A Survey of Brill-Noether Theory on Algebraic Curves

Alastair D. King, University of Liverpool, U.K.

These notes are based on a talk given at the Trento meeting "Vector Bundles in Algebraic Geometry" in September 1993. They are intended as a companion to the V.B.A.C. Problems List.

1. Basic Geography of Special Bundles and Special Linear Series

Notation.

C : a smooth projective curve, genus $g \geq 2$,

$\mathcal{O}$, K : the trivial line bundle (i.e. structure sheaf of C) and the canonical bundle,

E : a vector bundle over C, with invariants $n = \operatorname{rk} E$, $d = \deg E$.

Definition.

i) The bundle E is 'special' if $h^0(E)h^1(E) > 0$.

ii) A 'linear series' (or 'linear system') $g^r_{n,d}$ consists of a vector bundle E of rank n and degree d, and a *linear* subspace $\Lambda \subseteq H^0(E)$ of dimension $r+1$. The linear series is 'complete' if $\Lambda = H^0(E)$ (write $|E|$ for this series) and is 'special' if $r+1 > d-n(g-1)$. When this inequality is satisfied, we will say that (r, n, d) is of 'special type'.

Note that E is special iff E admits a special linear series iff the complete linear series $|E|$ is special.

It seems to be more appropriate to describe linear series in vector bundles in terms of the rational invariants $\mu = \frac{d}{n}$, $\lambda = \frac{r+1}{n}$ and the auxilliary invariant $\lambda' = \lambda - \mu + g - 1$, which is chosen so that the linear series is special iff $\lambda' > 0$. When this inequality is satisfied, we say that (λ, μ) is of 'special type'. Note also that, for $|E|$, $\lambda' = h^1(E)$, by Riemann-Roch. We can then talk about a linear series as being a $g(\lambda, \mu)$, without specifying the rank of the underlying bundle.

Basic Results.

i) By Serre duality, E is special iff $E^\vee \otimes K$ is special.

ii) Vanishing Lemma: If E is *semistable* and special then $0 \leq \mu \leq 2g-2$.

iii) Clifford's Theorem: If E is *semistable* and special, then $\lambda \leq 1 + \frac{1}{2}\mu$ (or $\lambda + \lambda' \leq g+1$), with equality iff $E = \mathcal{O}^n$ or K^n or C is hyperelliptic. Indeed, a line bundle with $h = 1 + \frac{1}{2}d$, must be a power of the hyperelliptic line bundle H, for which $|H|$ is the g^1_2 giving the $2:1$ map $C \to \mathbf{P}^1$.

Conclusion. The 'special region' in (λ, μ) space, where special linear series are forced to lie by the above basic results, is the same for semistable vector bundles as it is for line bundles. (See the attached 'map'.) We should then ask the finer questions of 'geography':

i) precisely which (λ, μ) are 'populated' by special linear series, for all C?

ii) for generic C?

iii) how does 'generic' depend on the rank?

2. Brill-Noether Varieties

We will describe the varieties (strictly, schemes) $W^r_{n,d}$, of semistable bundles admitting a $g^r_{n,d}$, and $G^r_{n,d}$, of linear series $g^r_{n,d}$.

We begin with the 'primitive' case, i.e. when n and d are coprime. In this case, all semistable bundles are stable, and they are parametrised by a fine moduli space $M_{n,d}(C)$, which is a smooth projective variety. Recall that 'fine' refers to the fact that there exists a universal family over $M_{n,d} \times C$. (In keeping with the spirit of this survey, it might be appropriate to denote this moduli space by J_μ, since it has all the important properties of the Jacobian J_d.)

Naively, i.e. as a set of points,

$$W^r_{n,d}(C) = \{E \in M_{n,d}(C) \mid h^0(E) \geq r+1\}$$

Note that, when (r,n,d) is not of special type, $W^r_{n,d} = M_{n,d}$.

To get the scheme structure on $W^r_{n,d}$ choose a universal bundle $\mathcal{E}$ over $M_{n,d} \times C$ and a fixed 'very positive' divisor N, i.e. so that $H^1(E(N)) = 0$, for all $E \in M_{n,d}$. We then have an exact sequence

$$0 \longrightarrow H^0(E) \longrightarrow H^0(E(N)) \xrightarrow{\phi_E} H^0(E|_N) \longrightarrow H^1(E) \longrightarrow 0$$

in which the middle two terms have dimension independent of E. If we 'globalise' this middle part, we obtain a map $\phi : V^0 \to V^1$ between the *vector bundles* $V^0 = \pi_{M*}\mathcal{E}(N)$ and $V^1 = \pi_{M*}\mathcal{E}|_N$.

