) = (\{3,1\};\{4,2\}) \qquad Y(\lambda) =$$



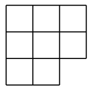
$$\lambda' = (4,3,1) = (\{4,2\};\{3,1\}) \qquad Y(\lambda') =$$



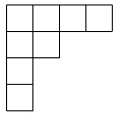
Note that there are different conventions on the precise form of Frobenius notation and, in particular, [5] uses a slightly different one.

In general, if $\lambda = (I; J)$, then the conjugate partition $\lambda' = (J; I)$, so that λ is self-conjugate if and only if $I = J$. If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ is a self-conjugate partition with no more than r rows (and columns), then the complementary partition $\lambda^c = (r - \lambda_r \geq \cdots \geq r - \lambda_1 \geq 0)$. If $\lambda = (I; I)$ in Frobenius notation, then $\lambda^c = (I^c; I^c)$, where $I^c = \{1, \ldots, r\} \smallsetminus I$ is the complementary subset. For example,

$$\lambda = (3,3,2,0) \qquad Y(\lambda) =$$
$$= (\{3,2\};\{3,2\})$$



$$\lambda^c = (4,2,1,1) \qquad Y(\lambda') =$$
$$= (\{4,1\};\{4,1\})$$



As a consequence, a self conjugate partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ may be recovered from Frobenius notation by writing $I = \{\alpha_1 > \cdots > \alpha_d\}$ in descending order and the complement $I^c = \{\alpha_{d+1} < \cdots < \alpha_r\}$ in ascending order and then putting

$$\lambda_i = \begin{cases} \alpha_i + i - 1 & 1 \leq i \leq d \\ i - \alpha_i & d < i \leq r \end{cases} \tag{8}$$

Now recall (e.g. [2] §15.5) that any partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_r \geq 0)$ with no more than r rows determines an irreducible tensor representation $R_\lambda(V)$ of $GL(V)$, where $r = \dim V$. Furthermore, the 'second Giambelli formula' ([2](24.11)) gives the character and hence the dimension of $R_\lambda(V)$ as a determinant. When $\lambda' = \lambda$ we have

$$\dim R_\lambda(V) = \det G_1(\lambda, r), \tag{9}$$

where the Giambelli matrix $G_1(\lambda, r)$ is the $r \times r$ matrix with entries

$$G_1(\lambda, r)_{ij} = \tbinom{r}{\lambda_i + j - i} \tag{10}$$

In other words, each row consists of r consecutive binomial coefficients chosen so that the ith row has $\binom{r}{\lambda_i}$ on the diagonal. For example

$$\lambda = (3,3,2,0) \qquad G_1(\lambda, 4) = \begin{pmatrix} \binom{4}{3} & \binom{4}{4} & 0 & 0 \\ \binom{4}{2} & \binom{4}{3} & \binom{4}{4} & 0 \\ \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ 0 & 0 & 0 & \binom{4}{0} \end{pmatrix}$$

Note that, to obtain the character of $R_\lambda(V)$, we simply need to interpret the symbol $\binom{r}{k}$ as the character of $\Lambda^k V$ rather than just as a binomial coefficient.

Using (8) we may also describe the Giambelli matrix $G_1(\lambda, r)$ in terms of Frobenius notation as follows. For $1 \leq i \leq d$ the ith row starts with $\binom{r}{\alpha_i}$, while for $d < i \leq r$ the ith row ends with $\binom{r}{r-\alpha_i}$. But note that $(\alpha_1, \ldots, \alpha_r)$ is a permutation of $(1, \ldots, r)$ and the sign of this permutation is $(-1)^{\#I_0}$, where $\#I_0$ is the number of even elements of I. Now $G_1(\lambda, r)$ is obtained by applying this permutation to the rows of the $r \times r$ matrix $G_2(I, r)$ with entries

$$G_2(I,r)_{ij} = \begin{cases} \binom{r}{j+i-1}, & \text{if } i \in I; \\ \binom{r}{j-i}, & \text{if } i \notin I. \end{cases} \tag{11}$$

