Ma10209 Algebra 1A : Exercises 4

Hand in answers to (H) questions on Moodle by 6pm on Tue 27 Oct.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W). (i) Show that every odd prime is of the form $4n \pm 1$, for $n \in \mathbb{Z}^+$.

(ii) Show that, if $m_1 \cdots m_k$ is of the form $4n-1$, then so is one of the factors m_i.

(iii) Show that this does not hold for the form $4n+1$.

2 (H). Prove that there are infinitely many primes numbers of the form $4n-1$ for $n \in \mathbb{Z}^+$.

3 (W). The **odd part** of an integer $m \in \mathbb{Z}^+$ is the largest odd divisor of m. Write down the odd parts of the 6 numbers $7, 8, 9, 10, 11, 12$. Write down the odd parts of 9 numbers $10, 11, \ldots, 18$. Do you notice anything interesting?

4 (H). Prove that the odd parts of the n successive integers $n+1, n+2, \ldots, 2n$ are precisely the first n odd numbers: $1, 3, \ldots, 2n-1$. Why is there always a power of 2 in the range $n+1, \ldots, 2n$?

5 (W). (i) Find $g = \gcd(75, 27)$ by using Euclid's algorithm.

(ii) Find some pair of integers λ and μ such that $g = 75\lambda + 27\mu$.

(iii) Find all pairs of integers λ, μ such that $g = 75\lambda + 27\mu$.

6 (H). (i) Find $g = \gcd(8633, 13439)$ by Euclid's algorithm.

(ii) Find integers λ and μ such that $g = 8633\lambda + 13439\mu = g$.

(iii) Find all pairs of integers λ, μ such that $g = 8633\lambda + 13439\mu$.

7 (W). Prove that $2^n + 1$ and $2^{n+1} + 1$ are coprime, for all $n \in \mathbb{Z}^+$.

8 (H). The Fibonacci sequence is defined recursively by $F_1 = F_2 = 1$ and $F_{n+1} = F_n + F_{n-1}$, for all integers $n \geqslant 2$. Prove that $\gcd(F_n, F_{n+1}) = 1$ for all $n \in \mathbb{Z}^+$.

9 (A). Suppose that $S = \{x_1, \ldots, x_{15}\}$ is a set of 15 pairwise coprime positive integers all in the range $1 < x_i \leqslant 2020$. Prove that S contains a prime number. [Hint: The 15th prime is 47.]

ADK 20 Oct 2020