Periodic algebras which are almost Koszul

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1 Introduction

Let k be a field. A k-algebra A is called *periodic* if it has a periodic projective resolution as an A, A-bimodule. This paper grew out of an attempt to find periodic resolutions for the trivial extension algebras of the path algebras of Dynkin quivers in bipartite orientation. Such an algebra A (if it has three or more simples) is quadratic and its quadratic dual is the preprojective algebra for the same Dynkin graph. Computations had indicated that the initial terms of a minimal projective bimodule resolution of A are given by its Koszul bimodule complex (see Subsection 3.4) and that the algebra A has $2(h-1)$ as a period, where h is the Coxeter number of the underlying Dynkin graph. In this paper we prove these results within a theory of what we call 'almost Koszul' algebras, which are algebras with properties very close to those of Koszul algebras. This new theory also enables us to establish periodicity for other families of algebras.

Theorem 2.1 in Section 2 contains our results for the trivial extension algebras of the path algebras of bipartite Dynkin quivers. Theorem 2.2 extends them to the trivial extensions of algebras derived equivalent to path algebras of Dynkin quivers; we are grateful to Jeremy Rickard for both its statement and its deduction from Theorem 2.1.

In Section 3 we define almost Koszul rings and develop a theory for them along the lines of the theory of Koszul rings in [5]. An almost Koszul ring A is positively graded and artinian and has two positive integral invariants p and q, where p is just the grade of the last non-zero component of A. One of the properties of q is that the top of A has a minimal projective resolution in which the first q maps are linear in the sense of Koszul theory. For $q = 1$ or $p = 1$ these rings turn out to be the truncated rings and are dealt with *ad hoc* in Subsection 3.2. Otherwise an almost Koszul ring is quadratic and has a quadratic dual which is also almost Koszul but with the p and q invariants interchanged. The Koszul bimodule complex of an almost Koszul ring has only $q + 1$ non-zero terms and the main result of this section, Theorem 3.15, includes a description of its $(q + 1)$-st syzygy. This section includes a brief discussion of the Yoneda algebra and Hilbert polynomial of an almost Koszul algebra.

Section 4 contains results, central to the proof of Theorem 2.1, about the preprojective algebra B of a Dynkin quiver. We show that B is an almost Koszul algebra with $p = h - 2$ and $q = 2$, so that its quadratic dual, the trivial extension algebra A for the corresponding Dynkin quiver, is almost Koszul with $p = 2$ and $q = h-2$. We also identify a Nakayama automorphism [29] β of B, with order 1 or 2, which will be used in Section 5 to determine the structure of the $(h - 1)$-st syzygy in the minimal projective bimodule resolution of A. Section 4 also includes a new proof that the third bimodule syzygy of B is its k-linear dual $\mathrm{D}B$, an unpublished result of Ringel and Schofield.

In Section 5 we complete the proof of Theorem 2.1 and show that the Yoneda algebra of the trivial extension algebra A is the graded twisted polynomial algebra $B[t; \beta]$ in an indeterminate t of degree $h-1$ (see Theorem 5.3).

Section 6 contains further examples of almost Koszul algebras. These are constructed using covering theory. Some are not self-injective. The others belong to infinite families of periodic finite-dimensional algebras which are also $\mathrm{D}Tr$-periodic. These families appear to include all the finite-dimensional $\mathrm{D}Tr$-periodic algebras discussed by Auslander and Reiten in [1] and Buchweitz in [8].

Most of our examples of almost Koszul algebras have either $p \leq 2$ or $q \leq 2$. The only others we know are described at the end of Section 6 and have $p = q = 3$. They derive from the two-point quivers with relations discussed in [4] and [18].

To conclude, we note that there are precedents for our use of one-sided projective resolutions to induce projective bimodule resolutions of an algebra. The idea underlies Cartan and Eilenberg's 'inverse process', given in Chapter X.6 of [10], for calculating Hochschild homology and cohomology of certain types of algebras from their one-sided homology and cohomology. Their examples include the group ring $\mathbb{Z}G$ of a finite group G and it is clear that their process converts the periodic resolutions of $\mathbb{Z}$ for cyclic and quaternionic groups, given in their Chapter XII, to periodic bimodule resolutions of $\mathbb{Z}G$. The philosophy of constructing projective bimodule resolutions from one-sided resolutions is also explicitly formulated by Sköldberg at the end of

Section 2 of [28].

Throughout the paper algebras are associative and have identities. For an A, A-bimodule M, we write ${}_\alpha M_\beta$ for the twist of M by automorphisms α and β of A; the action of A on ${}_\alpha M_\beta$ is given by $(a, m, b) \mapsto \alpha(a)m\beta(b)$. Graded algebras, unless otherwise stated, are graded by the non-negative integers. Lastly, we use the term 'Dynkin graph' or 'Dynkin quiver' for the simply laced Dynkin graphs or quivers $\mathbb{A}_n$, $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$. Their Coxeter numbers, h, which are $n+1$, $2(n-1)$, 12, 18 and 30, respectively, appear throughout this paper.

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2 A periodicity theorem

We now formulate our results on the periodicity of the trivial extension algebras of the path algebras of Dynkin quivers. For any vector space U over the fixed basefield k, we write $\mathrm{D}U$ for the dual space $\mathrm{Hom}_k(U, k)$. The *trivial extension algebra*, $T(E)$, of a finite-dimensional k-algebra, E, has underlying vector space $E \oplus \mathrm{D}E$, where $\mathrm{D}E$ is viewed as an E,E-bimodule, and $T(E)$ has multiplication given by $(e+f)(e'+f') = ee'+ef'+fe'$. It is a symmetric algebra with non-degenerate symmetric bilinear form $\langle e+f, e'+f'\rangle = f(e')+f'(e)$.

The following theorem will be proved in Section 5.

Theorem 2.1. *Let k be a field and Q a Dynkin quiver in bipartite orientation and with Coxeter number h. Then $T(kQ)$ is periodic and has $2(h-1)$ as a period. This period is minimal unless k has characteristic 2 and the graph of Q is one of $\mathbb{A}_1$, $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, in which case the minimal period is $h-1$.*

Let $A = T(kQ)$, with Q as in Theorem 2.1. We actually do more than the theorem states: we give an explicit construction of a minimal projective bimodule resolution of A. For the cases $Q = \mathbb{A}_1$ and $Q = \mathbb{A}_2$, the algebra A is truncated and the construction of the resolution and the proof of the theorem are covered by Subsection 3.2. When Q is a Dynkin quiver with three or more vertices, A is a quadratic almost Koszul algebra with quadratic dual the preprojective algebra B for Q. Note that B is a graded algebra with

$h-1$ non-zero graded components which we denote $B_0, \ldots, B_{h-2}$. In this case we will use the results of Sections 3 and 4 to show that there is an exact sequence with minimal maps

$$0 \to \Sigma \to P(\mathrm{D}B_{h-2}) \to \cdots \to P(\mathrm{D}B_1) \to P(\mathrm{D}B_0) \to A \to 0 \qquad (1)$$

where $P(X)$ is a graded projective A, A-bimodule with top X and $\Sigma = {}_1A_\alpha$ for an automorphism α of A of order at most 2.

The automorphism α induces an automorphism $M \mapsto {}_1M_\alpha$ of the category of A, A-bimodules which preserves projectives. Hence the image of (1) is the first $h-1$ terms of the minimal projective bimodule resolution of ${}_1A_\alpha$ and has $(h-1)$-st syzygy ${}_1A_{\alpha^2}$ which is isomorphic to ${}_1A_1$ We may splice this onto (1) to obtain a complete period, with $2(h-1)$ terms, of the minimal projective bimodule resolution of A.

The following theorem is an extension of Theorem 2.1 to a much wider class of symmetric algebras.

Theorem 2.2 (Rickard)**.** *Let k be a field and Q a Dynkin quiver with Coxeter number h. Suppose that E is a k-algebra which is derived equivalent to the path algebra kQ. Then $T(E)$ is periodic and has $2(h-1)$ as a period. This period is minimal unless k has characteristic 2 and the graph of Q is one of $\mathbb{A}_1$, $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, in which case the minimal period is $h-1$.*

Proof. If E is derived equivalent to the path algebra kQ of a Dynkin quiver (or for that matter any tree), then in particular it is derived equivalent to $\Lambda = kQ'$, where Q' has bipartite orientation and the same underlying graph as Q. By Theorem 2.1 the result holds for Λ. Suppose that the derived equivalence between E and Λ is given by a tilting complex $P^\bullet$ for Λ. Write $Q^\bullet = P^\bullet \otimes_\Lambda T(\Lambda)$. Then by [23], Theorem 3.1, $Q^\bullet$ is a tilting complex for $T(\Lambda)$ with endomorphism ring $T(E)$, so that $T(\Lambda)$ and $T(E)$ are also derived equivalent. Next, Proposition 2.5 in [24] shows that $\mathrm{Hom}_{T(\Lambda)}(Q^\bullet, T(\Lambda)) \otimes_k Q^\bullet$ is a tilting complex for the enveloping algebra $T(\Lambda)^{\mathrm{e}} = T(\Lambda)^{op} \otimes_k T(\Lambda)$ of $T(\Lambda)$ with endomorphism algebra the enveloping algebra $T(E)^{\mathrm{e}}$ of $T(E)$, and that the bimodules ${}_{T(\Lambda)}T(\Lambda)_{T(\Lambda)}$ and ${}_{T(E)}T(E)_{T(E)}$ correspond under the resulting equivalence of derived categories. Finally, since both enveloping algebras are self-injective, it is a consequence of Corollary 2.2 in [23] and of its proof, that the last equivalence induces an equivalence of the stable categories of $T(\Lambda)$- and $T(E)$-bimodules under which the bimodules ${}_{T(\Lambda)}T(\Lambda)_{T(\Lambda)}$ and ${}_{T(E)}T(E)_{T(E)}$ correspond. Since we are dealing with self-injective algebras, the fact that $T(\Lambda)$ has period n is equivalent to the formula $\Omega^n(T(\Lambda)) \simeq T(\Lambda)$ in the stable bimodule category for $T(\Lambda)$; here Ω denotes the syzygy functor for the latter category. Since $T(\Lambda)$ and $T(E)$ correspond under an equivalence of stable bimodule categories, it follows that n must also be a period for $T(E)$, as required.

3 Almost Koszul rings

Throughout this section S will be a semisimple artinian ring and the tensor product $\otimes_S$ will be written as $\otimes$ without a suffix. In any graded left or right S-module, M, we assume that the components, M_n, are artinian. In Section 3.4 we require also that S is separable over a field k.

3.1 Definition

In [5], Definitions 1.1.2 and 1.2.1, a positively graded ring $A = \oplus_{n\in\mathbb{N}} A_n$ with $A_0 = S$ is called a left Koszul ring if A_0 considered as a graded left A-module has a graded projective resolution

$$\cdots \to P^2 \to P^1 \to P^0 \to A_0 \to 0$$

in which each P^i is generated by its component of degree i and hence $P^i = A \otimes P^i_i$.

Definition 3.1. Let A be a graded ring with $A_0 = S$. We call A a *left almost Koszul ring* if there exist integers $p, q \geq 1$ such that

1. $A_n = 0$ for all $n > p$, and
2. there is a graded complex

$$P^\bullet: \quad 0 \to P^q \to \cdots \to P^1 \to P^0 \to 0 \tag{2}$$

of projective left A-modules such that each P^i is generated by its component P^i_i of degree i, and the only non-zero homology is A_0 in degree 0, and $W = A_p \otimes P^q_q$ in degree $p + q$.

We say then that A is a *left* (p, q)*-Koszul ring.*

Note that, in the above definition, W is semisimple. If $W = 0$, then A is left Koszul and of global dimension q at most. If $W \neq 0$ and $p \geq 2$, then A is definitely not left Koszul because the $(q+2)$-nd projective P^{q+1} in the minimal resolution of A_0 is generated in degree $q + p > q + 1$; if $p = 1$, A may or may not be left Koszul.

The definition of almost Koszul implies, in the first place, that the graded components of the projective P^i are of the form $A_j \otimes P^i_i$. Thus the complex (2) decomposes into $p + q + 1$ homogeneous subcomplexes as follows:

$$\begin{array}{ccccccccc}
A_0 \otimes P^q_q & & \cdots & & A_0 \otimes P^2_2 & & A_0 \otimes P^1_1 & & A_0 \otimes P^0_0 \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
A_1 \otimes P^q_q & & \cdots & & A_1 \otimes P^2_2 & & A_1 \otimes P^1_1 & & A_1 \otimes P^0_0 \\
& \searrow & & \searrow & & \searrow & & \searrow & \\
A_2 \otimes P^q_q & & \cdots & & A_2 \otimes P^2_2 & & A_2 \otimes P^1_1 & & A_2 \otimes P^0_0 \\
& \searrow & & & & \searrow & & \searrow & \\
\vdots & & & & \vdots & & \vdots & & \vdots \\
& & & \searrow & & \searrow & & \searrow & \\
A_p \otimes P^q_q & & \cdots & & A_p \otimes P^2_2 & & A_p \otimes P^1_1 & & A_p \otimes P^0_0
\end{array} \qquad (3)$$

where, to save space, we have omitted zeros from each end of each of the subcomplexes. The definition now amounts to the requirement that all the subcomplexes are exact except the one term complexes at the top right and bottom left corners, that is, the complexes in degrees 0 and $p + q$. For example, since $P^0_0 = A_0 = S$, the exactness of the degree 1 complex gives $P^1_1 \simeq A_1$.

The definition also determines a minimal projective resolution of S as either a graded or an ungraded left A-module. Since each projective P^i is generated in the least possible degree, the complex (2) provides the first $q+1$ terms of such a resolution. Also the right action of S on A_0 lifts uniquely to make (2) a complex of A, S-bimodules. Let W^n be the n-fold tensor power of W over S, so that $W^0 = S = A_0$. In the complex

$$P^\bullet \otimes W^n : 0 \to P^q \otimes W^n \to \cdots \to P^0 \otimes W^n \to 0$$

the terms are projective left A-modules and the non-zero homology occurs only at the ends; at the left-hand end it is W^{n+1} and at the right-hand end it is W^n. Thus these complexes may be spliced together to form a projective A-module resolution of S which, since W is semi-simple, is minimal.

The minimality of (2) also implies that the complex $\mathrm{Hom}_A(P^\bullet, S)$ has trivial differentials. Hence

$$\mathrm{Ext}^i_A(S,S) = (P^i_i)^* \quad \text{for } 0 \leq i \leq q$$

and $\mathrm{Ext}^{q+1}_A(S,S) = W^*$, where for any left S-module X, we define X^* to be the right S-module $\mathrm{Hom}_S(X,S)$.