Hence we may use (2) to write the Poincaré polynomial

$$\begin{aligned} P(\mathcal{N}_r; t) &= \sum_{I \subset \{1,\ldots,r\}} (-1)^{\#I_0} t^{\Sigma(I)} \det G_2(I, r) && (12) \\ &= \det G_3(r; t) && (13) \end{aligned}$$

where $G_3(r;t)$ is the $r \times r$ matrix with entries

$$G_3(r;t)_{ij} = \binom{r}{j-i} - (-t)^i \binom{r}{j+i-1}.$$

The equality of (12) and (13) simply follows from the fact that det is linear in rows. In particular, we have a formula for the total homology

$$T(r) = \dim H_*(\mathcal{N}_r) = \det G_3(r;1) \tag{14}$$

For example,

$$G_3(4;1) = \begin{pmatrix} \binom{4}{0} & \binom{4}{1} & \binom{4}{2} & \binom{4}{3} \\ 0 & \binom{4}{0} & \binom{4}{1} & \binom{4}{2} \\ 0 & 0 & \binom{4}{0} & \binom{4}{1} \\ 0 & 0 & 0 & \binom{4}{0} \end{pmatrix} + \begin{pmatrix} \binom{4}{1} & \binom{4}{2} & \binom{4}{3} & \binom{4}{4} \\ -\binom{4}{2} & -\binom{4}{3} & -\binom{4}{4} & 0 \\ \binom{4}{3} & \binom{4}{4} & 0 & 0 \\ -\binom{4}{4} & 0 & 0 & 0 \end{pmatrix}$$

Note that by interpreting $\binom{r}{k}$ as the character of $\Lambda^k V$ we may use (14) as a formula for the $GL(V)$ character of $H_*(\mathcal{N}_r)$ and not just its dimension. This is because we have so far been careful not to use any binomial identities, such as $\binom{r}{k} = \binom{r}{r-k}$, which do not hold at the level of $GL(V)$ characters. However, if we do allow ourselves to use such an identity, which would still hold at the level of $SO(V)$ characters, then we notice that the matrix $G_3(r;1)$ has odd rows which are symmetric under reversal and even rows which are antisymmetric. The determinant of such a matrix can always be simplified as follows.

LEMMA 2.1. *Let Z be an $r \times r$ matrix whose odd rows are symmetric and whose even rows are antisymmetric. Let $r_0 = \lfloor r/2 \rfloor$ be the number of even rows and $r_1 = \lceil r/2 \rceil$ be the number of odd rows. Then*

$$\det Z = 2^{r_0} \det X_0 \det X_1 \tag{15}$$

where X_0 and X_1 are the $r_0 \times r_0$ and $r_1 \times r_1$ matrices consisting of the final parts of the even and odd rows, that is

$$\begin{aligned} X_0[i,j] &= Z[2i, r_1 + j] \\ X_1[i,j] &= Z[2i-1, r_0 + j] \end{aligned}$$

Proof. The proof uses elementary row and column operations. We describe the general case, while showing the case $r = 5$ as an illustration. Here $X_1 = \big(x_{ij}\big)$ and $X_0 = \big(y_{ij}\big)$.

$$\det Z = \det \begin{pmatrix} x_{13} & x_{12} & x_{11} & x_{12} & x_{13} \\ -y_{12} & -y_{11} & 0 & y_{11} & y_{12} \\ x_{23} & x_{22} & x_{21} & x_{22} & x_{23} \\ -y_{22} & -y_{11} & 0 & y_{21} & y_{22} \\ x_{33} & x_{32} & x_{31} & x_{32} & x_{33} \end{pmatrix}$$

We reverse the order of the first r_1 columns and rearrange the rows so that all the odd rows precede all the even rows. Note that this may be done with $\binom{r_1}{2}$ column transpositions and $\binom{r_1}{2}$ row transpositions so that the sign of the determinant is unchanged.

$$\det Z = \det \begin{pmatrix} x_{11} & x_{12} & x_{13} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} & x_{32} & x_{33} \\ 0 & -y_{11} & -y_{12} & y_{11} & y_{12} \\ 0 & -y_{11} & -y_{22} & y_{21} & y_{22} \end{pmatrix}$$

We now add column $r + 1 - j$ to column $r_1 + 1 - j$, for $j = 1, \ldots, r_0$ to obtain a block upper triangular matrix in which r_0 columns have a factor of 2, which may be removed to give the required answer.

$$\begin{aligned} \det Z &= \det \begin{pmatrix} x_{11} & 2x_{12} & 2x_{13} & x_{12} & x_{13} \\ x_{21} & 2x_{22} & 2x_{23} & x_{22} & x_{23} \\ x_{31} & 2x_{32} & 2x_{33} & x_{32} & x_{33} \\ 0 & 0 & 0 & y_{11} & y_{12} \\ 0 & 0 & 0 & y_{21} & y_{22} \end{pmatrix} \\ &= 2^{r_0} \det X_1 \det X_0 \end{aligned}$$

▮

Combining Lemma 2.1 with (14) we obtain the following.