Then, $W^r_{n,d}$ is the determinantal locus where ϕ drops rank by an appropriate amount. The expected codimension at a 'typical' point E (i.e. $h^0(E) = r+1$) is $h^0(E)h^1(E)$. Hence, the expected dimension of $W^r_{n,d}$ is the 'Brill-Noether number'

$$\rho^r_{n,d} = 1 + n^2(g - 1 - \lambda\lambda')$$

From this we see that, in our map, there is a natural region defined by $\widetilde{\rho} \geq 0$, where $\widetilde{\rho} = g - 1 - \lambda\lambda'$. In this region one expects the Brill-Noether loci to have dimension at least 1. Outside, the region there are certain 'exceptional' points where $\lambda, \mu \in \frac{1}{n}\mathbf{Z}$ and $\widetilde{\rho} = -\frac{1}{n^2}$. At these points one expects to find isolated special bundles.

The tangent space to $W^r_{n,d}$ at a typical point E is the kernel of

$$p^* : \mathrm{Ext}^1(E,E) \to H^0(E)^* \otimes H^1(E)$$

which is dual to the 'Petri map'

$$p : H^0(E) \otimes H^0(E^\vee \otimes K) \to H^0(\mathrm{End}(E) \otimes K)$$

given by multiplication of sections. Thus, $W^r_{n,d}$ has the expected dimension at E iff the Petri map is injective.

To define $G^r_{n,d}$, we pull back $\phi : V^0 \to V^1$ to the relative Grassmannian

$$\mathrm{Gr}(r+1, V^0) \xrightarrow{\pi} M_{n,d}$$

and consider the subvariety over which the universal subbundle $s : S \hookrightarrow \pi^* V^0$ is contained in the kernel of $\pi^*\phi$, i.e. $G^r_{n,d}$ is the vanishing locus of $(\pi^*\phi) \cdot s$.

In the non-primitive case, i.e. $(n, d) \neq 1$, the construction described above breaks down because there are semistable points in the moduli space and there is no universal bundle. We can still define $W^r_{n,d}$ in the open set of stable points (even as a scheme, because of the existence of a local universal family there) and then take its closure in the whole moduli space. A better way to define $W^r_{n,d}$ would be as the image of the corresponding determinantal locus in the quot scheme, whose GIT quotient is $M_{n,d}$, since a universal bundle does exist over the quot sheme. With this definition, $W^r_{n,d}$ may have more components than with the previous one. Defining $G^r_{n,d}$ is more difficult, and we return to that later.

3. Principal results of Brill-Noether Theory.

First, we recall the central results of Brill-Noether Theory for line bundles, some of which were 'known' classically, but which were give 'modern' proofs by Kempf, Kleiman & Laksov, Fulton & Lazarsfeld, Griffiths & Harris, Gieseker (1972-1982). See [**ACGH**] for full details.

For all curves C,

$$\begin{aligned} \rho^r_d \geq 0 \Rightarrow\ & G^r_d \text{ and } W^r_d \text{ are non-empty,} \\ \rho^r_d \geq 1 \Rightarrow\ & G^r_d \text{ and } W^r_d \text{ are connected.} \end{aligned}$$

The varieties may be reducible, but each component has dimension at least ρ^r_d.

For generic curves C, G^r_d is smooth of dimension exactly ρ^r_d, and hence W^r_d is irreducible, and, if $r \geq d - g$, of the same dimension. Furthermore, the singular locus of W^r_d is exactly W^{r+1}_d and there is a formula for the fundamental class of W^r_d in $H^*(J_d)$. This generalises the formula of Castelnuovo, for the number of special linear series in the case $\rho^r_d = 0$, namely

$$g! \prod_{i=0}^{r} \frac{i!}{(g-d+r+i)!}.$$

Based on the results for line bundles, the results for vector bundles fall into two classes, 'expected' and 'unexpected'.

Results of Expected Type.

i) [**Su**] For all curves: if $0 < \mu \leq g - 1$, then $W^0_{n,d}$ is irreducible of dimension $\rho^0_{n,d}$.

ii) [**Te**] For generic curves: suppose that $a, b \in \mathbf{Z}$ and that the point $(\lambda, \lambda') = (a, b)$ is in $\{\widetilde{\rho} \geq 0\}$. Then, for any $(\lambda, \lambda') \leq (a, b)$, there is a semistable bundle E, of any rank, with a linear series with invariants (λ, λ'). Further, if $\mu \notin \mathbf{Z}$ or the point (a, b) is in $\{\widetilde{\rho} > 0\}$, then there exists a stable E with the appropriate linear series.

