MA10209 ALGEBRA 1A : EXERCISES 6

Hand in answers to (H) questions on Moodle by 6pm on Tue 10 Nov.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional,

1 (W). Suppose that $\alpha \in \mathbb{C}$ is a non-zero complex number.

(i) Find the two square roots of α.

(ii) In particular, what are they when $\alpha = \pm i$?

(iii) Factorise the polynomial $x^4 + 1$ over $\mathbb{C}$ and over $\mathbb{R}$.

2 (H). Suppose that $\alpha \in \mathbb{C}$ is a non-zero complex number.

(i) Find the three cube roots of α.

(ii) In particular, what are they when $\alpha = -1$ and when $\alpha = 8$?

(iii) Factorise the polynomials $x^3 + 1$ and $x^3 - 8$ over $\mathbb{C}$ and over $\mathbb{R}$.

3 (W). (i) For $z \in \mathbb{C}$, show that $z = \overline{z}$ if and only if $z \in \mathbb{R}$.

(ii) For any polynomial $f(x) = a_n x^n + \cdots + a_1 x + a_0$, with all coefficients $a_j \in \mathbb{R}$, show that, for $\alpha \in \mathbb{C}$, if $f(\alpha) = 0$, then $f(\overline{\alpha}) = 0$.

(iii) Show that, if $\alpha \in \mathbb{C}$, then both $\alpha + \overline{\alpha} \in \mathbb{R}$ and $\alpha\overline{\alpha} \in \mathbb{R}$.

4 (H). (i) For any $\alpha \in \mathbb{C}$, find a quadratic polynomial $f(x) = ax^2 + bx + c$, with $a, b, c \in \mathbb{R}$, such that $f(\alpha) = 0$.

(ii) For any $a \in \mathbb{R}$ with $a > 0$, find all $z \in \mathbb{C}$ such that $z^4 + a^4 = 0$.

(iii) Hence write $x^4 + a^4$ as a product of two quadratic polynomials with real coefficients.

5 (W). Recall that the **order** of a root of unity ω is the smallest $m \in \mathbb{Z}^+$ such that $\omega^m = 1$, in which case we call ω a **primitive** m-th root of unity.

(i) Write down all the 8th roots of unity.

(ii) Write down their orders and hence identify the primitive 8th roots of unity.

(iii) Find the polynomial $\Phi_8(x)$ whose roots are precisely the primitive 8th roots of unity.

6 (H). (i) Show that $\omega = e^{2\pi i k/n}$ is a primitive n-th root of unity if and only if k is coprime to n.

(ii) Write down the primitive n-th roots of unity, for $n = 5$ and $n = 12$.

(iii) Find polynomials $\Phi_n(z)$ whose roots are precisely the primitive n-th roots of unity, for $n = 5$ and $n = 12$.

7 (W)**.** Show that, if $\alpha \in \mathbb{C}$ is nonzero and not a root of unity, then its powers α^k, for $k \in \mathbb{Z}$, are all different.

8 (H)**.** Let ω be a primitive n-th root of unity.

(i) Show that its powers ω^k, for $k \in \{1, \ldots, n\}$, are all different

(ii) Deduce that they are precisely all the n-th roots of unity.

9 (A)**.** Prove that, for any $n \geqslant 2$,

$$\sum_{k=1}^{n-1} e^{2\pi i k/n} = -1.$$

Can you give a geometric justification for this identity?

ADK 3 Nov 2020