Proposition 3.2. *If A is a left (p,q)-Koszul ring with $p > 1$, then*

$$B = \bigoplus_{i=0}^{q} \mathrm{Ext}^i_A(S,S) \tag{4}$$

is a subalgebra of $\mathrm{Ext}^\bullet_A(S,S)$. Furthermore, B is generated by B_0 and B_1 and $\mathrm{Ext}^\bullet_A(S,S)$ is generated by B and $W^* = \mathrm{Ext}^{q+1}_A(S,S)$.

Proof. The fact that B is generated by B_0 and B_1 follows as in the Koszul case (cf. [5] Section 2.10). To show then that B is a subalgebra, it is sufficient to prove that $B_1 \cdot B_q = 0$ and $B_q \cdot B_1 = 0$, which may easily be verified, making use of the fact that $p > 1$. Finally, the fact that B and W^* generate follows by routine arguments from the fact that W is semi-simple.

We may also use the properties of the graded ext algebra to characterize almost Koszul rings, by direct analogy with [5] Proposition 2.1.3. This proposition states that a graded ring A is Koszul if and only if, for every $i, n \geq 0$, the graded ext groups $\mathrm{ext}^i(A_0, A_0[n])$ are 0 unless $n = i$. Our analogue is as follows.

Proposition 3.3. *Let A be a graded ring with $A_0 = S$ and with non-zero graded components only in degrees at most p, where $p \geq 1$. Also, let $q \geq 1$. Then A is a left (p,q)-Koszul ring if and only if*

1. *for* $0 \leq i \leq q$, $\mathrm{ext}^i(A_0, A_0[n]) = 0$ *unless* $n = i$, *and*

2. $\mathrm{ext}^{q+1}(A_0, A_0[n]) = 0$ *unless* $n = p + q$.

We next observe that the proof in [5], Proposition 2.2.1, that the opposite ring of a left Koszul ring is left Koszul, also goes through for almost Koszul rings.

Proposition 3.4. *A left* (p, q)*-Koszul ring is also a right* (p, q)*-Koszul ring.*

Proof. The idea of the proof is that, if A is a left (p, q)-Koszul ring, then application of the exact functor $X \mapsto {}^*X = \mathrm{Hom}_S(-, S_S)$ from mod-S to S-mod converts the defining sequence into the first $q + 1$ terms of a graded injective resolution of the *right* A-module $A_0 \cong {}^*A_0$,

$$0 \leftarrow {}^*P^q \leftarrow \cdots \leftarrow {}^*P^0 \leftarrow 0$$

with $(q + 1)$-st cosyzygy ${}^*W$ in degree $-p - q$. One may then use this resolution to calculate the graded $\mathrm{ext}^i(A_0, A_0[n])$ groups needed, according to Proposition 3.3, to prove that A is a right (p, q)-Koszul ring.

3.2 Example: truncated rings

The notation $V^{\otimes}$ is used to denote the tensor ring over S of the S, S-bimodule V, V^r to denote the r-fold tensor power of V, and (V^r) the ideal it generates. A ring of the form $A = V^{\otimes}/I$, where $I = (V^{p+1})$, is called a truncated ring. For such a ring, clearly $A = A_0 + A_1 + \cdots + A_p$, where $A_r = V^r$, and there is an obvious exact sequence of graded left A-modules

$$0 \to A_p \otimes V \to A \otimes V \to A \to A_0 \to 0.$$

This shows that A is left $(p, 1)$-Koszul. In fact the following proposition accounts for all (p, q)-Koszul rings with either $p = 1$ or $q = 1$.

Proposition 3.5. *Any left* $(p, 1)$*-Koszul ring is a truncated ring and any left* $(1, q)$*-Koszul ring is truncated of the form* $V^{\otimes}/(V^2)$.

Proof. Let A be a left (p,1)-Koszul ring. Then the diagram (3) has only two columns and all the maps are isomorphisms. Thus $A_{i+1} \simeq A_i \otimes A_1 \simeq (A_1)^i$

for $i = 0, 1, \cdots, p-1$ and so A is truncated. Any left $(1, q)$-Koszul ring A is trivially truncated, for then

$$A = A_0 + A_1 = {A_1}^{\otimes}/({A_1}^2).$$

Remark 3.6. Note that a truncated ring of the form $V^{\otimes}/(V^2)$ is actually Koszul with dual $V^{\otimes}$. Hence the full Ext algebra $\mathrm{Ext}_A^{\bullet}(S, S)$ is generated in degrees 0 and 1 and B as defined in Proposition 3.2 is not a subalgebra. Thus the condition $p > 1$ is necessary in Proposition 3.2 and the case $p = 1$ is degenerate.

A projective bimodule resolution of a truncated ring A (as above) may be constructed starting from the fundamental four-term exact sequence

$$0 \to \Sigma \to A \otimes V \otimes A \to A \otimes A \to A \to 0 \tag{5}$$

where $\Sigma = I/I^2$ (see formula (1.2) of [9]). Now $\Sigma \simeq A \otimes W$ as an A, S-bimodule, where $W = V^{p+1}$, so we obtain A, S-bimodule isomorphisms

$$\Sigma \otimes_A (A \otimes X) \simeq A \otimes W \otimes X$$

for all S, S-bimodules X. Hence, we may apply $\Sigma \otimes_A -$ repeatedly to (5) to obtain four-term exact sequences

$$0 \to \Sigma^{(n+1)} \to A \otimes (W^n \otimes V) \otimes A \to A \otimes W^n \otimes A \to \Sigma^{(n)} \to 0 \tag{6}$$

where $\Sigma^{(n)}$ denotes the n-fold tensor power of Σ over A. On splicing these sequences together we obtain a projective bimodule resolution which is minimal because it has the same terms as the minimal resolution constructed in [9] Example 5.3.

A special case of a truncated ring is provided by the trivial extension algebra $A = T(kQ)$, where $Q = \mathbb{A}_n$ with linear ordering. In this case, A is the quotient of the path algebra of the cyclically oriented $\widetilde{\mathbb{A}}_{n-1}$ by the ideal of all paths of length at least $n+1$. The arbitrary truncation of such a cyclically oriented quiver was studied by Liu and Zhang [21]. In particular, they show that the minimal projective bimodule resolution is periodic.

In the case $A = T(kQ)$, with $Q = \mathbb{A}_n$ as above, we have $\Sigma = \Omega^2(A) = {}_1A_\alpha$ where α is the automorphism of A given by rotating the quiver one place. Since α has order n, $2n$ is a period and $\Omega^{2r}(A) \not\simeq A$ for $r < n$. For $n \geq 2$ the odd syzygies of A have dimension greater than that of A and so $2n$ is the least period. For $n = 1$ one verifies directly that $\Omega(A) \simeq A$ if and only if $\operatorname{char} k = 2$. In particular we see that Theorem 2.1 holds for the quivers $\mathbb{A}_1$ and $\mathbb{A}_2$.

3.3 Quadratic duality

In Section 2.3 of [5] it is proved that a Koszul ring is a *quadratic ring*, that is, it is the quotient of a tensor ring $V^{\otimes} = S \oplus V \oplus V^2 \oplus \cdots$ over S of the S,S-bimodule, V, by an ideal (R) generated by an S,S-sub-bimodule $R \subset V^2$. However the truncated rings already show that $(p,1)$-Koszul rings are not quadratic (unless $p = 1$). Taking account of these exceptions, the arguments in [5] give the following result.

Proposition 3.7. *A (p,q)-Koszul ring A is generated by A_0 and A_1, and, if $q \geq 2$, then A is quadratic.*

The following lemma shows that, for any quadratic ring, the first three terms of a minimal graded projective resolution of A_0 are generated in degrees 0, 1 and 2, respectively.

Lemma 3.8. *Let $A = V^{\otimes}/(R)$ be a quadratic ring. The graded complex of projective left A-modules*

$$A \otimes R \xrightarrow{\iota} A \otimes V \xrightarrow{m} A \to 0, \tag{7}$$

in which m is the multiplication map and ι is induced by the inclusion of R in $V \otimes V$ followed by multiplication, provides the first three terms of the minimal projective A-module resolution of A_0 as a left A-module.

Proof. The complex clearly has homology A_0 in degree 0, and in degree 1 reduces to the exact sequence $0 \to A_0 \otimes V \to A_1 \to 0$. It therefore suffices to show that for each $n \geq 2$, there is an exact sequence of S,S-bimodules

$$A_{n-2} \otimes R \to A_{n-1} \otimes V \to A_n \to 0. \tag{8}$$

For each $n \geq 2$ there is an exact sequence

$$0 \to R_n \to V^n \to A_n \to 0$$

in which R_n is the degree n component of the ideal (R). Let $R_0 = R_1 = 0$. Then, for $n \geq 2$, $R_n = R_{n-1}V + V^{n-2}R$. We can form the commutative diagram with exact rows

$$\begin{array}{ccccccccc}
0 & \to & R_{n-2} \otimes R & \to & V^{n-2} \otimes R & \to & A_{n-2} \otimes R & \to & 0 \\
 & & \downarrow \iota_R & & \downarrow \iota_V & & \downarrow \iota & & \\
0 & \to & R_{n-1} \otimes V & \to & V^{n-1} \otimes V & \to & A_{n-1} \otimes V & \to & 0 \\
 & & \downarrow m_R & & \downarrow m_V & & \downarrow m & & \\
0 & \to & R_n & \to & V^n & \to & A_n & \to & 0
\end{array}$$

in which the three upper vertical maps are induced by the inclusion of R in V^2 and the three lower ones are multiplication maps. Since m_V is an isomorphism, m is certainly surjective. Given that $R_n = \text{Im } m_R + \text{Im } \iota_V m_V$, exactness at the middle term of the right-hand column is easily verified by diagram chasing. This completes the proof of the lemma.

From now on we assume that A is a graded quadratic ring. As in [5], Section 2.6, we define S, S-sub-bimodules K^i of V^i by the formulae $K^0 = S$, $K^1 = V$, $K^2 = R$ and $K^{i+1} = VK^i \cap K^i V$ for all $i \geq 2$. *Warning: our* K^i *are denoted* K^i_i *in* [5]. The traditional left Koszul complex of A is

$$\cdots \to A \otimes K^n \to \cdots \to A \otimes K^1 \to A \otimes K^0 \to 0. \tag{9}$$

The n-th term is so graded that $(A \otimes K^n)_{n+m} = A_m \otimes K^n$ and the differential out of this term is the composite of the following sequence of natural maps

$$A \otimes K^n \hookrightarrow A \otimes VK^{n-1} \to AA_1 \otimes K^{n-1} \hookrightarrow A \otimes K^{n-1}.$$

Theorem 2.6.1 in [5] shows that A is a Koszul ring if and only if its left Koszul complex is a projective resolution of A_0. Its proof depends on Proposition 2.1.3 of that paper and on the fact that, for any n, the sequence

$$A_0 \otimes K^{n+1} \to A_1 \otimes K^n \to A_2 \otimes K^{n-1}$$

is exact. The same proof, but with the first proposition replaced by our Proposition 3.3, gives the following result.

Proposition 3.9. *A quadratic ring A is left (p, q)-Koszul (with $p, q \geq 2$) if and only if*

1. *$A_n = 0$ for all $n > p$,*
2. *$K^m = 0$ for all $m > q$ and*
3. *the only non-zero homology modules of its left Koszul complex are A_0 in degree 0, and $A_p \otimes K^q$ in degree $p + q$.*

In this case the left Koszul complex for A provides the first $q+1$ terms of the minimal projective resolution of A_0.

Next we consider the alternative versions described in Section 2.8 of [5] of the left and right Koszul complexes associated with the quadratic ring A, which makes use of the left and right quadratic dual rings of A. The *left quadratic dual ring* of A, denoted $A^!$, is defined by the formula

$$A^! = (V^*)^{\otimes}/(R^{\perp})$$

where $V^* = \mathrm{Hom}_S({}_SV, {}_SS)$ and $R^{\perp} \subset (V^*)^2 = (V^2)^*$ is the annihilator of $R \subset V^2$. Similarly, the *right quadratic dual ring* of A, denoted ${}^!A$, is defined by the formula

$${}^!A = ({}^*V)^{\otimes}/({}^{\perp}R)$$

where ${}^*V = \mathrm{Hom}_S(V_S, S_S)$ and ${}^{\perp}R \subset ({}^*V)^2 = {}^*(V^2)$ is the annihilator of $R \subset V^2$, this time in ${}^*(V^2)$. Their connection with the Koszul complexes (left and right) of A is made via the formulae

$${}^*(A^!_i) = K^i = ({}^!A_i)^*. \tag{10}$$

Remark 3.10. The left Koszul complex (9) of A is isomorphic to

$$\cdots \to A \otimes {}^*(A^!_2) \to A \otimes {}^*(A^!_1) \to A \otimes {}^*(A^!_0) = A \to 0$$

where the differential

$$A \otimes {}^*(A^!_m) \to A \otimes {}^*(A^!_{m-1})$$

is induced by the left dual of the multiplication map $A^!_{m-1} \otimes A^!_1 \to A^!_m$ in $A^!$ and the identification of ${}^*(A^!_1)$ with A_1.

Suppose now that A is also a left (p, q)-Koszul ring. From the last proposition it follows that $A^!_m = 0$ for $m > q$ and that the only non-zero homology of the complex

$$0 \to A \otimes {}^*(A^!_q) \to \cdots \to A \otimes {}^*(A^!_0) \to 0 \tag{11}$$

is A_0 in degree 0 and $A_p \otimes {}^*(A^!_q)$ in degree $p + q$.

Also the total space of this complex can be viewed as a bigraded $A, A^!$-bimodule with a differential of bidegree $(1, -1)$ which has non-zero homology only in bidegrees $(0, 0)$ and (p, q). On applying to it the exact functor $\mathrm{Hom}_S(-, {}_SS)$, we obtain a bigraded $A^!, A$-bimodule, with a differential of bidegree $(1, -1)$ with non-zero homology only in bidegrees $(0, 0)$ and (q, p), and which is the total space of the complex

$$0 \to A^! \otimes A_p^* \to \cdots \to A^! \otimes A_0^* \to 0.$$

This complex has non-zero homology only in degree 0, where it is $A_0^!$, and in degree $p+q$, where it is $A_q^! \otimes A_p^*$. Together with the fact that $A_n^! = 0$ for all $n > q$, this proves the following result.

Proposition 3.11. *If A is a left (p, q)-Koszul ring with $p, q \geq 2$, then $A^!$ is a left (q, p)-Koszul ring.*

Remark 3.12. The fact that the complex (11) provides the first $q+1$ terms of the minimal projective resolution of S as a left A-module means that $A_i^! = \operatorname{Ext}_A^i(S, S)$. In fact a stronger statement is true: $A^! = B$, where B is the 'initial' sub-algebra of the Ext algebra defined in Proposition 3.2.