Proposition 2.1.

$$T(2n+1) = 2^{n+1}\det C(n)\det B(n) \qquad (16)$$
$$T(2n+2) = 2^{n+1}\det D(n)\det A(n+1) \qquad (17)$$

where $A(n)$, $B(n)$, $C(n)$ and $D(n)$ are $n\times n$ matrices with coefficients

$$A(n)_{ij} = \tbinom{2n}{n+1+j-2i} + \tbinom{2n}{n-2+j+2i}$$
$$B(n)_{ij} = \begin{cases} \tbinom{2n+1}{n+2-2i} & j=1 \\ \tbinom{2n+1}{n+1+j-2i} + \tbinom{2n+1}{n-2+j+2i} & j>1 \end{cases}$$
$$C(n)_{ij} = \tbinom{2n+1}{n+1+j-2i} - \tbinom{2n+1}{n+j+2i}$$
$$D(n)_{ij} = \tbinom{2n+2}{n+1+j-2i} - \tbinom{2n+2}{n+j+2i}$$

Proof. Apply Lemma 2.1 with $Z = G_3(r;1)$ and note two additional special features. Firstly, the last part of the last row of $G_3(r;1)$ is $(0,\ldots,0,1)$. This means that, when $r = 2n+1$, we take $C(n)$ to be X_0 and $B(n)$ to be X_1 with the last row and column omitted, while when $r = 2n+2$ we take $A(n+1)$ to be X_1 and $D(n)$ to be X_0 with the last row and column omitted. Secondly, when $r = 2n+1$ the middle coloumn of Z is divisible by 2, which would mean that the first entry of the ith row of $B(n)$ would be

$$\tbinom{2n+1}{n+2-2i} + \tbinom{2n+1}{n-1+2i} = 2\tbinom{2n+1}{n+2-2i}$$

Therefore we may remove this factor of 2 from the first column of $B(n)$ and get the extra factor of 2 in (16). ▮

We now make a closer analysis of the determinants in Proposition 2.1 to prove our main result.

Proof (of Theorem 1.1). First we find that $\det B(n) = \det C(n)$, by applying to $B(n)$ successively the operations

$$\mathrm{Row}_i \mapsto \mathrm{Row}_i - \mathrm{Row}_{i+1} \qquad i = 1,\ldots,n-1$$
$$\mathrm{Col}_j \mapsto \mathrm{Col}_j + \mathrm{Col}_{j-2} \qquad j = 3,\ldots,n$$

Next we find that $\det A = \det B$ and $\det C = \det D$ by applying to A or C the column operations

$$\mathrm{Col}_j \mapsto \mathrm{Col}_j + \mathrm{Col}_{j-1} \qquad j = n,\ldots,2$$

and using the fact that $\binom{r}{k} + \binom{r}{k-1} = \binom{r+1}{k}$.

Finally we make the surprising observation that $\det B(n)$ is one of the Giambelli-type determinant formulae ([2] Corollary 24.35) for the dimension/character of the irreducible $SO(2n{+}1)$ representation W_a with highest weight $a = (1, \ldots, 1, 2)$. Here we use the basis of fundamental weights and the coefficient 2 goes at the end of the Dynkin diagram with the short simple root. Then the Weyl dimension formula ([2] Corollary 24.6 & Exercise 24.30) gives

$$\dim W_a = \frac{\prod_{1\le i<j\le n} 2(j-i)(2(i+j)-1) \prod_{1\le j\le n} (4j-1)}{\prod_{1\le j\le n} (2j-1)!}$$

This expression may be simplified to (5), which is also a simplification of (6). ▮

Remark 2. 1. In proving both Proposition 2.1 and the equality between $\det B(n)$ and $\det C(n)$, the only property of the binomial coefficients we use is that $\binom{r}{k} = \binom{r}{r-k}$, which means that the formulae (16) and (17) hold at the level of $SO(r)$ characters, rather than just dimensions. In particular, this implies that, as an $SO(2n+1)$ representation, the homology $H_*(\mathcal{N}_{2n+1})$ is isomorphic to the direct sum of 2^{n+1} copies of $W_a \otimes W_a$. It is less clear how to interpret (17) at the level of representations, although one can say that $\det A(n+1)$ is the restriction to $SO(2n+2)$ of the irreducible $SO(2n+3)$ character $\det B(n+1)$. On the other hand, $\det D(n)$ is a virtual character of $SO(2n+2)$, whose restiction to $SO(2n+1)$ is the irreducible character $\det B(n)$.

