Ma10209 Algebra 1A : Exercises 8

Hand in answers to (H) questions on Moodle by 6pm on Tue 24 Nov.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W)**.** For $A, B, C \in \mathrm{M}_2(\mathbb{F})$, with coefficients $A = (a_{ij})$ etc., show by direct computation that the distributive law $A(B+C) = AB + AC$ holds. Identify which field axioms you are using. [Hint: calculate the ij coefficient of both sides.]

2 (H)**.** For $A, B, C \in \mathrm{M}_2(\mathbb{F})$, with coefficients $A = (a_{ij})$ etc., show by direct computation that the associative law $(AB)C = A(BC)$ holds. Identify which field axioms you are using. [Hint: calculate the ij coefficient of both sides.]

3 (W)**.** Show that the area $V(v, w)$ of the parallelogram spanned by vectors $v, w \in \mathbb{R}^2$ is $|v_1 w_2 - v_2 w_1|$, i.e. show that $V^2 = (v_1 w_2 - v_2 w_1)^2$. [Hint: recall that $V = |v||w| \sin\theta$, where θ is the angle between v and w, while the scalar product $v \cdot w = |v||w| \cos\theta$.]

4 (H)**.** For $A, B \in \mathrm{M}_2(\mathbb{F})$, show by direct computation that the product rule $\det(AB) = (\det A)(\det B)$ holds. Hint: to economise on indices, write

$$A = \begin{pmatrix} x_1 & x_2 \\ y_1 & y_2 \end{pmatrix}, \qquad B = \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \end{pmatrix}.$$

5 (W)**.** Consider the subset

$$\mathcal{C} = \left\{ Z(a,b) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} : a, b \in \mathbb{R} \right\} \subseteq \mathrm{M}_2(\mathbb{R}).$$

Show that $\mathcal{C}$ is closed under addition and multiplication in $\mathrm{M}_2(\mathbb{R})$ and that multiplication is commutative for elements of $\mathcal{C}$.

6 (H)**.** For $Z(a, b)$ and $\mathcal{C}$ as in Ex 5.

(i) Show that $Z(a, b)$ is invertible unless $(a, b) = (0, 0)$.

(ii) Show that $\mathcal{C}$ is a field, which can be identified with $\mathbb{C}$.

(iii) Under what condition on (a, b) is $Z(a, b)$ orthogonal?

7 (W)**.** Recall that the Euclidean distance between two points $x, y \in \mathbb{R}^3$ is $|x - y|$, where $|z|^2 = z^{\mathrm{T}} z$, for any $z \in \mathbb{R}^3$, thought of as a column vector.

(i) If $A \in \mathrm{M}_3(\mathbb{R})$ is orthogonal, show that the map $\phi_A \colon \mathbb{R}^3 \to \mathbb{R}^3 \colon x \mapsto Ax$ preserves Euclidean distance, that is, $|Ax - Ay| = |x - y|$, for all $x, y \in \mathbb{R}^3$.

(ii) If $v \in \mathbb{R}^3$ show that the translation map $\tau_v \colon \mathbb{R}^3 \to \mathbb{R}^3 \colon x \mapsto x+v$ preserves Euclidean distance.

8 (H). A **Euclidean transformation** $\psi_{A,v} \colon \mathbb{R}^3 \to \mathbb{R}^3$ is given by $\psi_{A,v}(x) = Ax + v$, for some orthogonal $A \in \mathrm{M}_3(\mathbb{R})$ and $v \in \mathbb{R}^3$.

(i) Show that a Euclidean transformation $\psi_{A,v} \colon \mathbb{R}^3 \to \mathbb{R}^3$ preserves Euclidean distance.

(ii) Show that the set of all Euclidean transformations is closed under composition, that the identity map $\mathrm{Id}_{\mathbb{R}^3}$ is a Euclidean transformation and that the inverse of a Euclidean transformation is also a Euclidean transformation.

9 (A). Let $\mathbb{F}$ be the two element field $\mathbb{Z}_2 = \{0, 1\}$.

(i) Determine the set S of all the invertible matrices in $\mathrm{M}_2(\mathbb{F})$.

(ii) Find $A, B \in S$ with the property that $AB \neq BA$.

(iii) Find all $C \in S$ with the property that $C^2 = I$, but $C \neq I$.

(iv) Find all $D \in S$ with the property that $D^3 = I$, but $D \neq I$.

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