Remark 3.13. Our main source for quadratic duals, [5], deals with algebras over an arbitrary semisimple base ring and so distinguishes left and right dualities into that base ring. Hence a quadratic ring A has two duals, $A^!$ and ${}^!A$. Each of our examples is a quotient of the path algebra over a base field k of a finite quiver $Q = (Q_0, Q_1)$ for which the base ring is $S = \sum_{i \in Q_0} ke_i$, where the e_i are a complete set of orthogonal primitive idempotents. We may then replace dualities into S by vector space duality $\mathrm{D}(-) = \operatorname{Hom}_k(-, k)$ into k. For example, if ${}_SV$ is a left S-module, we may identify the right S-module $\mathrm{D}V$ with its right S dual $V^* = \operatorname{Hom}_S(V, S)$ by associating with $\theta \in \mathrm{D}V$ the map $\theta^* \colon v \mapsto \sum_{i \in Q_0} \theta(e_i v) e_i$ of V to S. Furthermore if V is an S, S-bimodule, then θ^* is an S, S-bimodule morphism. Such identifications enable us to identify the left and right duals $A^!$ and ${}^!A$ of a quadratic algebra A. In particular, the degree m component of each of these algebras may be identified with the vector space dual $\mathrm{D}K^m$ of the m-th term of the Koszul complex of A.

We conclude this subsection with a brief discussion of Hilbert series of almost Koszul algebras. Let A be a quotient of the path algebra of a quiver Q by a homogeneous ideal, graded as usual by pathlength. Its Hilbert series is defined, for example in [5], to be the $|Q_0| \times |Q_0|$ matrix $P(A, t)$ whose ij-th matrix entry is the formal series $P(A, t)_{ij} = \sum_{n \geq 0} t^n \dim_k(e_i A_n e_j)$. We will give an analogue for almost Koszul algebras of the formula

$$P(A, t) P(A^!, -t)^T = I$$

satisfied by any Koszul algebra A.

For any A and any natural number q, let $N(A,q)$ be the matrix with ij-th entry the multiplicity of the i-th simple A-module Se_i as a composition factor in $\Omega^{q+1}(Se_j)$. If A is (p,q)-Koszul, then $\Omega^{q+1}(Se_j) = A_p \otimes \mathrm{D}(A^!_q)e_j$; it is semisimple and

$$N(A,q)_{ij} = \dim_k e_i \Omega^{q+1}(Se_j) = \sum_{l \in Q_0} \dim_k(e_i A_p e_l) \dim_k(e_j A^!_q e_l).$$

Applying the Euler-Poincaré Principle to the left Koszul complex (11) of A, just as in the proof of Lemma 2.11.1 of [5], we obtain the following proposition.

Proposition 3.14. *Let A be a left (p,q)-Koszul algebra with $p,q \geq 2$. Then*

$$P(A,t)P(A^!,-t)^T = I + (-1)^q t^{p+q} N(A,q)$$

and hence $N(A^!,p) = N(A,q)^T$.

Suppose that, in addition, A is self-injective. Then $\Omega^{q+1}(Se_j)$ is simple and $N(A,q)$ is a permutation matrix. As a permutation it is the product $\nu_A \nu_{A^!}^{-1}$, where ν_A and $\nu_{A^!}$ are the Nakayama permutations for A and $A^!$.

3.4 Koszul bimodule complexes

To any quadratic ring, one may associate a Koszul complex of projective bimodules which is a (necessarily minimal) resolution of the ring precisely when it is Koszul ([7] Section 3.6, [9] Section 9). In this section, we show that if the ring is almost Koszul, then this Koszul bimodule complex provides an initial segment of a minimal projective resolution.

The Koszul bimodule complex $(P(K^\bullet), d)$ of a quadratic ring A is given as follows. Its terms are the graded projective A,A-bimodules

$$P(K^m) = A \otimes K^m \otimes A,$$

and its differential is a sum $d = d_l + d_r$ of anticommuting differentials $d_l, d_r \colon P(K^m) \to P(K^{m-1})$ given by $d_l = d' \otimes 1_A$ and $d_r = (-1)^m 1_A \otimes d''$, where d' and d'' are the differentials in the left and right Koszul complexes $A \otimes K^\bullet$ and $K^\bullet \otimes A$ of A, respectively. The term of total degree t in the projective bimodule $P(K^m)$ is

$$P(K^m)_t = \bigoplus_{i+m+j=t} A_i \otimes K^m \otimes A_j. \tag{12}$$

Now assume that A is a (p,q)-Koszul ring with $p, q \geq 2$. Then $A_n = 0$ for $n > p$ and $K^m = 0$ for $m > q$ and the Koszul bimodule complex

$$0 \to P(K^q) \to P(K^{q-1}) \to \cdots \to P(K^0) \to 0$$

has just $q+1$ non-zero terms.

The theorem below provides a partial description of the last kernel

$$\Sigma = \operatorname{Ker}\left(P(K^q) \to P(K^{q-1})\right).$$

Clearly Σ is a graded A, A-bimodule with degrees bounded below by q and above by $2p+q$. We write $Z = \Sigma_{p+q}$ so that

$$Z \subseteq \bigoplus_{i=0}^{p} A_i \otimes K^q \otimes A_{p-i}.$$

Let $\pi_l \colon Z \to A_p \otimes K^q \otimes A_0$ and $\pi_r \colon Z \to A_0 \otimes K^q \otimes A_p$ be the canonical projections. Note that the codomain of π_l is the $(q+1)$-st syzygy W that occurs in Definition 3.1 of a left almost Koszul ring, while the codomain of π_r is the corresponding syzygy for a right almost Koszul ring.

Theorem 3.15. *Let A be a (p,q)-Koszul ring with $p, q \geq 2$. The Koszul bimodule complex $(P(K^\bullet), d)$ provides the first $q+1$ terms of the minimal projective A, A-bimodule resolution of the ring A. The $(q+1)$-st syzygy bimodule Σ is generated by its degree $p+q$ component Z. More precisely, the inclusion of Z in Σ induces a left A-module isomorphism of $A \otimes Z$ with Σ and a right A-module isomorphism of $Z \otimes A$ with Σ. Moreover, the S, S-bimodule maps π_l and π_r are isomorphisms.*

Proof. Note that $P(K^1) = A \otimes V \otimes A$, $P(K^0) = A \otimes A$ and the differential d maps $1 \otimes v \otimes 1$ to $v \otimes 1 - 1 \otimes v$. Thus the cokernel of d at $P(K^0)$ is just the multiplication map $A \otimes A \to A$.

To discuss the homology of the complex at $P(K^m)$ for $m \geq 1$, we use the fact that the complex $(P(K^\bullet), d)$ is the direct sum of subcomplexes $(P(K^\bullet)_t, d)$, one for each t, $0 \leq t \leq 2p+q$, with terms given by (12). For each t we shall compute the homology of $(P(K^\bullet)_t, d)$ by viewing it as the total complex of the finite double complex

$$
\begin{array}{ccccc}
 & \vdots & & \vdots & \\
\cdots & A_i \otimes K^m \otimes A_j & \xrightarrow{d_l} & A_{i+1} \otimes K^{m-1} \otimes A_j & \cdots \\
 & \downarrow d_r & & \downarrow d_r & \\
\cdots & A_i \otimes K^{m-1} \otimes A_{j+1} & \xrightarrow{d_l} & A_{i+1} \otimes K^{m-2} \otimes A_{j+1} & \cdots \\
 & \vdots & & \vdots &
\end{array}
$$

which we denote by $(P(K^\bullet)_t, d_l, d_r)$.

Let $m \geq 1$. Since A is left (p,q)-Koszul, the j-th row of this double complex is exact at the term containing K^m unless $i = p$ and $m = q$ and so $t = p + q + j$; similarly since A is right (p,q)-Koszul, the i-th column of this double complex is exact at the term containing K^m unless $j = p$ and $m = q$ and so $t = p + q + i$.

We now make use of a familiar argument for computing the total homology of a double complex. First, for $1 \leq m < q$, the contribution of each total degree t is zero. Hence our complex is exact at $P(K^m)$ for $1 \leq m < q$. Second, when $m = q$, the degree t component of Σ is zero for $t < p+q$. Consider next the case $t = p + q$ corresponding to the calculation of $Z = \Sigma_{p+q}$. All the rows of $(P(K^\bullet)_t, d_l, d_r)$ are exact except the row with the single entry $A_p \otimes K^q \otimes A_0$, as are all columns except that with the single entry $A_0 \otimes K^q \otimes A_p$. Now any element of Z has the form $z = z_0 + z_1 + \cdots + z_p$ where z_i is in $A_i \otimes K^q \otimes A_{p-i}$ and $d_l z_i + d_r z_{i+1} = 0$ for $i = 0, \cdots, p-1$. That the projections $\pi_r \colon z \mapsto z_0$ and $\pi_l \colon z \mapsto z_p$ are both isomorphisms is now an easy consequence of the exactness properties noted above.

Since Σ is the $(q+1)$-st syzygy for A as a bimodule, then $\Sigma/(\text{rad}A)\Sigma$ is the $(q+1)$-st syzygy for A_0 as a right A-module. Since A is a right (p,q)-Koszul ring, it follows that $\Sigma/(\text{rad}A)\Sigma$ is isomorphic to $K^q \otimes A_p$. Hence the inclusion of Z in Σ induces a commutative diagram

$$
\begin{array}{ccc}
Z & \rightarrow & \Sigma/(\text{rad}A)\Sigma \\
\downarrow \pi_r & & \downarrow \simeq \\
A_0 \otimes K^q \otimes A_p & \simeq & K^q \otimes A_p
\end{array}
$$

and so the map $Z \rightarrow \Sigma/(\text{rad}A)\Sigma$ is an isomorphism. We deduce that $A \otimes Z$ is isomorphic to Σ. Similarly $Z \otimes A$ is isomorphic to Σ.

This completes the proof of the theorem.

4 The preprojective algebras

Let $Q = (Q_0, Q_1)$ be a connected finite quiver with more than one vertex and without oriented cycles. Let $B = B(Q)$ be the preprojective algebra of Q over a field k. In Subsection 4.1 we show that, if Q is a Dynkin quiver with Coxeter number h, then B is $(h-2, 2)$-Koszul. This complements the known result [22] that, if Q is not Dynkin, then B is a Koszul algebra of global dimension 2, for which we give a short new proof.

The rest of this section is concerned only with the preprojective algebras of Dynkin quivers. In Subsection 4.2, we develop formulae relating some of the multiplication constants for B (deferring a case by case determination of a certain constant C to Subsection 4.4). These formulae are used to construct explicitly an element $\hat{u}$ of $\mathrm{D}B$ such that $\mathrm{D}B = B\hat{u} = \hat{u}B$; hence B is self-injective. We also determine a Nakayama automorphism β of B. The formulae are used again in Section 5, in association with Theorem 3.15, to calculate the $(h-1)$-st syzygy bimodule of the quadratic dual A of B. Subsection 4.3 contains a new proof of the unpublished result of Ringel and Schofield that the third syzygy bimodule of B is $\mathrm{D}B$.

4.1 Definition and first properties

We make use of two equivalent descriptions of the preprojective algebras.

In the first description B is the quotient $k\overline{Q}/\rho$ of the path algebra of a quiver $\overline{Q}$ by an ideal ρ of relations. The quiver $\overline{Q}$ has vertex set Q_0 and arrows $y_{ij}\colon i \to j$ and $y_{ji}\colon j \to i$ corresponding to each arrow $i \to j$ in Q_1. The ideal ρ is generated by the quadratic relations

$$\rho_i = \sum_{j \in \mathcal{N}(i)} \epsilon_{ij} y_{ij} y_{ji}, \tag{13}$$

one for each i in Q_0, where $\mathcal{N}(i)$ is the set of neighbours of i in Q and

$$\epsilon_{ij} = \begin{cases} 1 & \text{if } i \to j \in Q_1, \\ -1 & \text{if } j \to i \in Q_1. \end{cases}$$

Thus B is a quadratic algebra with a natural path-length grading, relative to which its graded components will be denoted by B_0, B_1, B_2, $\cdots$. (Our simplistic notation for arrows suffices since we are really only interested in quivers without multiple arrows.)

The second description is due to Baer, Geigle and Lenzing [3]. We call it the BGL view and refer to [26] and [11] for proofs of its equivalence to the quiver with relations description. The idea behind the BGL view is that a preprojective algebra should be the direct sum of the indecomposable preprojective modules of an hereditary algebra. Let $\Lambda = kQ$. Since Λ is hereditary, the Auslander translate τ^{-1} on the category of left Λ-modules is the endofunctor $\mathrm{Ext}^1_\Lambda(\mathrm{D}\Lambda, -)$. In the BGL view, the preprojective algebra of Λ is defined to be the graded algebra

$$B = B^{(0)} \oplus B^{(1)} \oplus B^{(2)} \oplus \cdots ,$$

in which $B^{(n)} = \mathrm{Hom}_\Lambda(\Lambda, \tau^{-n}\Lambda)$, with multiplication $B^{(n)} \times B^{(m)} \to B^{(n+m)}$; $(u, v) \mapsto uv$, where uv is the composite map

$$\Lambda \xrightarrow{u} \tau^{-n}\Lambda \xrightarrow{\tau^{-n}v} \tau^{-m-n}\Lambda.$$

Evidently B is isomorphic as a left Λ-module to the direct sum of one copy of each indecomposable preprojective left Λ-module. This makes it clear that B is finite dimensional if and only if Q is a Dynkin quiver.

To relate the two points of view, observe that a basis element e_i of B_0, where $i \in Q_0$, corresponds to the canonical projection of Λ onto Λe_i in $B^{(0)}$, and a basis element y_{ij} of B_1 corresponds to an irreducible map from Λe_i to Λe_j if $i \to j \in Q_1$ (and is thus an element of $B^{(0)}$) and to an irreducible map from Λe_i to $\tau^{-1}(\Lambda e_j)$ if $j \to i \in Q_1$ (and is thus an element of $B^{(1)}$).

When Q is a Dynkin quiver, B is not only finite dimensional, but is also self-injective. The Nakayama functor on left Λ-modules [16, section 2.1] induces a permutation ν of the vertices of Q such that $\mathrm{D}(e_i\Lambda) = \tau^{-n}(\Lambda e_{\nu(i)})$ for some $n = n(i)$. Note that ν is independent of the orientation of Q but n is not. Furthermore, for all $i \in Q_0$, the degree of the map from Λe_i to $\mathrm{D}(e_{\nu(i)}\Lambda)$ is $h - 2$, where h is the Coxeter number of Q. Thus $B_{h-2} \neq 0$ and, since the composite of any $h - 1$ irreducible maps is zero, $B_n = 0$ for $n \geq h - 1$.

