MA10209 ALGEBRA 1A : EXERCISES 7

Hand in answers to (H) questions on Moodle by 6pm on Tue 17 Nov.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W)**.** Using the extended field axioms (F1)-(F9) from lectures, prove that the following hold, for any x, y in a field $\mathbb{F}$. Say explicitly at each step which axiom you are using.

(i) If $x + y = 0$, then $y = -x$.

(ii) $(-1)x = -x = x(-1)$.

(iii) $(-x)(-y) = xy$.

2 (H)**.** Using the extended field axioms (F1)-(F9) from lectures, prove that the following hold, for any x, y in a field $\mathbb{F}$. Say explicitly at each step which axiom you are using.

(i) If $xy = 1$, then $x \neq 0$ and $y = x^{-1}$.

(ii) If $x \neq 0$ and $y \neq 0$, then $xy \neq 0$ and $(xy)^{-1} = y^{-1}x^{-1}$.

(iii) If $x \neq 0$, then $-x \neq 0$ and $(-x)^{-1} = -x^{-1}$.

3 (W)**.** (i) For $f, g \in \mathbb{F}[X]$, show that $\deg(f + g) \leqslant \max(\deg(f), \deg(g))$. When is this a strict inequality?

(ii) Use Euclid's Algorithm to find the monic gcd of $X^2 + 5X + 6$ and $2X^2 + 3X - 2$.

4 (H)**.** Use Euclid's Algorithm to find the monic gcds of

(i) $X^3 + 3X^2 + 4X + 2$ and $2X^2 + 7X + 5$,

(ii) $X^4 + X^3 + X^2 + X + 1$ and $X^4 + 1$. In this case, say why the result is not unexpected.

5 (W)**.** Let $f, g, h \in \mathbb{F}[X]$ with f and g coprime. Show that

(i) if $f \mid gh$, then $f \mid h$,

(ii) if $f \mid h$ and $g \mid h$, then $fg \mid h$.

6 (H)**.** Let $f, g, h \in \mathbb{F}[X]$ with h a greatest common divisor of f and g. Show that

(i) f/h and g/h are coprime,

(ii) fg/h is a least common multiple of f and g.

7 (W). (i) If $\alpha, \beta \in \mathbb{F}$ with $\alpha \neq \beta$, show that $X - \alpha$ and $X - \beta$ are coprime in $\mathbb{F}[X]$.

(ii) Show that a polynomial $f \in \mathbb{F}[X]$ of degree n can have at most n roots in $\mathbb{F}$.

(iii) Deduce that a polynomial $f \in \mathbb{R}[X]$ of degree n can have at most n roots in $\mathbb{C}$.

(iv) Show that a polynomial $f \in \mathbb{F}[X]$ of degree 2 or 3 is irreducible if and only if f has no roots in $\mathbb{F}$. Is the same true when f has degree 4?

8 (H). Which of the following polynomials are irreducible in $\mathbb{F}[X]$?

(i) $f = X^2 + X + 1$, $g = X^4 + 1$, when $\mathbb{F} = \mathbb{R}$.

(ii) $f = X^2 + X - 1$, $g = X^4 + 1$, when $\mathbb{F} = \mathbb{Q}$.

(iii) $f = X^2 + X + 3$, $g = X^3 + X + 1$, when $\mathbb{F} = \mathbb{Z}_5$.

9 (A). Let $\mathbb{F}$ be the field $\mathbb{Z}_7$. Thinking of $\mathbb{F}^2$ as $\{x + iy : x, y \in \mathbb{F}\}$ we can make $\mathbb{F}^2$ into a commutative ring, by setting $(x + iy) + (a + ib) = (x + a) + i(b + y)$ and $(x + iy)(a + ib) = (xa - yb) + i(xb + ya)$, just as for the construction of $\mathbb{C}$ from $\mathbb{R}$.

(i) Show that, for any $(x, y) \in \mathbb{F}^2$, if $x^2 + y^2 = 0$, then $(x, y) = (0, 0)$.

(ii) Deduce that $\mathbb{F}^2$ is a field, with the operations given above.

(iii) Factorise $X^4 + 1$ into irreducibles in $\mathbb{F}[X]$ and in $\mathbb{F}^2[X]$.

ADK 10 Nov 2020