Results of Unexpected Type.

i) [**GN**] In the 'bottom triangle' $0 \leq \mu < \lambda \leq 1$, all stable bundles, except $\mathcal{O}$ lie to the left of the tangent line to $\widetilde{\rho} = 0$ at $(\lambda, \mu) = (1, 1)$. Thus, in particular there exist points with $\rho^r_{n,d} \geq 1$, but with $W^r_{n,d}$ empty, for all curves. This phenomenon can be compared with Drezet & Le Potier's 'fractal mountain range' which excludes the existence of some stable bundles on $\mathbf{P}^2$, which should exist for purely dimensional reasons [**DL**].

ii) [**BF**] There are contexts in which the Petri map has extra symmetry which may prevent it from being injective. One such is when $E \cong E^\vee \otimes K$. Such bundles E may be thought of as generalisations of theta characteristics. For such bundles, the Petri map is symmetric and hence definitely not injective, when $h^0(E) > 1$. In this case, there is an alternative Petri map $S^2 H^0(E) \to H^0(S^2 E)$ and this gives rise to a different expected dimension.

When $\operatorname{rk} E = 2$ and $\det E = K$, then one automatically has $E \cong E^\vee \otimes K$. The expected dimension for $W^r_{2,K} \subseteq M_{2,K}$ (the subscript K means that the determinant is fixed to be K) is

$$\sigma^r_{2,K} = 3(g-1) - \frac{(r+1)(r+2)}{2}$$

It is clearly possible to have $\sigma^r_{2,K} \geq 0 > \rho^r_{2,2g-2}$. Hence, there will exist special stable bundles, for which the ordinary Brill-Noether number is negative, e.g. $g = 6$, $r = 4$.

[**BF**] For $g \leq 9$, $W^r_{2,K}$ is non-empty iff $\sigma^r_{2,K} \geq 0$.

These moduli spaces have arisen naturally in the context of Fano varieties and curves in K3 surfaces. On one hand [**Mu**], for generic C, with $g = 10$, the locus $W^6_{2,K}$ is empty. On the other hand [**Vo**], for C generic amongst curves in K3 surfaces, with $g = 2s \geq 6$, the locus $W^{s+1}_{2,K}$ is non-empty. Since curves in a K3 surface can be generic from the point of view of Brill-Noether for line bundles, this provides an example of speciality of curves being seen only by higher rank bundles.

4. Construction of non-primitive $G^r_{n,d}$s

When one cannot construct $G^r_{n,d}$ using a relative Grassmannian associated to a universal family (and even when one can), these spaces can be contructed directly as moduli spaces of linear series. This has been done in [**Be**], [**RV**] and [**KN**].

For any positive $\alpha \in \mathbf{R}$, one defines the α-slope of a linear system $E, \Lambda \subseteq H^0(E)$ to be

$$\mu_\alpha = \mu + \alpha\lambda = \frac{\deg E + \alpha \dim \Lambda}{\operatorname{rk} E}$$

A subsystem F, Π of E, Λ consists of $F \subseteq E$ and $\Pi \subseteq \Lambda \cap H^0(F)$. One then defines α-stable, α-semistable and S-equivalence in the usual way and shows that one can construct a moduli space of (S-equivalence classes of) α-semistable linear systems, containing an open set of (isomorphism classes of) α-stable systems.

We will denote this moduli space by $G^r_{n,d}(\alpha)$. One can show that, for all but a discrete set of 'critical' values of α, all semistable systems are stable and that, in the intervals between critical values, the stability condition and hence the moduli space is independent of α. Furthermore, in the 'first interval' $0 < \alpha <$ *first critical value*, the underlying bundle is necessarily semistable. Hence, the correct candidate for $G^r_{n,d}$ is the moduli space $G^r_{n,d}(\alpha)$, for α in this first interval. There is a map $G^r_{n,d} \to M_{n,d}$ whose image is $W^r_{n,d}$.

[**RV**] For all curves C and $0 \leq \mu \leq g-1$, the space $G^0_{n,d}$ is smooth.

5. Singular Curves.

The construction of the $G^r_{n,d}$s goes through for singular curves, provided one is careful with the definitions.

Firstly, a curve is any scheme of pure dimension 1 (or locally Cohen-Macaulay), so curves with embedded points are not allowed, but reducible and non-reduced curves are. Secondly, one should equip the curve with a fixed ample polarisation $\mathcal{O}_C(1)$, which will not be uniquely determined if the curve is not integral. For an arbitrary (coherent) sheaf (of $\mathcal{O}_C$-modules) one can define the rank and degree by using the Hilbert polynomial

$$\chi(E(n)) = \operatorname{rk} E \chi(\mathcal{O}(n)) + \deg E$$

These invariants will be integers for locally free sheaves (rk > 0) and for sheaves supported in dimension 0 (rk $= 0$), but will be rational with bounded denominator, for general sheaves. To obtain projective moduli spaces, one must consider not just locally free sheaves but also sheaves of pure dimension 1 (or locally Cohen-Macaulay, or depth 1) as carrying linear systems.

One should also be aware that for sufficiently bad curves, i.e. some non-reduced ones and reducible ones with badly 'balanced' polarisations, the structure sheaf $\mathcal{O}_C$ is not stable, which alters most of the numbers occurring in the various inequalities.

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