Remark 2. 2. Some of the determinant manipulations above may be applied to the formula $\det G_3(r;t)$ for the Poincaré polynomial to show that $(1+t)^{\lceil r/2\rceil}$ divides $P(\mathcal{N}_r;t)$ just as $2^{\lceil r/2\rceil}$ divides $T(r)$. By setting $t = 1$ and recalling that $T(r)/2^{\lceil r/2\rceil}$ is always odd, we see that no higher power of $(1+t)$ divides $P(\mathcal{N}_r;t)$.

Remark 2. 3. Lemma 2.1 may be refined to provide a simplification of the determinant of a matrix in which either the odd rows are symmetric or the even rows are antisymmetric. This leads to some refinements of Proposition 2.1, which also hold at the level of $SO(V)$ characters and shed some light on the multiplicity 2^{n+1}.

For any set $I \subset \mathbb{N}$, let I_0 denote the set of even numbers in I and I_1 denote the set of odd numbers. We will now write $H_I(\mathcal{N}_r)$ as $H_{[I_1,I_0]}(\mathcal{N}_r)$

and, for any $K \subset \{1,\ldots,r\}_1$ and $L \subset \{1,\ldots,r\}_0$, will define

$$H_{[K,*]}(\mathcal{N}_r) = \bigoplus_{J\subset\{1,\ldots,r\}_0} H_{[K,J]}(\mathcal{N}_r)$$
$$H_{[*,L]}(\mathcal{N}_r) = \bigoplus_{J\subset\{1,\ldots,r\}_1} H_{[J,L]}(\mathcal{N}_r)$$

What can be shown is that

$$\begin{aligned}
\mathrm{Char}_{SO(2n+1)}\, H_{[K,*]}(\mathcal{N}_{2n+1}) &= \det B(n)\det C(n) \\
\mathrm{Char}_{SO(2n+2)}\, H_{[K,*]}(\mathcal{N}_{2n+2}) &= \det D(n)\det A(n+1) \\
\mathrm{Char}_{SO(2n+1)}\, H_{[*,L]}(\mathcal{N}_{2n+1}) &= 2\det B(n)\det C(n) \\
\mathrm{Char}_{SO(2n+2)}\, H_{[*,L]}(\mathcal{N}_{2n+2}) &= \det D(n)\det A(n+1)
\end{aligned}$$

In other words, the partial sums $H_{[K,*]}(\mathcal{N}_r)$ are all isomorphic as representations of $SO(r)$, independent of K. Furthermore, the partial sums $H_{[*,L]}(\mathcal{N}_r)$ are all isomorphic as representations of $SO(r)$, independent of L, and this representation is the same as the one above, when r is even, and twice the one above, when r is odd.

We illustrate this result by giving all the representation dimensions for $r = 4$ arranged in a table with rows indexed by I_1 and columns by I_0. As predicted, all rows and columns have the same sum, which in this case is $\beta(1)\beta(2) = 105$.

$I_1\backslash I_0$	$\{\}$	$\{4\}$	$\{2\}$	$\{2,4\}$
$\{\}$	1	20	20	64
$\{1\}$	4	45	20	36
$\{3\}$	36	20	45	4
$\{1,3\}$	64	20	20	1

This table also displays two entertaining properties, which appear to hold for the dimensions of $H_I(\mathcal{N}_r)$. These are observed empirically for $r \le 20$ but not proved in general. Firstly, there is an involution σ of $\{1,\ldots,r\}$ with the property that $H_I(\mathcal{N}_r)$ has odd dimension if and only if $\sigma(I) = I$. This involution is defined by partitioning $\{1,\ldots,r\}$, when r is even, or $\{2,\ldots,r\}$, when r is odd, into certain even intervals and reversing each interval. Secondly, the largest dimension of $H_I(\mathcal{N}_r)$ occurs when $I = \{1,\ldots,r\}_0$ or $I = \{1,\ldots,r\}_1$. One easily computes that this dimension is 2^z, where $z = r(r-1)/2$ is the dimension of the centre of $\mathcal{N}_r$.

