Ma10209 Algebra 1A : Exercises 9

Hand in answers to (H) questions on Moodle by 6pm on Tue 1 Dec.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W)**.** In each case, determine whether the subset H is a subgroup of the group G.

(i) $G = \mathbb{Q}$, under addition, $H = \{m/n : m, n \in \mathbb{Z}, 0 < n < 10\}$.

(ii) $G = C_n$, the n-th roots of unity, H is the subset of primitive n-th roots.

(iii) $G = \mathrm{GL}_2(\mathbb{F})$, under matrix multiplication, $H = \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \mathbb{F} \right\}$.

2 (H)**.** Let G be a group, and suppose that $H \leqslant G$ and $K \leqslant G$.

(i) Prove that $H \cap K \leqslant G$.

(ii) Prove that $H \cup K \leqslant G$ only if either $H \leqslant K$ or $K \leqslant H$.

(iii) The **centre** of a group G is the subset $Z(G) = \{a \in G : ag = ga \text{ for all } g \in G\}$. Show that $Z(G) \leqslant G$.

3 (W)**.** (i) Show that the additive groups $\mathbb{Z}_n$ and the multiplicative groups C_n are cyclic. Which elements are generators?

(ii) Write down the Cayley table for $\mathbb{Z}_8^*$. Is this a cyclic group?

4 (H)**.** (i) Write down the Cayley tables for $\mathbb{Z}_5^*$ and $\mathbb{Z}_{12}^*$. Which of these are cyclic groups? If so, which elements are generators.

(ii) Show that a group of order 4 cannot have an element of order 3. [Hint: if it did, then, calling the elements e, a, a^2, b with $a^3 = e$, deduce a contradiction using the cancellation law.]

(iii) Suppose $G = \{e, a, b, c\}$ is a non-cyclic group with e the identity element. Show that there is only one possible Cayley table for G. Deduce that there are essentially only two groups of order 4.

5 (W)**.** Let G be a group, with operation $\circ$ and identity e_G and H be a group, with operation $\bullet$ and identity e_H Show that the product $G \times H$ is a group with operation

$$(g_1, h_1) * (g_2, h_2) = (g_1 \circ g_2, h_1 \bullet h_2)$$

and identity (e_G, e_H).

6 (H). (i) Let G be a group, with $x, y \in G$. Prove that x and $y^{-1}xy$ have the same order. Deduce that $\operatorname{ord} xy = \operatorname{ord} yx$.

(ii) For (g, h) in the product group $G \times H$, prove that $\operatorname{ord}(g, h) = \operatorname{lcm}(\operatorname{ord} g, \operatorname{ord} h)$, taking care to say how this should be interpreted when either g or h has infinite order.

7 (W). Let $\sigma = (2\ 4)$, $\tau = (3\ 1\ 5)$ and $\rho = (3\ 2\ 1\ 4)$ in S_5.

(i) What are the orders of σ, τ and ρ?

(ii) What are the orders of $\sigma\tau$, $\tau\rho$ and $\rho\sigma$? Which pairs commute?

8 (H). (i) Write down all the elements of S_3 together with their orders and signs.

(ii) Show that S_3 has no subgoup of order 4.

(iii) Find all the subgroups of order 4 in S_4. Are any of these subgroups of A_4?

9 (A). Let G be a group, with identity element e.

(i) Suppose that $g^2 = e$ for all $g \in G$. Prove that G is an abelian group

(ii) Suppose the H is a non-empty subset of G that satisfies $ab^{-1} \in H$ for all $a, b \in H$. Prove that H is a subgroup of G.

(iii) Suppose that G has even order. Prove that G contains an element of order 2. [Hint: consider the subsets $\{x, x^{-1}\}$, for $x \in G$.]

ADK 24 Nov 2020