Later on it will be convenient to choose special orientations of Q; the following lemma gives us sufficient freedom to do this.

Lemma 4.1. *Let $v \in Q_0$ and $Q^{(v)}$ be the quiver with the same vertices as Q and arrow set obtained from that of Q by reversing all arrows incident on v. Then $B(Q^{(v)}) \simeq B(Q)$.*

Proof. Using the quiver with relations presentation, we observe that the mapping $e_i \mapsto e_i$, $y_{ij} \mapsto y^{(v)}_{ij}$ for $i \neq v$ and $y_{vj} \mapsto -y^{(v)}_{vj}$ extends to an isomorphism $k\overline{Q} \simeq k\overline{Q}^{(v)}$ of tensor algebras. The lemma follows because, for each $i \in Q_0$, this isomorphism maps ρ_i to $\rho_i^{(v)}$.

As a first application we obtain the projective presentations of the simple right B-modules associated with the vertices of Q. The proof was suggested by ideas in some manuscript notes kindly provided by Bill Crawley-Boevey.

Proposition 4.2. *For each vertex i in Q_0, there is an exact sequence of right B-modules*

$$e_iB \to \bigoplus e_jB \to e_iB \to S_i \to 0, \tag{14}$$

where j runs through all neighbours of i in Q and S_i is the one dimensional B-module on which e_i acts as the identity.

If Q is not Dynkin, then the left hand map is injective.

If Q is Dynkin, then the kernel of the left hand map is $S_{\nu(i)}$.

Proof. Choose an orientation of Q so that the left projective Λ-module Λe_i is simple. Then the almost split sequence starting at Λe_i has the form

$$0 \to \Lambda e_i \to \bigoplus_{i \to j \in Q_1} \Lambda e_j \to \tau^{-1}(\Lambda e_i) \to 0. \tag{15}$$

Applying the functor $\mathrm{Hom}_\Lambda(- , B)$ to (15), we obtain the exact sequence of right B-modules

$$0 \to \mathrm{Hom}_\Lambda(\tau^{-1}(\Lambda e_i), B) \to \bigoplus_{i \to j \in Q_1} e_jB \to e_iB \to \mathrm{Ext}^1_\Lambda(\tau^{-1}(\Lambda e_i), B) \to 0. \tag{16}$$

Now ${}_\Lambda B$ is the direct sum of one copy of each indecomposable preprojective Λ-module. Hence the right B-module $\mathrm{Ext}^1_\Lambda(\tau^{-1}(\Lambda e_i), B)$ is one-dimensional and is not annihilated by e_i and so is the simple module S_i described in the statement of the proposition. The middle terms of (16) are the first two terms of the minimal projective presentation of S_i.

Also $\mathrm{Hom}_\Lambda(\tau^{-1}(\Lambda e_i), B) \simeq \mathrm{Hom}_\Lambda(\Lambda e_i, \tau B)$. If Q is not Dynkin, then $\tau B = B$ and so this term is the projective module e_iB and the projective dimension of S_i is 2.

If Q is Dynkin, then there is a split exact sequence of left Λ-modules

$$0 \to \mathrm{D}\Lambda \to B \to \tau B \to 0 \tag{17}$$

in which the first map is inclusion and the second is the projection of B onto its Λ-module direct summand τB. Since Λe_i is simple, $\mathrm{Hom}_\Lambda(\Lambda e_i, \mathrm{D}\Lambda) \simeq S_{\nu(i)}$ and we obtain from (16) and (17) the exact sequence of right B-modules

$$0 \to S_{\nu(i)} \to e_i B \to \bigoplus_{i \to j \in Q_1} e_j B \to e_i B \to S_i \to 0, \tag{18}$$

as required.

The first aim of this section is achieved by the following corollary.

Corollary 4.3. *If Q is a Dynkin quiver, then B is $(h-2, 2)$-Koszul. Otherwise B is Koszul.*

Proof. Lemma 3.8 shows that the first three terms of a minimal presentation of a simple module over a quadratic algebra are necessarily generated in degrees 0, 1 and 2. Thus the fact that the left-hand map in (14) is injective if Q is not a Dynkin quiver, suffices to show that B is Koszul. If Q is a Dynkin quiver, then B_{h-2} is the last non-zero component of B and the third syzygies of the simple B-modules are themselves simple. Hence B is $(h-2, 2)$-Koszul. This completes the proof.

4.2 The Dynkin case

For the remainder of this section Q is a Dynkin quiver and we denote $h-2$ by q.

The next proposition expresses a relationship between the spaces B_1, B_{q-1} and B_q. It is conveniently formulated using the BGL description of B as an algebra of maps from projective left Λ-modules. Because Q is Dynkin, each of the above spaces is a direct sum of canonical one-dimensional subspaces which we use to obtain bases. For B_1 we choose the y_{ij} as a basis. The space B_q is the direct sum of one-dimensional subspaces $e_i B_q e_{\nu(i)} = \mathrm{Hom}_\Lambda(\Lambda e_i, \mathrm{D}(e_i\Lambda))$ and from the i-th summand we choose a generator $u_{i\nu(i)}$. Furthermore B_{q-1} is a direct sum of one-dimensional subspaces $e_i B_{q-1} e_{\nu(j)}$, the sum being taken over all pairs of neighbouring vertices i and j. Specifically

$$e_i B_{q-1} e_{\nu(j)} = \begin{cases} \mathrm{Hom}_\Lambda(\Lambda e_i, \tau\mathrm{D}(e_j\Lambda)) & \text{if } \epsilon_{ij} = 1, \\ \mathrm{Hom}_\Lambda(\Lambda e_i, \mathrm{D}(e_j\Lambda)) & \text{if } \epsilon_{ij} = -1. \end{cases}$$

From each of these subspaces we choose a generator $v_{i\nu(j)}$.

We now observe that, for each $i \in Q_0$ and each neighbour j of i, the products $y_{ij}v_{j\nu(i)}$ and $v_{i\nu(j)}y_{\nu(j)\nu(i)}$ are non-zero elements of $e_i B_q e_{\nu(i)}$.

Definition 4.4. For each pair of neighbours i and j in Q, let λ_{ij} and μ_{ij} denote the non-zero scalars such that

$$\lambda_{ij} y_{ij} v_{j\nu(i)} = u_{i\nu(i)} = \mu_{ij} v_{i\nu(j)} y_{\nu(j)\nu(i)}. \tag{19}$$

Also let

$$\eta_{ij} = \lambda_{ij}\lambda_{ji}/\mu_{ij}\mu_{ji}. \tag{20}$$

Proposition 4.5. *(a) For the Dynkin quiver Q, there is a constant C such that, for each pair of neighbours i and j, $\eta_{ij} = C\epsilon_{ij}\epsilon_{\nu(i)\nu(j)}$.*
(b) $C = -1$.

Proof of Proposition 4.5, part (a). The basis elements of $\mathrm{D}B_{q-1}$ and $\mathrm{D}B_q$ dual to the basis elements v of B_{q-1} and u of B_q will be denoted $\hat{v}$ and $\hat{u}$, with suffices reversed. Formula (19) dualises to give the formulae

$$\lambda_{ij} \hat{u}_{\nu(i)i} y_{ij} = \hat{v}_{\nu(i)j} = \mu_{ji} y_{\nu(i)\nu(j)} \hat{u}_{\nu(j)j}. \tag{21}$$

We multiply this equation on the right by $(\epsilon_{ij}/\lambda_{ij})y_{ji}$ and sum over the neighbours j of i. Using (13), and (21) with i and j interchanged, we find that $0 = \sigma\hat{u}_{\nu(i)i}$, where

$$\sigma = \sum_{j\in\mathcal{N}(i)} \frac{\epsilon_{ij}}{\eta_{ij}} y_{\nu(i)\nu(j)} y_{\nu(j)\nu(i)} \in e_{\nu(i)} B_2 e_{\nu(i)}$$

may be regarded as a map from $\tau\mathrm{D}(e_i\Lambda)$ to $\mathrm{D}(e_i\Lambda)$. Any such map, if non-zero, composes with some map from Λe_i to $\tau\mathrm{D}(e_i\Lambda)$ to give $u_{i\nu(i)}$, which would imply $\sigma\hat{u}_{\nu(i)i} \neq 0$. Hence $\sigma = 0$. Since $\rho_{\nu(i)}$ is the only quadratic relation in $e_{\nu(i)}\rho\, e_{\nu(i)}$, it follows that $\epsilon_{ij}/\eta_{ij} = \eta_i\epsilon_{\nu(i)\nu(j)}$ for some constant η_i. Now $\eta_{ij} = \eta_{ji}$, $\epsilon_{ij} = -\epsilon_{ji} = \pm 1$ and Q is connected. Hence the η_i have a common value which we denote by $1/C$. This completes the proof of (a).

The proof of part (b) of Proposition 4.5 proceeds by cases; it is lengthy and is given in Subsection 4.4 at the end of this section.

Definition 4.6. Let β denote the automorphism of B defined on the generators of B by $\beta(e_i) = e_{\nu(i)}$ and

$$\beta(y_{ij}) = \begin{cases} y_{\nu(i)\nu(j)} & \text{if } \epsilon_{ij} = 1, \\ \epsilon_{\nu(i)\nu(j)} y_{\nu(i)\nu(j)} & \text{if } \epsilon_{ij} = -1. \end{cases}$$

(Of course one needs to verify that $\beta(\rho_i) = -\rho_{\nu(i)}$.)

Note that $\beta^2 = \mathrm{id}$ but $\beta = \mathrm{id}$ if and only if $\nu = \mathrm{id}$ and $\mathrm{char}\, k = 2$.

The following corollary shows that β is a Nakayama automorphism of B in the sense defined by Yamagata in [29].

Corollary 4.7. *The B, B-bimodule $\mathrm{D}B$ contains an element $\hat{u}$ such that $\mathrm{D}B = B\hat{u} = \hat{u}B$ and $\hat{u}b = \beta(b)\hat{u}$ for all $b \in B$.*

Proof. Any $\hat{u} = \sum \xi_i \hat{u}_{\nu(i)i}$ with the ξ_i non-zero scalars will generate $\mathrm{D}B$ as both a left module and as a right module. It therefore suffices to show that we may choose the ξ_i so that $\hat{u}y_{ij} = \beta(y_{ij})\hat{u}$ for each y_{ij}. It follows from (21) and Definition 4.6 that we need to choose the ξ_i so that they satisfy

$$\frac{\mu_{ji}}{\lambda_{ij}} \cdot \frac{\xi_i}{\xi_j} = \begin{cases} 1 & \text{if } \epsilon_{ij} = 1, \\ \epsilon_{\nu(i)\nu(j)} & \text{if } \epsilon_{ij} = -1. \end{cases} \tag{22}$$

Given a vertex i, a neighbour j and a non-zero scalar ξ_j, we can use (22) to define ξ_i. Then it follows from Proposition 4.5 that (22) with i and j interchanged is also satisfied. Thus we may consider a maximal connected subquiver Q' of Q such that, for all $i \in Q'_0$ and all $j \in \mathcal{N}(i) \cap Q'_0$, equation (22) is satisfied for a choice of non-zero ξ_i. Suppose $Q' \neq Q$, so that Q'_0 contains a vertex k which has a neighbour $l \notin Q'_0$. If $\epsilon_{lk} = 1$, choose $\xi_l = \frac{\lambda_{lk}}{\mu_{kl}} \xi_k$. Then the equations (22) are satisfied for all $i, j \in Q''_0$, where Q'' is obtained from Q' by adjoining the vertex l and the arrow joining it to k. If, on the other hand, $\epsilon_{lk} = -1$, choose $\xi_l = \frac{\lambda_{lk}}{\mu_{kl}} \epsilon_{\nu(i)\nu(j)} \xi_k$. Then the equations (22) hold for the same choice of Q''. In either event, this contradicts the maximality of Q' and we conclude that $Q' = Q$. This completes the proof.

Theorem 4.8. *The preprojective algebra $B = B(Q)$ of a Dynkin quiver Q is self-injective and ${}_B\mathrm{D}B_B \simeq {}_1B_\beta$, where β is given in Definition 4.6. Also B is symmetric if and only if $\mathrm{char}\, k = 2$ and Q is $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$.*

Proof. The first statement follows immediately from Corollary 4.7. The conditions on $\mathrm{char}\, k$ and Q in the second statement are precisely the conditions for β to be the identity, in which case B is certainly symmetric. Symmetry requires $\nu = \mathrm{id}$, in which case $\beta(y_{ij}) = \epsilon_{ij} y_{ij}$. In characteristic other than 2 it is easy to verify that ${}_1B_\beta$ contains no central generator to play the rôle that the identity of B plays in the bimodule ${}_1B_1$ and so ${}_1B_\beta \not\simeq {}_1B_1$.

4.3 The bimodule resolution

In this subsection we use Theorem 3.15 to give a new proof that the third syzygy of the preprojective algebra B of a Dynkin quiver is isomorphic to $\mathrm{D}B$ and hence B is periodic with period dividing 6. Though this is not relevant to our results on trivial extension algebras, our method of proof will be used in more general situations in Section 6.

Let B be the preprojective algebra of a Dynkin quiver with 2 or more vertices. Since B is $(h-2,2)$-Koszul, we can use the analysis in Subsection 3.4 of the Koszul bimodule complex $(P(K^\bullet), d)$ to express its third syzygy as a sub-bimodule of $P(K^2) = B \otimes K^2 \otimes B$. The complex $(P(K^\bullet), d)$ is

$$0 \to P(K^2) \xrightarrow{d^2} P(K^1) \xrightarrow{d^1} P(K^0) \to 0,$$

where $P(K^0) = B \otimes B$, K^1 is the S,S-bimodule with basis the generators y_{ij} of B_1 and K^2 is the S,S-bimodule spanned by the relators ρ_i for B given by (13). The maps d^1 and d^2 are given by

$$d^1(1 \otimes y_{ij} \otimes 1) = y_{ij} \otimes e_j - e_i \otimes y_{ij}$$

and

$$d^2(1 \otimes \rho_i \otimes 1) = \sum_{j \in \mathcal{N}(i)} \epsilon_{ij}(y_{ij} \otimes y_{ji} \otimes 1 + 1 \otimes y_{ij} \otimes y_{ji}).$$

Theorem 4.9. *Let B be the preprojective algebra of a Dynkin quiver Q. Its third syzygy $\Omega^3(B)$ is isomorphic to ${}_1B_\beta$, where β is given in Definition 4.6.*

As an immediate consequence of Theorem 4.8 we now have the following corollary.

Corollary 4.10 (Ringel and Schofield)**.** $\Omega^3(B) \simeq \mathrm{D}B$.

