Ma10209 Algebra 1A : Exercises 10

Hand in answers to (H) questions on Moodle by 6pm on Tue 8 Dec.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W). Show that the following maps are group homomorphisms. In each case, find the kernel and image of the homomorphism.

(i) $U\colon \mathbb{R} \to \mathbb{C}^*\colon \theta \mapsto \cos\theta + i\sin\theta$

(ii) $m\colon \mathbb{C}^* \to \mathbb{R}^*\colon z \mapsto |z|$

(iii) $c\colon \mathbb{C} \to \mathbb{C}\colon z \mapsto \bar{z}$

(iv) $\gamma\colon \mathbb{C}^* \to \mathbb{C}^*\colon z \mapsto \bar{z}$

2 (H). Show that the following maps are group homomorphisms. In each case, find the kernel and image of the homomorphism.

(i) $q\colon \mathbb{Z} \to \mathbb{Z}_n\colon x \mapsto [x]$

(ii) $\alpha\colon \mathbb{Z}_{12} \to \mathbb{Z}_3 \times \mathbb{Z}_4\colon [x]_{12} \mapsto \big([x]_3, [x]_4\big)$

3 (W). Let $\alpha\colon G \to H$ and $\beta\colon H \to K$ be homomorphisms of groups.

(i) Show that $\beta \circ \alpha\colon G \to K$ is also a homomorphism.

(ii) For $g \in G$ and $n \in \mathbb{Z}$, show that $\alpha(g^n) = \alpha(g)^n$.

4 (H). Suppose $\alpha\colon G \to H$ is a homomorphism of groups and $g \in G$ has finite order.

(i) Show that $\alpha(g)$ has finite order and that the order of $\alpha(g)$ divides the order of g.

(ii) If α is injective, show that the order of $\alpha(g)$ is equal to the order of g.

5 (W). Let G be the group of rotational symmetries of a tetrahedron (the regular solid with four equilateral triangles as faces).

(i) Describe all elements of G geometrically and say what their orders are. [Hint: $|G| = 12$.]

(ii) Labelling the vertices of the tetrahedron with the set $\{1, 2, 3, 4\}$, also write down the elements of G as permutations. Which subgroup of S_4 do you get this way?

6 (H). Consider the following subgroups of S_3

$$H = \{\, e, (1\,2) \,\}, \qquad K = \{\, e, (1\,2\,3), (1\,3\,2) \,\}.$$

Compute the subsets gH, Hg, gK and Kg, for each $g \in S_3$, and observe, in each of these four cases, that the cosets do indeed partition the group.

7 (W). Let $G \times X \to X\colon (g,x) \mapsto g \cdot x$ be a group action. Given $x \in X$, the **stabiliser** of x is

$$\mathrm{Stab}(x) = \{g \in G : g \cdot x = x\}.$$

(i) Show that $\mathrm{Stab}(x)$ is a subgroup of G.

(ii) Show that $g \cdot x = h \cdot x$ if and only if g and h are in the same left coset of $\mathrm{Stab}(x)$.

8 (H). A subgroup $H \leqslant G$ is **normal** if $ghg^{-1} \in H$, for all $h \in H$ and $g \in G$.

(i) Show that every subgroup of an abelian group is normal.

(ii) Let $\alpha\colon G \to H$ be a homomorphism. Show that $\operatorname{Ker}\alpha$ is a normal subgroup of G. [Note: first prove that $\operatorname{Ker}\alpha$ is a subgroup.]

(iii) Show that $H \leqslant G$ is normal if and only if $gH = Hg$, for all $g \in G$.

9 (A). For a group G, there is an action of G on G given by $g \cdot h = ghg^{-1}$, for $g, h \in G$. This is called **conjugation** and the orbits are called the **conjugacy classes** of G.

(i) Show that this does indeed define an action.

(ii) This action is not necessarily faithful; what is the kernel of the associated homomorphism $\phi\colon G \to \mathrm{Sym}(G)\colon g \mapsto \phi_g$, where $\phi_g(h) = ghg^{-1}$?

(iii) For $G = S_n$, show that the conjugacy classes consist of all permutations with the same cycle shape, that is, whose cycles (including 1-cycles) give partitions of the same shape (see solutions to Ex 3.6).

(iv) How many conjugacy classes are there in S_4. What are their sizes?

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