Ma10209 Algebra 1A : Exercises 2

Hand in answers to (H) questions on Moodle by 6pm on Tue 13 Oct.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

Note: the words "function" and "map" are used interchangeably in these questions.

1 (W). Let $F\colon \mathbb{Q} \to \mathbb{Q}$ be defined by $F(x) = 2x$. Is F a bijection? If so, write down its two-sided inverse.

Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$. Is f injective, surjective or bijective? If so, write down an appropriate inverse.

2 (H). Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 2x$.

(i) Write down infinitely many functions $g\colon \mathbb{Z} \to \mathbb{Z}$ such that $g \circ f = \mathrm{Id}_{\mathbb{Z}}$.

(ii) Show that there is no function $h\colon \mathbb{Z} \to \mathbb{Z}$ such that $f \circ h = \mathrm{Id}_{\mathbb{Z}}$.
(Give a direct argument; don't just quote a result from lectures.)

3 (W). Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^3$ for all $x \in \mathbb{R}$ and let $g\colon \mathbb{R} \to \mathbb{R}$ be defined by $g(x) = 2x + 1$ for all $x \in \mathbb{R}$.

Give formulae for the functions $f \circ g$ and $g \circ f$, and also $f^2 = f \circ f$ and g^2. Which of these functions are bijections? Justify your answers.

4 (H). Let $f\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = x^2$ for all $x \in \mathbb{Z}$ and let $g\colon \mathbb{Z} \to \mathbb{Z}$ be defined by $g(x) = x + 2$ for all $x \in \mathbb{Z}$.

For $n \in \mathbb{Z}^+$, give formulae for the functions f^n and g^n. Which of these functions are bijections? Justify your answers.

5 (W). For each $n \in \mathbb{Z}^+$, let $J_n = \{1, \ldots, n\}$ be the set which consists of the first n positive integers.

(i) How many maps $f\colon J_n \to J_n$ are there?

(ii) How many bijections $f\colon J_n \to J_n$ are there?

(iii) If we set $J_0 = \varnothing$, what are the answers to (i) and (ii) for $n = 0$?

6 (H). Let J_n, for $n \in \mathbb{Z}^+$, be as in the previous exercise.

(i) How many maps $f\colon J_n \to J_3$ are there?

(ii) How many injections $f\colon J_n \to J_3$ are there?

(iii) How many surjections $f\colon J_n \to J_3$ are there?

(iv) If we set $J_0 = \varnothing$, what are the answers to (i), (ii) and (iii) for $n = 0$?

7 (W). Let $f\colon A \to B$ and $g\colon B \to C$ be functions

(i) If $g \circ f$ is injective, does it follow that f is injective?

(ii) If $g \circ f$ is injective, does it follow that g is injective?

Give a proof or counterexample in each case.

8 (H). Let $f\colon A \to B$ and $g\colon B \to C$ be functions

(i) If $g \circ f$ is surjective, does it follow that f is surjective?

(ii) If $g \circ f$ is surjective, does it follow that g is surjective?

(iii) If $g \circ f$ is bijective, does it follow that f and g are both bijective?

(iv) If $g \circ f$ and $f \circ g$ are both bijective, does it follow that f and g are both bijective?

Give a proof or counterexample in each case.

9 (A). Suppose that S is a finite set and $f\colon S \to S$ is a map.

(i) Suppose f is a bijection. For any $s \in S$, show that $f^k(s) = s$, for some $k \in \mathbb{Z}^+$.

(ii) Show that f is a bijection if and only if $f^n = \mathrm{Id}_S$ for some $n \in \mathbb{Z}^+$.

ADK 6 Oct 2020