Proof of Theorem. Theorem 3.15 shows that $\Omega^3(B)$ is generated both as a left and as a right B-module by its degree h component

$$Z \subseteq P(K^2)_h = \sum_{0 \le r \le q} B_r \otimes K^2 \otimes B_{q-r}$$

and that $Z \to B_0 \otimes K^2 \otimes B_q$ is an S,S-bimodule isomorphism. Hence $Z \simeq {}_1S_\nu$ since $B_0 \simeq K^2 \simeq S$ and $B_q \simeq {}_1S_\nu$. Now Z has a basis of elements $w_{i\nu(i)}$

which project onto $e_i \otimes \rho_i \otimes u_{i\nu(i)}$ in $B_0 \otimes K_2 \otimes B_q$. Since $d^2 w_{i\nu(i)} = 0$, a short calculation using formula (19) shows that

$$w_{i\nu(i)} = e_i \otimes \rho_i \otimes u_{i\nu(i)} + \sum_{l \in \mathcal{N}(i)} \lambda_{il} y_{il} \otimes \rho_l \otimes v_{l\nu(i)} + \cdots \qquad (23)$$

where the omitted terms are the uniquely determined contributions from the summands with $r \geq 2$.

Let $w = \sum \zeta_i w_{i\nu(i)}$, where the ζ_i are non-zero scalars. Then $Z = Sw = wS$. So $\Omega^3(B) = Bw = wB$ and there is an automorphism γ of B such that $wb = \gamma(b)w$ for each element b of B. Since w is homogeneous we may and shall assume that γ has degree 0. On taking $b = e_i$, we see from (23) that $\gamma(e_i) = e_{\nu(i)}$ and hence that $\gamma(y_{ij}) = \gamma_{ij} y_{\nu(i)\nu(j)}$ for some non-zero scalar γ_{ij}. Next, on taking $b = y_{\nu(i)\nu(j)}$, we obtain $\zeta_i w_{i\nu(i)} y_{\nu(i)\nu(j)} = \zeta_j \gamma_{\nu(i)\nu(j)} y_{ij} w_{j\nu(j)}$. We substitute for the w's from (23) and use (19) to simplify products of the form yv. On comparing coefficients of terms in $B_1 \otimes K^2 \otimes B_q$ we then find that

$$\gamma_{\nu(i)\nu(j)} = \frac{\lambda_{ij}}{\mu_{ji}} \cdot \frac{\zeta_i}{\zeta_j}.$$

An argument similar to that used in the proof of Corollary 4.7 to show that the equations (22) can be satisfied, shows that the ζ_i may be chosen to give

$$\gamma_{\nu(i)\nu(j)} = \begin{cases} 1 & \text{if } \epsilon_{\nu(i)\nu(j)} = 1, \\ \epsilon_{ij} & \text{if } \epsilon_{\nu(i)\nu(j)} = -1. \end{cases}$$

With this choice of w, $\gamma = \beta$ as required.

Remark 4.11. Since $\beta^2 = \mathrm{id}$, Theorem 4.9, shows that $\Omega^6(B) \simeq {}_1B_1$ and so, as Ringel and Schofield pointed out, 6 is always a period for B. As is shown by Erdmann and Snashall in [13] and [14], the minimal period is 6 unless either $Q = \mathbb{A}_2$, in which case it is 2, or $\operatorname{char} k = 2$ and Q is one of $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, in which case it is 3. We clearly disagree with the implied assertion in [13] that for $Q = \mathbb{E}_6$, there is no automorphism γ of B of order 2 such that $\Omega^3(B) \simeq {}_1B_\gamma$.

Remark 4.12. In an unpublished manuscript, Ringel and Schofield gave an elegant direct proof that $\Omega^3(B) \simeq \mathrm{D}B$. In our terms, the essential idea of their proof is the construction of an isomorphism between d^2 and $\mathrm{D}d^1$, where d^1 and d^2 are the maps in the Koszul bimodule complex for B.

4.4 Calculation of C

We shall make use of the universal covering $\widetilde{B}$ of the preprojective algebra B of the Dynkin quiver Q in order to provide a convenient way of presenting the grading of B. The quiver $\widetilde{\Gamma}$ of $\widetilde{B}$ has vertex set $\mathbb{Z} \times Q_0$ and arrows $\tilde{y}^r_{ij}\colon (r,i) \to (r,j)$ and $\tilde{y}^r_{ji}\colon (r,j) \to (r+1,i)$ corresponding to each arrow $i \to j$ in Q. The defining relations for $\widetilde{B}$ are

$$\rho_{r,i} = \sum_{i\to j\in Q_1} \tilde{y}^r_{ij}\tilde{y}^r_{ji} - \sum_{k\to i\in Q_1} \tilde{y}^r_{ik}\tilde{y}^{r+1}_{ki}.$$

(These are equivalent to the usual mesh relations for $(-1)^r\tilde{y}^r_{ij}$.) The covering map $\widetilde{\Gamma} \to \overline{Q}$, $(r,i) \mapsto i$, $\tilde{y}^r_{ij} \mapsto y_{ij}$ induces a covering map $\widetilde{B} \to B$. We make use of the facts that, for given $i, j \in Q_0$ and $s \in \mathbb{N}$, there is at most one integer $t = t_{ij,s}$ such that there is a path in $\widetilde{\Gamma}$ of length s from $(0,i)$ to (t,j), and that the covering map induces an isomorphism from the vector space $e_{(0,i)}\widetilde{B}_s e_{(t,j)}$ to $e_i B_s e_j$.

To prove that $C = -1$ for a given Dynkin quiver Q, we first fix a vertex i with a unique neighbour j in Q. Choose a directed path $\tilde{u}$ of length q in $\widetilde{\Gamma}$ from $(0,i)$ to $(t_{i\nu(i),q}, \nu(i))$, which passes through $(t_{ij,1}, j)$ and $(t_{i\nu(j),q-1}, \nu(j))$, and which is not zero modulo the relations. Denote the element of $e_i B_q e_{\nu(i)}$ which corresponds to $\tilde{u}$ by u. Thus u is a possible choice for the element we earlier called $u_{i\nu(i)}$. We may now choose paths $\tilde{v}_{j\nu(i)}$ and $\tilde{v}_{i\nu(j)}$ so that $\tilde{u} = \tilde{y}_{ij}\tilde{v}_{j\nu(i)} = \tilde{v}_{i\nu(j)}\tilde{y}_{\nu(j)\nu(i)}$, where for simplicity we have dropped the upper indices from the $\tilde{y}_{ij}$. These map to $u = y_{ij}v_{j\nu(i)} = v_{i\nu(j)}y_{\nu(j)\nu(i)}$ in B and correspond to taking (for our fixed i and j) $\lambda_{ij} = 1 = \mu_{ij}$ in equation (19). Now form the paths $\tilde{v}_{j\nu(i)}\tilde{y}_{\nu(i)\nu(j)}$ and $\tilde{y}_{ji}\tilde{v}_{i\nu(j)}$ from $(0,j)$ to $(t_{j\nu(j),q}, \nu(j))$. The corresponding elements of $e_j B_q e_{\nu(j)}$ are non-zero and, using the same symbols without tildes for these, we may choose $u_{j\nu(j)} = v_{j\nu(i)}y_{\nu(i)\nu(j)} = C\epsilon_{ij}\epsilon_{\nu(i)\nu(j)}y_{ji}v_{i\nu(j)}$, see equation (20) and Proposition 4.5.

We now need to consider cases.

For $\mathbb{A}_n$, we label the points in order, starting from one end. Choose $i = n$ and $j = n-1$ so that $\nu(i) = 1$ and $\nu(j) = 2$. It is easy to see from Figure 1 that we have to deform $\tilde{v}_{j\nu(i)}\tilde{y}_{\nu(i)\nu(j)}$ across $n-2$ two-meshes (meshes corresponding to relations which are the sum of 2 monomials) to a multiple of $\tilde{y}_{ji}\tilde{v}_{i\nu(j)}$. Let ζ be the number of sources and sinks, other than the end points, of Q. Then there are ζ meshes for which the two terms of the relation have the same sign (and $n-2-\zeta$ in which they have opposite

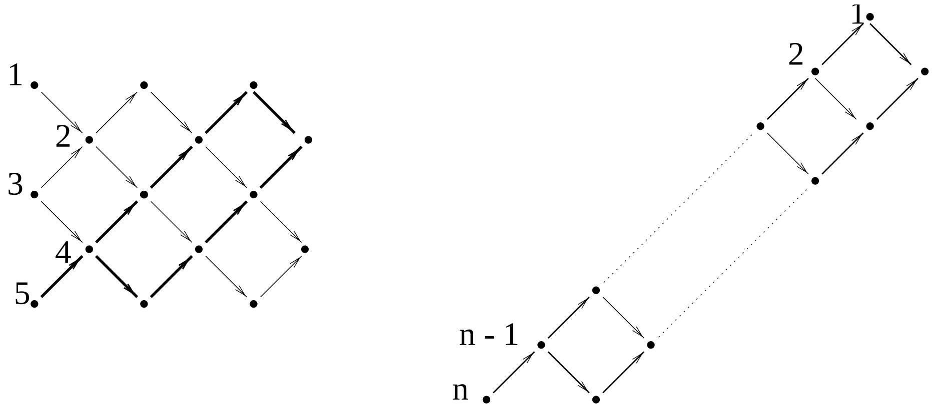

Figure 1: Paths in $\mathbb{A}_5$ and $\mathbb{A}_n$

signs) so that, modulo the relations, $\tilde{v}_{j\nu(i)}\tilde{y}_{\nu(i)\nu(j)} = (-1)^{\zeta}\tilde{y}_{ji}\tilde{v}_{i\nu(j)}$, and hence $v_{j\nu(i)}y_{\nu(i)\nu(j)} = (-1)^{\zeta}y_{ji}v_{i\nu(j)}$ in B. Now $\epsilon_{\nu(i)\nu(j)} = -(-1)^{\zeta}\epsilon_{ij}$ and hence $C = -1$ for $\mathbb{A}_n$.

For $\mathbb{D}_n$, $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$, we label the end of (one of) the short arm(s) 1 and the remaining points in order, starting with 2 at the end of the next shortest arm.

Notice that, since $\widetilde{\Gamma} \subset \mathbb{Z}Q$ is directed and there is at most one arrow between any pair of points of $\widetilde{\Gamma}$, any path in $\widetilde{\Gamma}$ which starts at $(0, i)$ may be written unambiguously as a sequence of vertices of Q. We shall write a path as a sequence of vertices enclosed in square brackets. From now on we shall drop the tildes, and also drop any distinction between elements of $\widetilde{B}$ and B.

Consider first $\mathbb{D}_n$. Let $i = n$; then $\nu(i) = n$ also and $j = n - 1 = \nu(j)$. We suppose that

$$\rho_3 = \alpha y_{31}y_{13} + \beta y_{32}y_{23} + \gamma y_{34}y_{43},$$

so that α, β and γ are each ± 1, depending on the orientation of Q. We shall see that the signs in the other relations are not needed for the calculation. We choose the path

$$u = [n(n-1)\cdots 43234\cdots(n-1)n].$$

Then

$$v_{(n-1)n}y_{n(n-1)} = [(n-1)(n-2)\cdots 43234\cdots(n-1)n(n-1)]$$

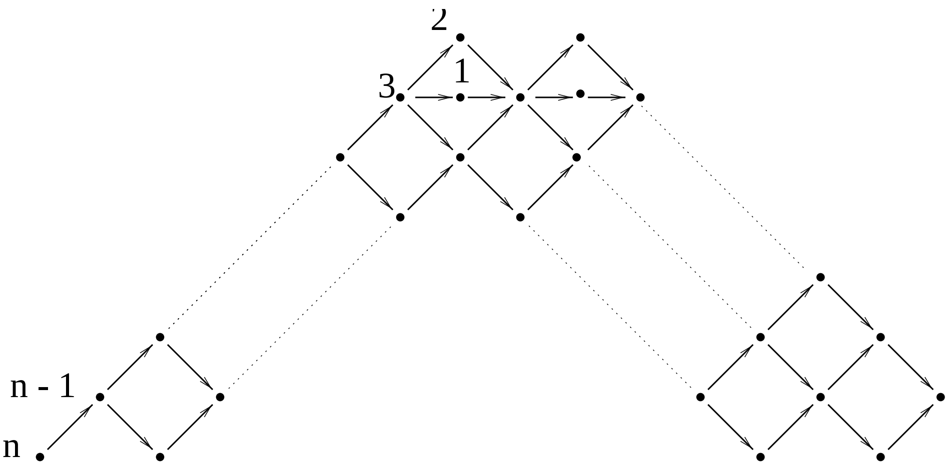

Figure 2: Paths in $\mathbb{D}_n$

and

$$y_{(n-1)n}v_{n(n-1)} = [(n-1)n(n-1)\cdots 43234\cdots(n-2)(n-1)].$$

The second of these can be deformed across $n-4$ two-meshes with relations $\rho_{n-1},\ldots,\rho_4$ to a multiple of

$$[(n-1)(n-2)\cdots 4343234\cdots(n-2)(n-1)].$$

Since the composite along the path [232] is $(\pm 1)\rho_2$ which is zero, this deforms through the three-mesh to

$$(-\alpha/\gamma)[(n-1)(n-2)\cdots 4313234\cdots(n-2)(n-1)].$$

A similar argument, but with a deformation through the three-mesh *followed by* deformations through $n-4$ two-meshes with relations $\rho_4,\ldots,\rho_{n-1}$, now gives

$$y_{(n-1)n}v_{n(n-1)} = (-\gamma/\beta)(-\alpha/\gamma)[(n-1)(n-2)\cdots 43134\cdots(n-1)n(n-1)],$$

since the two deformations through each of the two-meshes give multiplication by 1, whatever the orientation. Observe now that the path $[(n-1)(n-2)\cdots 434343\cdots(n-1)n(n-1)]$ can be deformed through the one-mesh with relation ρ_n and so is zero. A further deformation through the three-mesh thus gives

$$\begin{aligned} y_{(n-1)n}v_{n(n-1)} &= (-\beta/\alpha)(-\gamma/\beta)(-\alpha/\gamma)[(n-1)(n-2)\cdots 432343 \\ &\qquad\qquad\qquad\qquad \cdots (n-2)(n-1)] \\ &= -[(n-1)(n-2)\cdots 432343\cdots(n-2)(n-1)] \\ &= -v_{(n-1)n}y_{n(n-1)}. \end{aligned}$$

Since neither n nor $n-1$ is moved by ν, it follows that $C=-1$ for $\mathbb{D}_n$, regardless of orientation.