3. ASYMPTOTICS AND BOUNDS FOR β AND T

To study the asymptotics of $\beta(n)$, and hence $T(r)$, we consider the expression (6) from Theorem 1.1, that is,

$$\beta(n) = \prod_{k=1}^{n} \frac{(4k)!k!^2}{(2k)!^3}.$$

We will make repeated use of Euler's summation formula ([1] Chap.12, Art.106–108), for the difference between the sum and the integral of a function. We start with one special case: Stirling's asymptotic series

$$\log n! = \left(n + \tfrac{1}{2}\right) \log n - n + \tfrac{1}{2} \log(2\pi) + \phi(n), \tag{18}$$

where

$$\phi(n) = 1/12n - 1/360n^3 + 1/1260n^5 - \cdots . \tag{19}$$

Hence

$$\log \left(\frac{(4k)!k!^2}{(2k)!^3} \right) = (2k - \tfrac{1}{2}) \log 2 + \tfrac{1}{16} \Phi(k), \tag{20}$$

where $\Phi(k) = 16\big(\phi(4k) + 2\phi(k) - 3\phi(2k)\big)$, so that

$$\Phi(k) = 1/k - 7/96k^3 + \varepsilon_1(k), \tag{21}$$

with $0 < \varepsilon_1(k) < 31/1280k^5 < 1/40k^5$, for simplicity. Thus,

$$\log \beta(n) = \big(n^2 + n/2\big) \log 2 + \tfrac{1}{16} \sum_{k=1}^{n} \Phi(k) \tag{22}$$

We can now estimate the last term using other cases of Euler's summation formula. Firstly,

$$\sum_{k=1}^{n} 1/k = \gamma + \log n + 1/2n - 1/12n^2 + \varepsilon_2(n),$$

where $\gamma \simeq 0.5772$ is Euler's constant and $0 < \varepsilon_2(n) < 1/120n^4$. Secondly,

$$\sum_{k=1}^{n} 1/k^3 = \zeta(3) - 1/2n^2 + 1/2n^3 - \varepsilon_3(n),$$

where $0 < \varepsilon_3(n) < 1/4n^4$. Finally, we have the simple estimate

$$\sum_{k=1}^{n} \varepsilon_1(k) = c - \varepsilon_4(n),$$

where $0 < \varepsilon_4(n) < 1/160n^4$ and c is a constant, with $0 < c < \zeta(5)/40$. Putting these three together with (21) gives

$$\sum_{k=1}^{n} \Phi(k) = C + \log n + 1/2n - 3/64n^2 - 7/192n^3 + \varepsilon_5(n) \qquad (23)$$

where $-1/160n^4 < \varepsilon_5(n) < 1/32n^4$ and $C = \gamma - 7\zeta(3)/96 + c$. Repeating the analysis above with one more term in the asymptotic series (19), we could show that $0.495 < C < 0.515$, but to get better accuracy we must use numerical experiments to show that $C \simeq 0.5055$.

From (23), we obtain the bounds

$$C + \log(n + 1/2) < \sum_{k=1}^{n} \Phi(k) < C + \log(n + 1/2 + 1/12n) \qquad (24)$$

and thus (22) yields

$$(n + 1/2)^{1/16} < \frac{\beta(n)}{2^{(n^2+n/2)} e^{C/16}} < (n + 1/2 + 1/12n)^{1/16}. \qquad (25)$$

In particular,

$$\beta(n) \sim 2^{(n^2+n/2)} n^{1/16} e^{C/16}. \qquad (26)$$

Now, if we put (26) into Theorem 1.1, then we immediately obtain Theorem 1.2, with

$$\kappa = 2^{3/8} e^{C/8} \simeq 1.3814. \qquad (27)$$

On the other hand, if we put (25) into Theorem 1.1, then a little manipulation yields the following.

Theorem 3.1. *When r is odd,*

$$r^{1/8} < \frac{T(r)}{2^{r^2/2}\kappa} < \left(r^2 + 1\right)^{1/16}. \qquad (28)$$

When r is even,

$$\left(r^2 - 1\right)^{1/16} < \frac{T(r)}{2^{r^2/2}\kappa} < r^{1/8}. \qquad (29)$$

A careful reader will note that to derive the upper bounds in (28) and (29) from (25) we need $r \geq 5$, but it may be checked directly that these inequalities also hold for $r < 5$.

