Ma10209 Algebra 1A : Exercises 3

Hand in answers to (H) questions on Moodle by 6pm on Tue 20 Oct.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W). Use the Pigeonhole Principle (PHP) to prove that, if 6 numbers are chosen from the set $S = \{0, \ldots, 9\}$, then there will be some pair whose sum is 9.

2 (H). Use PHP to prove the following.

(i) Given 17 points in a 4×4 square, there are two points at most $\sqrt{2}$ apart.

(ii) If a group of people meet and some shake hands with each other, then there must be two people who shake hands with the same number of people.

3 (W). Show that, if A is a finite set and X is a proper subset of A, then $|X| < |A|$

4 (H). Let A, B be finite sets with $|A| = |B|$.

(i) Prove that, if $f\colon A \to B$ is injective, then f is also surjective and thus a bijection. [Hint: use PHP.]

(ii) Deduce that, conversely, if $f\colon A \to B$ is surjective, then f is actually a bijection. [Hint: use Cor. 1.22.]

5 (W). Let $J_n = \{1, \ldots, n\}$, for $n \in \mathbb{Z}^+$. For $n = 1, 2, 3$ determine the number of partitions of J_n.

6 (H). Let J_n, be as above.

(i) For $n = 4, 5$, determine the number of partitions of J_n.

(ii) How many ways are there to partition J_n into two subsets?

7 (W). Consider the relation on $\mathbb{Z}^+ \times \mathbb{Z}^+$ given by $(a, b) \sim (c, d)$ iff $ad = bc$.

(i) Show directly that this is an equivalence relation.

(ii) Observe that this is in fact the equivalence relation $f(x) = f(y)$ for the function $f\colon \mathbb{Z}^+ \times \mathbb{Z}^+ \to \mathbb{Q}^+ \colon (a, b) \mapsto a/b$.

Note that, if we did not already know about (positive) rational numbers, then (i) enables us to construct them as equivalence classes for this relation.

8 (H)**.** Determine whether the following relations are reflexive, symmetric or transitive.

(i) The relation $\mid$ (pronounced 'divides') on the set $\mathbb{Z}$, that is, $u \mid v$ if and only if there is $x \in \mathbb{Z}$ such that $ux = v$.

(ii) For any fixed $n \in \mathbb{Z}^+$, the relation on $\mathbb{Z}$ given by $a \equiv b$ iff $n \mid a - b$.

9 (A)**.** Let A be any set. Write down an injection $f\colon A \to \mathcal{P}(A)$.
For any map $F\colon A \to \mathcal{P}(A)$, consider the subset

$$B = \{x \in A : x \notin F(x)\}.$$

Show that B is not in the image of F. Conclude that there is no surjection $A \to \mathcal{P}(A)$ and thus, in particular, that $A \ncong \mathcal{P}(A)$.

ADK 13 Oct 2020