For $\mathbb{E}_6$, $\mathbb{E}_7$ and $\mathbb{E}_8$ we write the relation for the three-mesh as

$$\rho_4 = \alpha y_{41}y_{14} + \beta y_{43}y_{34} + \gamma y_{45}y_{54}.$$

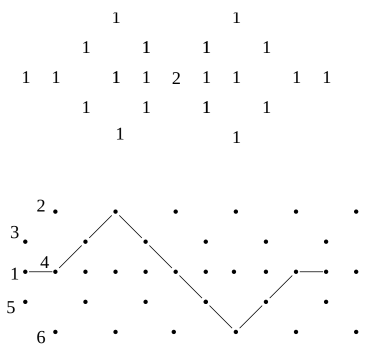

Figure 3: The function d_{is} and the path u in the Auslander-Reiten quiver for $\mathbb{E}_6$

Consider first $\mathbb{E}_6$. For this $\nu(6)=2$ and $\nu(5)=3$ and the vertices 1 and 4 are not moved by ν. We take $i=1$ and $j=4$. The upper part of Figure 3 shows the dimension d_{is} of the vector space $e_{(0,1)}\widetilde{B}_s e_{(t_{1i,s},i)}$, whenever this is non-zero. The bottom part of the figure shows the path

$$u = [14323456541]$$

which is not zero modulo the relations. With this choice for u

$$v_{41}y_{14} = [43234565414] \quad \text{and} \quad y_{41}v_{14} = [41432345654].$$

We first note that the path [454545] can be deformed through the one-mesh with relation ρ_6, and the path [434343] can be deformed through the one-mesh with relation ρ_2, and so the composite along each is zero. The path [141] is one-mesh corresponding to ρ_1 and so the composite along it is also zero. We set out the calculation in Table 1, the first line of which is $y_{41}v_{14}$. Each subsequent line is derived from the previous one by deformation

	[41432345654]	
$\langle 3\rangle, \langle 5\rangle$	[4143**4**345**4**54]	
$-\gamma/\alpha$	[4**5**434345454]	$[434343] = 0$
$-\alpha/\beta$	[45434**1**45454]	$[454545] = 0$
	[454**5**414**3**454]	$[141] = 0$ (twice)
$-\beta/\alpha$	[45454**3**43454]	$[454545] = 0$
$-\alpha/\gamma$	[454**1**4343454]	$[434343] = 0$
$-\beta/\gamma$	[4**3**414343454]	$[141] = 0$
	[434**5**434341**4**]	$[434343] = 0$ (twice)
$-\gamma/\beta$	[43454345**5**414]	$[141] = 0$
$-\alpha/\beta$	[43454**1**45414]	$[454545] = 0$
$-\beta/\gamma$	[434**3**4145414]	$[141] = 0$
$-\gamma/\alpha$	[43434**5**45414]	$[434343] = 0$
$\langle 3\rangle, \langle 5\rangle$	[43**2**345**6**5414]	$[434343] = 0.$

Table 1: Calculation of C for $\mathbb{E}_6$

through one or more meshes and possibly making use of the relations above. The last line contains $v_{41}y_{14}$. We have indicated changes in boldface and listed on the right hand side any relation used; on the left hand side we have noted any factors obtained on deforming through the three-mesh and written $\langle n\rangle$ for the factor ± 1 obtained on deforming through the two-mesh corresponding to ρ_n ($n = 3$ or 5).

For both $n = 3$ and for $n = 5$, there are exactly 2 deformations through two-meshes corresponding to ρ_n. Thus these make no net contribution to the relationship of $y_{41}v_{14}$ to $v_{41}y_{14}$. The product of the factors coming from deformations through the three-mesh is -1 and so $C = -1$ for $\mathbb{E}_6$, irrespective of orientation.

For both $\mathbb{E}_7$ and $\mathbb{E}_8$ the Nakayama permutation is the identity. We now give the proof that $C = -1$ for $\mathbb{E}_8$ in any orientation; the proof for $\mathbb{E}_7$ is similar, and is left to the reader.

We choose $i = 8$ (the end of the long arm) and $j = 7$.

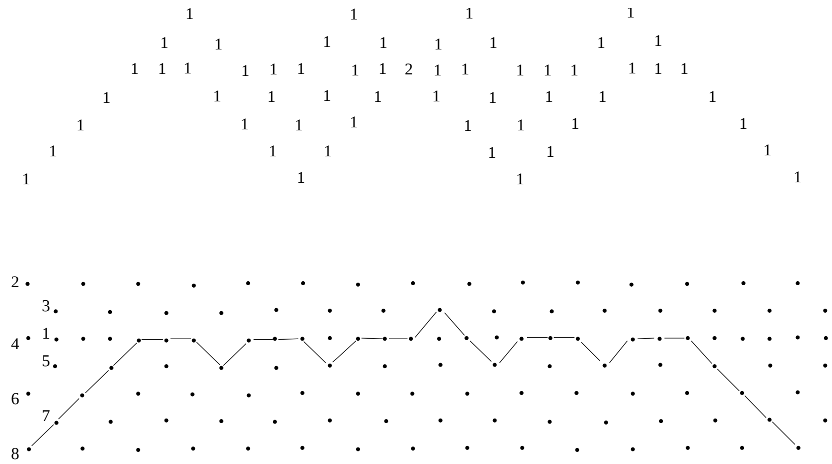

Figure 4: The function d_{is} and the path u in the Auslander-Reiten quiver for $\mathbb{E}_8$

The upper part of Figure 4 shows the dimension d_{is} of the vector space $e_{(0,8)}\widetilde{B}_s e_{(t_{8i,s},i)}$, whenever this is non-zero. The bottom part of the figure shows the path

$$u = [876541454145414345414541456 78]$$

which is not zero modulo the relations. With this choice for u,

$$v_{78}y_{87} = [76541454145414345414541456787]$$

and

$$y_{78}v_{87} = [78765414541454143454145414567].$$

First note that the paths [545454545], [7654545] and [5454567] can each be deformed through a one-mesh with relation ρ_8 and so the composite along each is zero. Similarly the composite along the path [34343] is zero since the path can be deformed through a one-mesh corresponding to ρ_2. The path $[141] = \rho_1$, so the composite along it is also zero. We set out this calculation in Table 2 using the same conventions as we used for $\mathbb{E}_6$ For $n = 7$, for $n = 6$ and for $n = 5$, there are exactly 2 deformations through two-meshes corresponding to ρ_n. Thus these make no net contribution to the relationship of $y_{78}v_{87}$ to $v_{78}y_{87}$. The product of the factors coming from

	[78765414541454143454145414567]	
$\langle 7\rangle$	[7**6**765414**3**4145414**5**454145414567]	$[141]=0$ (twice)
$\langle 6\rangle, (-\beta/\alpha)$	[76**5**654143414541454541454**3**4567]	$[5454567]=0$
$\langle 5\rangle, (-\beta/\gamma)$	[765**4**541434145414545414**3**434567]	$[141]=0$
$(-\beta/\alpha)$	[765454**3**4341454145454143434567]	$[7654545]=0$
	[7654**1**43434**5**454145454143434567]	$[34343]=0$ (twice)
$(-\gamma/\beta)$	[765414**5**4345454145454143434567]	$[141]=0$
$(-\beta/\alpha)$	[76541454345454**3**45454143434567]	$[545454545]=0$
$(-\beta/\gamma)$	[765414543454543454**3**4143434567]	$[141]=0$
$(-\alpha/\gamma)$	[7654145434545434**1**434143434567]	$[34343]=0$
$(-\gamma/\beta)$	[76541454345454**5**41434143434567]	$[141]=0$
$(-\alpha/\beta)$	[76541454**1**45454541434143434567]	$[545454545]=0$
$(\beta/\gamma)^2$	[7654145414**3**454**3**41434143434567]	$[141]=0$ (twice)
	[765414541434**1**434**5**434143434567]	$[34343]=0$ (twice)
$(\gamma/\beta)^4$	[7654145414**5**414**5**454**5**414**5**434567]	$[141]=0$ (4 times)
$(-\beta/\alpha)$	[76541454145414545454**3**45434567]	$[545454545]=0$
$(-\alpha/\gamma)$	[7654145414541454545434**1**434567]	$[34343]=0$
	[76541454145414**3**454543414**5**4567]	$[141]=0$ (twice)
$(-\beta/\alpha)$	[7654145414541434545434**3**454567]	$[5454567]=0$
$(-\alpha/\gamma), \langle 5\rangle$	[765414541454143454**1**434345**6**567]	$[34343]=0$
$(-\gamma/\beta), \langle 6\rangle$	[76541454145414345414**5**43456**7**67]	$[141]=0$
$(-\alpha/\beta), \langle 7\rangle$	[7654145414541434541454**1**4567**8**7]	$[5454567]=0$

Table 2: Calculation of C for $\mathbb{E}_8$

deformations through the three-mesh is -1 and so $C=-1$ for $\mathbb{E}_8$, irrespective of orientation.

This completes the proof of Proposition 4.5.

5 The trivial extension algebras

5.1 Proof of Theorem 2.1

In this section we prove Theorem 2.1 by constructing the minimal projective bimodule resolution of the trivial extension algebra $A = T(kQ)$ and computing its period. The cases $Q = \mathbb{A}_1, \mathbb{A}_2$ have been covered at the end of

Subsection 3.2. Thus we shall assume that Q, in addition to being Dynkin with bipartite orientation (as required by Theorem 2.1), *has at least* 3 *vertices*. It is then easy to show that $A = A_0 \oplus A_1 \oplus A_2$, where $A_0 = S$ has a basis $\{e_i\}_{i \in Q_0}$, A_1 has a basis $\{x_{ij}, x_{ji}\}_{i \to j \in Q_1}$ and A_2 has a basis $\{f_i\}_{i \in Q_0}$ where $f_i = x_{ij}x_{ji}$ for each neighbour j of i. The defining relations are $x_{ij}x_{ji} - x_{ih}x_{hi} = 0$ for j and h neighbours of i and $x_{ij}x_{jh} = 0$ whenever $h \neq i$. The algebra A is quadratic and its quadratic dual is the preprojective algebra $B = B(Q)$.

We let $q = h - 2$ as in Section 4. Since we have shown that B is $(q, 2)$-Koszul in Corollary 4.3, it follows from Proposition 3.11 that A is $(2, q)$-Koszul.

Theorem 3.15 shows that the Koszul bimodule complex $(P(K^\bullet), d)$ provides the first $q+1$ terms of the minimal projective A, A-bimodule resolution of A. In this case, by Remark 3.13, we have $K^m = \mathrm{D}B_m$ so the resolution begins as follows.

$$0 \to \Sigma \to P(\mathrm{D}B_q) \to P(\mathrm{D}B_{q-1}) \to \cdots \to P(\mathrm{D}B_0) \to A \to 0 \tag{24}$$

where $P(X) = A \otimes X \otimes A$. We shall show in Proposition 5.1 that the $(q+1)$-st syzygy Σ is isomorphic to ${}_1A_\alpha$ where α is the automorphism of A induced by the Nakayama automorphism β of B. It follows from Definition 4.6 that α is defined on the generators of A by the formulae $\alpha(e_i) = e_{\nu(i)}$ and

$$\alpha(x_{ij}) = \begin{cases} x_{\nu(i)\nu(j)} & \text{if } \epsilon_{ij} = 1, \\ \epsilon_{\nu(i)\nu(j)} x_{\nu(i)\nu(j)} & \text{if } \epsilon_{ij} = -1. \end{cases}.$$

So $\alpha^2 = \mathrm{id}$ and $\alpha = \mathrm{id}$ if and only if $\nu = \mathrm{id}$ and $\operatorname{char} k = 2$. Note that $\nu = \mathrm{id}$ exactly when Q is $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$.

Proposition 5.1. *The* $(q+1)$*-st syzygy module* Σ *is isomorphic to* ${}_1A_\alpha$. *Moreover* Σ *is isomorphic to* ${}_1A_1$ *only when* $\alpha = \mathrm{id}$.

Proof. We shall need explicit bases for some of the terms in the Koszul resolution of A. Since B is the quadratic dual of A, we can interpret the basis of B_1 consisting of the y_{ij} as the set of linear functionals on A_1 given by $y_{ij}(x_{kl}) = \delta_{jk}\delta_{il}$. We take over from Section 4 the notation $v_{i\nu(j)}$ and $u_{i\nu(i)}$ for basis elements of B_{q-1} and B_q. Further to the notation $\hat{v}$ and $\hat{u}$ for basis elements of $\mathrm{D}B_{q-1}$ and $\mathrm{D}B_q$ dual to v and u, the basis element of $\mathrm{D}B_1 = A_1$ dual to y will be denoted $\hat{y}$, again with suffices reversed. Note that $\hat{y}_{ij} = x_{ij}$.

Now $d = d_l + d_r$, where d_l and d_r are induced by the duals d' and d'' of the multiplication maps in B, as described at the beginning of Subsection 3.4 and in Remark 3.10. In particular, the last differential in (24) is induced by

$$d' \colon \mathrm{D}B_q \to \mathrm{D}B_1 \otimes \mathrm{D}B_{q-1}, \qquad \hat{u}_{\nu(i)i} \mapsto \sum_{j \in \mathcal{N}(i)} \frac{1}{\mu_{ij}} \hat{y}_{\nu(i)\nu(j)} \otimes \hat{v}_{\nu(j)i} \tag{25}$$

and

$$d'' \colon \mathrm{D}B_q \to \mathrm{D}B_{q-1} \otimes \mathrm{D}B_1, \qquad \hat{u}_{\nu(i)i} \mapsto \sum_{j \in \mathcal{N}(i)} \frac{1}{\lambda_{ij}} \hat{v}_{\nu(i)j} \otimes \hat{y}_{ji},$$

where μ_{ij} and λ_{ij} are the multiplication constants in Definition 4.4. Since $\mathrm{D}B_1 = A_1$ we find that

$$d_l(e_{\nu(i)} \otimes \hat{u}_{\nu(i)i} \otimes e_i) = \sum_{j \in \mathcal{N}(i)} \frac{1}{\mu_{ij}} x_{\nu(i)\nu(j)} \otimes \hat{v}_{\nu(j)i} \otimes e_i,$$

$$d_r(e_{\nu(i)} \otimes \hat{u}_{\nu(i)i} \otimes e_i) = (-1)^q \sum_{j \in \mathcal{N}(i)} \frac{1}{\lambda_{ij}} e_{\nu(i)} \otimes \hat{v}_{\nu(i)j} \otimes x_{ji}.$$

A straightforward calculation now shows that, for each $i \in Q_0$

$$z_{\nu(i)i} = e_{\nu(i)} \otimes \hat{u}_{\nu(i)i} \otimes f_i + \sum_{j \in \mathcal{N}(i)} \frac{\mu_{ji}}{\lambda_{ij}} x_{\nu(i)\nu(j)} \otimes \hat{u}_{\nu(j)j} \otimes x_{ji} + (-1)^{q-1} f_{\nu(i)} \otimes \hat{u}_{\nu(i)i} \otimes e_i \tag{26}$$

belongs to $\operatorname{Ker} d$. Since the first terms form a basis of $A_0 \otimes K^q \otimes A_2$, the $z_{\nu(i)i}$ form a basis of Z.