Remark 3. 1. Javier Cilleruehlo has shown us a more direct derivation of the asymptotic formula for $T(r)$ in Theorem 1.2. He starts by noting the following formulae.

$$\beta(n) \;=\; \prod_{1\leq k\leq n}\left(\frac{(4k-1)(4k-3)}{(2k-1)^2}\right)^{n+1-k} \tag{30}$$

$$T(r) \;=\; 2^{r(r+1)/2}\prod_{\substack{1\leq j\leq r\\ j\text{ odd}}}\left(1-\frac{1}{4j^2}\right)^{r-j} \tag{31}$$

He then makes a direct asymptotic analysis of (31), using amongst other things the formulae

$$\prod_{j\text{ odd}}\left(1-\frac{1}{4j^2}\right)^{r}=2^{-r/2}\quad\text{and}\quad\prod_{\substack{j>r\\ j\text{ odd}}}\left(1-\frac{1}{4j^2}\right)^{-r}\sim e^{1/8}.$$

As a consequence, he recovers Theorem 1.2, together with the formula

$$\kappa=e^{1/8}\prod_{j\text{ odd}}\left(1-\frac{1}{4j^2}\right)^{-j}\left(1+\frac{2}{j}\right)^{-1/8}. \tag{32}$$

With a more detailed analysis he obtains the following refinement with the same order of extra control as Theorem 3.1

$$T(r)=2^{r^2/2}r^{1/8}\kappa\left(1+cr^{-2}+O(r^{-3}))\right). \tag{33}$$

where $c=5/128$, when r is odd, and $c=-3/128$, when r is even.

4. APPLICATION

Let $\mathfrak{g}$ be any finite dimensional 2-step nilpotent Lie algebra and $\mathfrak{a}$ a finite dimensional abelian Lie algebra. Then

$$H_*\big(\mathfrak{g}\oplus\mathfrak{a}\big)=H_*\big(\mathfrak{g}\big)\otimes\Lambda^*\mathfrak{a}$$

and thus $\big|H_*\big(\mathfrak{g}\oplus\mathfrak{a}\big)\big|=2^{|\mathfrak{a}|}\big|H_*\big(\mathfrak{g}\big)\big|$, where $|W|$ is short-hand for $\dim W$. Assume now that $\mathfrak{g}$ has no abelian factors, let $\mathfrak{z}=\text{centre}(\mathfrak{g})$ and let V be any direct complement of $\mathfrak{z}$. The minimum number of generators of $\mathfrak{g}$ is $r=\dim V$ and $\mathfrak{g}$ is a homomorphic image of $\mathcal{N}_r$. In [3] (Theorem 2.1) it is proved that one may degenerate $\mathcal{N}_r$ to $\mathfrak{g} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is abelian with $|\mathfrak{a}| = |\mathcal{N}_r| - |\mathfrak{g}|$. Since under degeneration the homology can only grow we have that

$$2^{|\mathfrak{a}|} \left| H_*(\mathfrak{g}) \right| \geq T(r). \tag{34}$$

Thus, we can improve the lower bounds given in [6] for the total homology of a 2-step nilpotent Lie algebra. Theorem 1.2 shows that we obtain essentially the best general lower bound available in a single formula.

PROPOSITION 4.1. *Let $\mathfrak{g}$ be any 2-step nilpotent Lie algebra of finite dimension. Let $\mathfrak{z}$ be its centre, $z = \dim(\mathfrak{z})$ and $r = \operatorname{codim}(\mathfrak{z})$. Then*

$$\dim H_*(\mathfrak{g}) \geq 2^{z+r/2}(r^2-1)^{1/16}\kappa. \tag{35}$$

Proof. First assume that $\mathfrak{g}$ has no abelian factors and combine (34) with Theorem 3.1 to obtain the required inequality. But now notice that if we replace $\mathfrak{g}$ by $\mathfrak{g} \oplus \mathfrak{a}$, then both sides of the inequality are multiplied by $2^{|\mathfrak{a}|}$ and so it remains valid. Thus the result follows for all $\mathfrak{g}$. ∎

Note that, because $\kappa > 2^{3/8}$, this result does always improve the old lower bound of $2^{z+\lceil r/2 \rceil}$. If we were willing to separate into cases, then we could make a small improvement by replacing the term $(r^2-1)^{1/16}$ in (35) by $r^{1/8}$ when r is odd.

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