Another short calculation then shows that the z and the x satisfy equations

$$\lambda_{ij} z_{\nu(i)i} x_{ij} = \mu_{ji} x_{\nu(i)\nu(j)} z_{\nu(j)j}$$

formally identical to the equations (21) satisfied by the $\hat{u}$ and the y. We may therefore repeat the argument used in the proof of Corollary 4.7 to show that the element $z \in \Sigma$ defined by $z = \sum \xi_i z_{\nu(i)i}$, with the ξ_i determined by equation (22), satisfies $z x_{ij} = \alpha(x_{ij}) z$. Since $\Sigma = Az = zA$, this shows that $\Sigma \simeq {}_1A_\alpha$.

Finally we need to show that if α has order 2, then Σ is not isomorphic to ${}_1A_1$. This is easily done by checking that Σ contains no central generator to play the rôle that the identity of A plays in the bimodule ${}_1A_1$.

The following lemma is used in the proof of Theorem 2.1.

Lemma 5.2. *For $1 \leq r < q$, $\dim B_r > |Q_0|$.*

Proof. The idea of the proof is to show that, for each $i \in Q_0$, $\dim e_i B_r \geq 1$, and that the inequality is strict if i has more than one neighbour. The dimension of $e_i B_r$ is equal to the sum of the multiplicities of the i-th simple Λ-module in the indecomposable Λ-modules which lie at a distance r from Λe_i in the Auslander-Reiten quiver of Λ. Since $|Q_0| \geq 3$, the lemma follows by inspection of these quivers.

Proof of Theorem 2.1. We use Proposition 5.1. Since the $2(q+1)$-th syzygy of A is ${}_1A_{\alpha^2}$ and $\alpha^2 = \mathrm{id}$, it follows that $2(h-1)$,which is equal to $2(q+1)$, is certainly a period of A. This is the first assertion of Theorem 2.1 and it implies that the minimum period is a divisor of $2(h-1)$.

For $r \leq q$, it follows from (24) that the bimodule top of $\Omega^r(A)$ is $\mathrm{D}B_r$ and so has dimension $\dim B_r$. For $r < q$, Lemma 5.2 shows that this is larger than $\dim \mathrm{top} A$, so we conclude that $\Omega^r(A) \not\simeq A$. If $\Omega^q(A) \simeq A$ then there is an exact sequence

$$0 \to {}_1A_\alpha \to A \otimes \mathrm{D}B_q \otimes A \to A \to 0$$

and hence an exact sequence

$$0 \to S \to A \otimes S \to S \to 0,$$

which is false on dimension grounds.

It now follows that $2(h-1)$ is the minimal period unless $\Sigma \simeq A$ which, by Proposition 5.1, is equivalent to $\alpha = \mathrm{id}$. Since this occurs only when $\mathrm{char}\, k = 2$ and Q is $\mathbb{D}_{2n}$, $\mathbb{E}_7$ or $\mathbb{E}_8$, this completes the proof of Theorem 2.1.

5.2 The Yoneda algebra

In this subsection we calculate the Yoneda algebra $E^\bullet$ of $A = T(kQ)$ under the assumptions in force in the last subsection. So $B = A^!$ is the corresponding preprojective algebra and by Proposition 3.2 and Remark 3.12 the algebra $E^\bullet$ is generated by $E^1 \simeq A_1^! = B_1$ and $E^{q+1} \simeq \mathrm{D}W$ where $W = A_2 \otimes \mathrm{D}B_q$.

Theorem 5.3. *For A as above, $E^\bullet$ is isomorphic as a graded algebra to the twisted polynomial algebra $B[t;\beta]$ in one indeterminate t of degree $q+1$, in which the grading of B is the length grading, β is the Nakayama automorphism of B defined in Definition 4.6, and $tb = \beta(b)t$ for each $b \in B$.*

Remark 5.4. The homogeneous components of $B[t;\beta]$ are $B_r t^n = t^n B_r \simeq B_r$, of degree $n(q+1)+r$, where $0 \leq r \leq q$ and $n \geq 0$. Since $\beta^2 = \mathrm{id}$, the elements of B commute with the even powers of t.

Proof. We start by making a plausible choice for an element, t', in $E^{q+1} = \mathrm{Ext}_A^{q+1}(S,S)$ corresponding to the generator t of degree $q+1$ in $B[t;\beta]$. We choose t' to be (the equivalence class of) the exact sequence

$$0 \to {}_1S_\alpha \to P^q \to \cdots \to P^1 \to P^0 \to S \to 0, \tag{27}$$

obtained from the Koszul bimodule complex (24) by applying the functor $- \otimes_A S$ and using Proposition 5.1. It is a first segment of the minimal projective resolution of S and the term P^r is $A \otimes \mathrm{D}B_r$. Also t' is an S-module generator of

$$\mathrm{Ext}_A^{q+1}(S,S) \simeq \mathrm{Hom}_A({}_1S_\alpha, S) \simeq {}_\alpha S_1 = {}_\nu S_1$$

so that $t' = \sum_{i \in Q_0} t'_{\nu(i)i}$, where $t'_{\nu(i)i} = e_{\nu(i)} t' e_i$.

To obtain generators of E^1 we use the formulae

$$E^1 = \mathrm{Ext}^1(S,S) \simeq \mathrm{Hom}_A(P^1, S) \simeq B_1$$

and recall that B_1 has the y_{ij} as a basis. Let $[y_{ij}]$ denote (the equivalence class of) the short exact sequence

$$[y_{ij}]\colon\ 0 \to S \to Y_{ji} \to S \to 0$$

obtained by pushout of the exact sequence

$$P^1 = A \otimes \mathrm{D}B_1 \to P^0 = A \otimes \mathrm{D}B_0 \to S \to 0 \tag{28}$$

along the map $\bar{y}_{ij}\colon P^1 \to S$ given by

$$1 \otimes \hat{y}_{lh} \mapsto \begin{cases} 0 & \text{if } (h,l) \neq (i,j), \\ e_j & \text{if } (h,l) = (i,j). \end{cases} \tag{29}$$

The main part of the proof of the Theorem lies in establishing the formulae

$$\lambda_{ij} t' [y_{ij}] = \mu_{ji} [y_{\nu(i)\nu(j)}] t'. \tag{30}$$

These formulae relate elements of E^{q+2}, so we need to extend the projective resolution (27) by two further terms to

$$P^{q+2} \xrightarrow{d^{q+2}} P^{q+1} \xrightarrow{d^{q+1}} P^q \xrightarrow{d^q} \cdots \to P^0 \to S \to 0. \tag{31}$$

Since $\ker d^q = W = A_2 \otimes \mathrm{D}B_q$, we may choose $P^{q+1} = A \otimes A_2 \otimes \mathrm{D}B_q$. It has A-module generators $r_{\nu(j)j} = 1 \otimes f_{\nu(j)} \otimes \hat{u}_{\nu(j)j}$ and d^{q+1} is the map $r_{\nu(j)j} \mapsto f_{\nu(j)} \otimes \hat{u}_{\nu(j)j}$.

We next choose $P^{q+2} = A \otimes \mathrm{D}B_1 \otimes A_2 \otimes \mathrm{D}B_q$. It has A-module generators

$$s_{\nu(l)h} = 1 \otimes \hat{y}_{\nu(l)\nu(h)} \otimes f_{\nu(h)} \otimes \hat{u}_{\nu(h)h}$$

and the map d^{q+2} is given by

$$s_{\nu(l)h} d^{q+2} = x_{\nu(l)\nu(h)} \otimes f_{\nu(h)} \otimes \hat{u}_{\nu(h)h} = x_{\nu(l)\nu(h)} r_{\nu(h)h}.$$

For the proof of the formula (30) we exhibit maps $\theta, \phi \colon P^{q+2} \to S$ representing the elements $[y_{\nu(i)\nu(j)}]t'$, $t'[y_{ij}]$, respectively.

Let $\tau \colon P^{q+1} \to A_2 \otimes \mathrm{D}B_q \simeq S$, $r_{\nu(h)h} \mapsto f_{\nu(h)} \otimes \hat{u}_{\nu(h)h}$, be the map which induces t'. It lifts to give a commutative diagram

$$\begin{array}{ccccccc} P^{q+2} & \xrightarrow{d^{q+2}} & P^{q+1} & = & P^{q+1} & & \\ \downarrow{\scriptstyle \tau^1} & & \downarrow{\scriptstyle \tau^0} & & \downarrow{\scriptstyle \tau} & & \\ P^1 & \xrightarrow{d^1} & P^0 & \to & S & \to & 0 \end{array}$$

in which τ^0 and τ^1 are given by

$$r_{\nu(h)h}\tau^0 = 1 \otimes \hat{e}_{\nu(h)} \quad \text{and} \quad s_{\nu(l)h}\tau^1 = 1 \otimes \hat{y}_{\nu(l)\nu(h)}.$$

We take θ to be the composite function $\theta = \tau^1 \bar{y}_{\nu(i)\nu(j)}$. It is given on the generators of P^{q+2} by

$$s_{\nu(l)h}\theta = \delta_{hi}\delta_{jl} e_{\nu(j)}. \tag{32}$$

The calculation of ϕ needs a little more preparation. By definition $[y_{ij}]$ is the pushout of (28) by $\bar{y}_{ij} \colon P^1 \to S$. Since the right multiplication map $y_{ij,\rho}$ on the Koszul complex $A \otimes \mathrm{D}B$ of A,B-bimodules commutes with the differentials, $\bar{y}_{ij}$ lifts to a morphism of complexes

$$\begin{array}{ccccccccc} P^q & \xrightarrow{d^q} & \cdots & \to & P^1 & = & P^1 & & \\ \downarrow{\scriptstyle y_{ij,\rho}} & & & & \downarrow{\scriptstyle y_{ij,\rho}} & & \downarrow{\scriptstyle \bar{y}_{ij}} & & \\ P^{q-1} & \xrightarrow{d^{q-1}} & \cdots & \to & P^0 & \to & S & \to & 0. \end{array}$$

We claim that this extends further to a morphism of complexes

$$\begin{array}{ccccc} P^{q+2} & \xrightarrow{d^{q+2}} & P^{q+1} & \xrightarrow{d^{q+1}} & P^{q} \\ \downarrow \eta^2 & & \downarrow \eta^1 & & \downarrow y_{ij,\rho} \\ P^{q+1} & \xrightarrow{d^{q+1}} & P^{q} & \xrightarrow{d^{q}} & P^{q-1} \end{array}$$

in which $r_{\nu(h)h}\eta^1 = \delta_{hi}\frac{\mu_{ji}}{\lambda_{ij}} x_{\nu(i)\nu(j)} \otimes \hat{u}_{\nu(j)j}$ and

$$s_{\nu(l)h}\eta^2 = \delta_{hi}\delta_{lj}\frac{\mu_{ji}}{\lambda_{ij}} r_{\nu(j)j}.$$

The verification requires the use of formula (21) in evaluating $y_{ij,\rho}$, and $d^q = d_l \otimes S$ where d_l is given in equation (25).

Now set $\phi = \eta^2\tau$. From the values of η^2 and τ, we see that $\lambda_{ij}\phi = \mu_{ji}\theta$. The formula (30) follows.

Finally (30) is equivalent to the formulae

$$\lambda_{ij} t'_{\nu(i)i}[y_{ij}] = \mu_{ji}[y_{\nu(i)\nu(j)}]t'_{\nu(j)j}$$

and these are formally identical with (21). So arguing as in the proof of Corollary 4.7, we can find non-zero scalars ξ_h such that $t = \sum_{h \in Q_0} \xi_h t'_{\nu(h)h}$ satisfies $t[y_{ij}] = \beta([y_{ij}])t$ for all y_{ij}. Since t generates E^{q+1} and the y_{ij} generate B, this proves the theorem.

6 More examples

The only examples of (p, q)-Koszul algebras so far exhibited in this paper are the truncated algebras, which are those with $q = 1$, and the preprojective algebras of Dynkin quivers and their quadratic duals, which have $q = 2$ and $p = 2$, respectively, and are self-injective. In this section we give some other examples of quadratic (p, q)-Koszul algebras. Some of them are not self-injective. Those which are self-injective are both periodic and DTr-periodic. Most of our examples are closely related to the preprojective and trivial extension algebras of Dynkin quivers and, in particular, have either $p = 2$ or $q = 2$. We conclude with some (closely related) families of examples in which $p = q = 3$.

Our simplest example of a non-self-injective quadratic almost Koszul algebra is the algebra C with quiver $\overline{Q}$ (see Subsection 4.1), where $Q = \mathbb{A}_5$

with bipartite orientation, and relations as in the trivial extension algebra $A(Q)$, except that the two zero-relations through the middle vertex, 3, are omitted. It is easy to verify that C is $(2,2)$-Koszul by calculating the minimal projective resolutions of the simple modules. This calculation shows that the third syzygy of the simple Se_3 is $Se_1 + Se_3 + Se_5$, so the algebra is certainly not self-injective. The quadratic dual $C^!$ of C is also a non-self-injective $(2,2)$-Koszul algebra; it is the preprojective algebra of $\mathbb{A}_5$, with additional zero-relations $2 \to 3 \to 4$ and $4 \to 3 \to 2$.

In the remainder of this section we use covering theory to present new examples as quotients by group actions of quivers of the form $\mathbb{Z}Q$, with Q a Dynkin graph, and with various relations. In covering theory k-algebras need to be replaced by locally bounded k-categories and we use the language and concepts of [6].

Let A be a positively graded, locally bounded k-category in which A_0 is semi-simple. By analogy with Definition 3.1, we call A a left (p,q)-Koszul category if,

1. $A_n = 0$ for $n > p$, and
2. for each simple left A-module M of degree 0 there is a graded complex of the form (2) of projective left A-modules such that (i) the i-th term P^i is generated by its component of degree i, and (ii) the only non-zero homology is M in degree 0 and the component of degree $p+q$ in P^q.

Let A and $\widetilde{A}$ be positively graded, locally bounded k-categories and $F\colon \widetilde{A} \to A$ be a covering functor ([6], Definition 3.1) which has degree 0 in the obvious sense. Then the pushdown functor F_λ ([6], Section 3.2) from $\widetilde{A}$-modules to A-modules induced by F maps graded modules to graded modules, simples of degree 0 to simples of degree 0, projectives generated in degree i to projectives generated in degree i and exact graded sequences to exact graded sequences. Hence A is left (p,q)-Koszul if and only if $\widetilde{A}$ is.

This idea may be applied to the almost Koszul algebras $A(Q)$ and $B(Q)$ discussed in Sections 5 and 4. They have universal coverings $\widetilde{A}(Q)$ and $\widetilde{B}(Q)$ which we describe below. These categories give rise to infinitely many new finite-dimensional almost Koszul algebras by quotienting out the actions of various groups of graded automorphisms. Similarly the non-self-injective algebras C and $C^!$ have universal covers which are $(2,2)$-Koszul and which give rise to infinitely many new finite-dimensional examples of non-self-injective $(2,2)$-Koszul algebras.

We now discuss bimodule resolutions, syzygies and periodicity for the new examples obtained from the universal coverings of the preprojective and trivial extension algebras of Dynkin quivers. We show that the new finite-dimensional examples are both periodic and DTr-periodic and obtain estimates for the minimal periods.

We start with the k-category $\widetilde{B} = \widetilde{B}(Q)$ where Q is a Dynkin quiver with Coxeter number h and bipartite orientation. As a graded k-category, $\widetilde{B}$ is generated by idempotents e_i^n in degree 0 for $(n, i) \in \mathbb{Z} \times Q_0$, and morphisms

$$y_{ij}^n \colon \begin{cases} (n,i) \to (n,j) & \text{if } \epsilon_{ij} = 1 \\ (n,i) \to (n+1,j) & \text{if } \epsilon_{ij} = -1, \end{cases}$$

of degree 1. (Note that these generators were denoted $e_{(n,i)}$ and $\tilde{y}_{ij}^n$ in Subsection 4.4.) Its defining relations are the inverse images under the covering functor $e_i^n \mapsto e_i$, $y_{ij}^n \mapsto y_{ij}$ of the defining relations (13) for the preprojective algebra $B(Q)$. Since we have chosen Q to have bipartite orientation, these relations are the mesh relations for $\mathbb{Z}Q$ viewed as a translation quiver. The group of covering automorphisms of $\widetilde{B}$ is the cyclic group generated by the translation t which increases the covering degree n by 1.

The category $\widetilde{B}$ inherits self-injectivity from B and we shall use this property to calculate DTr-periods for its finite-dimensional quotients. The appropriate dual of $\widetilde{B}$ is the $\widetilde{B}, \widetilde{B}$-bimodule $\mathrm{D}\widetilde{B}$ defined by

$$(\mathrm{D}\widetilde{B})((m,i),(n,j)) = \mathrm{D}(e_j^n \, \widetilde{B} \, e_i^m).$$

Our main assertions, justified briefly below, are as follows.

1. The Nakayama automorphism β of $B(Q)$ lifts naturally over the covering functor to a Nakayama automorphism of $\widetilde{B}$, that is, an automorphism $\widetilde{\beta}$ of $\widetilde{B}$ such that $\mathrm{D}\widetilde{B}$ is isomorphic as a $\widetilde{B}, \widetilde{B}$-bimodule to ${}_1\widetilde{B}_{\widetilde{\beta}}$.

2. The automorphism $\widetilde{\beta}$ commutes with t and satisfies $\widetilde{\beta}^2 = t^q$, where $q = h - 2$.

3. The third syzygy of $\widetilde{B}$ as a $\widetilde{B}, \widetilde{B}$-bimodule is isomorphic to ${}_1\widetilde{B}_{\widetilde{\gamma}}$ where $\widetilde{\gamma} := \widetilde{\beta}^{-1} t^{-1} = \widetilde{\beta} t^{-h-1}$ and so satisfies $\widetilde{\gamma}^2 = t^{-h}$.

For the definition of $\widetilde{\beta}$ we need to relate the path-length grading and the covering grading on the quiver $\mathbb{Z}Q$ and for this purpose make use of the function q^* defined for a bipartite quiver by

$$q^*(i) = \begin{cases} q/2 & \text{if } q \text{ is even,} \\ (q-1)/2 & \text{if } q \text{ is odd and } i \text{ is a source in } Q, \\ (q+1)/2 & \text{if } q \text{ is odd and } i \text{ is a sink in } Q. \end{cases}$$

Note that $q^*(i) = q/2$ unless Q is $\mathbb{A}_{2n}$. In any case,

$$q^*(i) + q^*(\nu(i)) = q.$$

We define $\widetilde{\beta}$ on the generators of $\widetilde{B}$ by the formulae

$$\widetilde{\beta}(e_i^n) = e_{\nu(i)}^{n+q^*(\nu(i))}, \quad \widetilde{\beta}(y_{ij}^N) = \epsilon_{ij}^{q-1} y_{\nu(i)\nu(j)}^{n+q^*(\nu(i))}.$$

The verifications that $\widetilde{\beta}$ is an automorphism of $\widetilde{B}$ and satisfies the conditions of the second assertion above are straightforward though one needs to use the formulae $\nu^2 = \mathrm{id}$ and $\epsilon_{ij}\epsilon_{\nu(i)\nu(j)} = (-1)^q$ for bipartite Dynkin quivers. The proof that $\mathrm{D}\widetilde{B} \simeq {}_1\widetilde{B}_{\widetilde{\beta}}$ follows closely the proof that $\mathrm{D}B \simeq {}_1B_\beta$ in Sections 4.2 and 4.4. A key point is that the socle elements $u_{i\nu(i)}^n$ of $\widetilde{B}$ are paths from (n, i) to $(n + q^*(i), \nu(i))$ and it is their canonical duals which generate $\mathrm{D}\widetilde{B}$.

For the third assertion, the calculation of $\Omega^3(\widetilde{B})$ may be made using the Koszul bimodule resolution of $\widetilde{B}$ exactly as in Subsection 4.3. This syzygy is generated by elements $w_{i\nu(i)}^n$ which map under the covering functor to the generators $w_{i\nu(i)}$ of $\Omega^3(B)$ constructed in the proof of Theorem 4.9. The automorphism $\widetilde{\gamma}$ differs from the Nakayama automorphism $\widetilde{\beta}$ in two ways. First, $\mathrm{D}\widetilde{B}$ and $\Omega^3(B)$ are of opposite variance which explains why $\widetilde{\gamma}$ decreases the covering index while $\widetilde{\beta}$ increases it. Second, as is apparent from the formula (23), the elements $w_{i\nu(i)}$ have length $q + 2 = h$, 2 longer than the length of the natural generators of $\mathrm{D}\widetilde{B}$.

Now let $\overline{B}$ be a positively graded locally bounded k-category for which there exists a graded covering functor

$$\widetilde{B} \to \overline{B},$$

where $\widetilde{B} = \widetilde{B}(Q)$, as above. Then the three assertions above show that the automorphisms $\overline{\beta}$, $\overline{\gamma}$ and $\bar{t}$ of $\overline{B}$ induced by $\widetilde{\beta}$, $\widetilde{\gamma}$ and t satisfy ${}_{\overline{B}}\mathrm{D}\overline{B}_{\overline{B}} \simeq {}_1\overline{B}_{\overline{\beta}}$

(so that $\overline{\beta}$ is a Nakayama automorphism of $\overline{B}$), $\overline{\beta}^2 = \bar{t}^q$, $\Omega^3(\overline{B}) \simeq {}_1\overline{B}_{\overline{\gamma}}$ and $\overline{\gamma}^2 = \bar{t}^{-h}$.

From now on, suppose that $\overline{B}$ is a finite dimensional algebra or, equivalently, that $\bar{t}$ has finite order, l say. Then the Nakayama automorphism $\overline{\beta}$ also has finite order, r say, which divides $2l/(q,l)$ and $\overline{\gamma}$ has finite order, s say,which divides $2l/(h,l)$. We conclude that $\overline{B}$ is periodic with $3s$ as a period. For one-sided modules, the Nakayama functor has the same finite order as $\overline{\beta}$ and the syzygy functor has order a divisor of the period of Ω. Thus the well-known formula "$\mathrm{D}Tr = \nu\Omega^2$" for self-injective algebras, implies that the least common multiple of r and $3s$ is a period of the Auslander translate $\mathrm{D}Tr$ of indecomposable, non-projective $\overline{B}$-modules. In particular, these algebras are $\mathrm{D}Tr$-periodic.

We now mention some algebras which arise in other contexts and are of the above type, and so periodic and $\mathrm{D}Tr$ periodic.

Let Λ be an indecomposable finite-dimensional standard self-injective algebra of finite representation type over an algebraically closed field. According to a famous theorem of Christine Riedtmann [25], the stabilised Auslander algebra Γ of Λ has a universal covering of the form $\widetilde{B}(Q)$ for some Dynkin graph Q. Thus Γ is of the form $\overline{B}$, as above, and so is periodic and $\mathrm{D}Tr$-periodic; of course it is usually wild. These examples include the group algebras of finite representation type noted by Auslander and Reiten in [1].

The remaining finite-dimensional examples in [1] are the stable Auslander algebras for one- and two-dimensional isolated hypersurface singularities of finite representation type. For the two-dimensional cases, the authors note that these algebras are the preprojective algebras of Dynkin quivers. They also draw attention to the description given by Dietrich and Wiedemann in [12] of the stable Auslander algebras for the one-dimensional case. Examination of these algebras shows that they too have universal covers of form $\widetilde{B}(Q)$ for a Dynkin quiver Q and so they are periodic and $\mathrm{D}Tr$-periodic. In most cases our estimates for the $\mathrm{D}Tr$-periods are not as sharp as those of [1].

We now turn to the algebras with universal covering $\widetilde{A} = \widetilde{A}(Q)$ with Q a Dynkin quiver with Coxeter number h, bipartite orientation and at least three vertices. The locally bounded k-category $\widetilde{A}$ is generated by idempotents e_i^n of degree 0 and morphisms x_{ij}^n of degree 1. Its defining relations are the inverse images under the covering functor of the defining relations of $A(Q)$ given in Subsection 5.1. We have already observed that $\widetilde{A}$ is an infinite dimensional $(2,q)$-Koszul category, where $q = h-2$. It is, in an obvious sense, the quadratic dual of the $(q,2)$-Koszul category $\widetilde{B}(Q)$ and so the automorphism $\widetilde{\gamma}$ of $\widetilde{B}$, being graded, induces an automorphism $\widetilde{\alpha}$ of $\widetilde{A}$. Note that $\widetilde{A}$ is the repetitive algebra of A, introduced by Hughes and Waschbüsch in [20]. Its Nakayama automorphism is just the translation t.

The assertions above about $\widetilde{B}$ are replaced by

1. The automorphism α of $A(Q)$ lifts naturally over the covering functor to the automorphism $\widetilde{\alpha}$ of $\widetilde{A}$.
2. The relation $\alpha^2 = \mathrm{id}$ lifts to $\widetilde{\alpha}^2 = t^{-h}$.
3. The $h-1$-st syzygy of $\widetilde{A}$ as an $\widetilde{A},\widetilde{A}$-bimodule is isomorphic to ${}_1\widetilde{A}_{\widetilde{\alpha}}$.

For the last assertion, the calculation of $\widetilde{\Sigma} = \Omega^{h-1}(\widetilde{A})$ can be made by formal modification of that of Σ in Proposition 5.1. This involves showing that the first $h-1$ terms of the minimal projective $\widetilde{A},\widetilde{A}$-bimodule resolution of $\widetilde{A}$ has the form

$$0 \to \widetilde{\Sigma} \to P(\mathrm{D}\widetilde{B}_q) \to P(\mathrm{D}\widetilde{B}_{q-1}) \to \cdots \to P(\mathrm{D}\widetilde{B}_0) \to \widetilde{A} \to 0 \qquad (33)$$

where $P(X) = \widetilde{A} \otimes X \otimes \widetilde{A}$ and the tensor products are over $\widetilde{A}_0$. Furthermore $\widetilde{\Sigma}$ is generated by its submodule $\widetilde{Z}$ of elements of length degree $q+2 = h$ and $\widetilde{Z}$ has a basis consisting of one non-zero element, denoted by $z^n_{\nu(i)i}$, from each of the one-dimensional spaces $e^n_{\nu(i)}\,\widetilde{Z}\,e_i^{n-q^*(\nu(i))-1}$. These elements are the analogues of the elements $z_{\nu(i)i}$ in equation (26).

We may now consider finite dimensional algebras $\overline{A}$ for which $\widetilde{A}$ is the universal covering. These algebras are $(2,q)$-Koszul. They are periodic with $(h-1)l$ as a period, where l is the order of the automorphism $\bar{t}$ of $\overline{A}$ induced by t. The Auslander translates also have $(h-1)l$ as a period. However, $\widetilde{A}$ is of locally finite representation type, and so all these finite dimensional quotients are of finite representation type and therefore automatically $\mathrm{D}Tr$-periodic.

Our final examples, which have $p = q = 3$, are based on the interesting algebras which were shown to be of tame representation type by Bekkert [4] and by Geiss and de la Peña [18]. Let Q be the quiver illustrated in Figure 5, I_λ be the ideal generated by the relations

$$x^2 = uv,\ xu = uy,\ yv = vx \text{ and } y^2 = \lambda vu, \qquad (34)$$

where $\lambda \neq 0, 1$, and $I^\dagger$ be the ideal generated by

$$x^2 = uv,\ xu = uy,\ yv = -vx \text{ and } y^2 = vu. \qquad (35)$$

Then $A_\lambda = kQ/I_\lambda$ is $(3,3)$-almost Koszul with $A_\lambda^! = A_{\lambda^{-1}}$. If $\operatorname{char} k \neq 2$, then $A^\dagger$ is also $(3,3)$-almost Koszul with $(A^\dagger)^! = A^\dagger$.

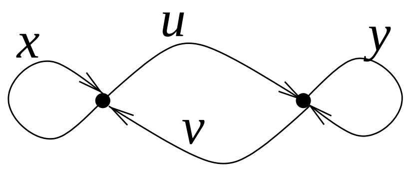

Figure 5: The quiver Q of the $(3,3)$-Koszul algebras A_λ and $A^\dagger$.

These algebras are finite dimensional quotients of a covering $k\widetilde{Q}/\widetilde{I}$, where $\widetilde{Q}$ is the quiver shown in Figure 6 and $\widetilde{I}$ is the ideal generated by the relations (34) or (35), appropriately interpreted.

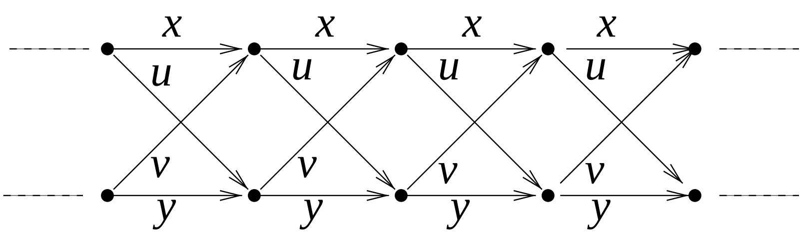

Figure 6: The quiver $\widetilde{Q}$ of the covering

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