Ma10209 Algebra 1A : Exercises 1

Hand in answers to (H) questions on Moodle by 6pm on Tue 6 Oct.
Homepage: `http://people.bath.ac.uk/masadk/ma209/`

(W) = Warmup, (H) = Homework, (A) = Additional

1 (W). Find a more economical way to write these sets:

(i) $\{u \in \mathbb{Z} : u^2 - 2u + 1 = 0\}$.

(ii) $\{w \in \mathbb{R} : w^2 + 1 = 0\}$.

(iii) $\{x \in \mathbb{Q} : x^2 \in \mathbb{Z}\}$. For discussion in the tutorial. No proof is required here, but try to decide what you think the answer is beforehand.

2 (H). Find a more economical way to write these sets:

(i) $\{v \in \mathbb{Z} : v^3 - 6v^2 + 11v - 6 = 0\}$.

(ii) $\{z \in \mathbb{C} : z^2 + 1 = 0\}$.

(iii) $\{(x, y) \in \mathbb{R}^2 : x + y = 6,\ x - y = 2\}$.

3 (W). Give a geometric description of the following subsets of $\mathbb{R}^2$.

(i) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$

(ii) $\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = -1\}$

(iii) $\{(x, y) \in \mathbb{R}^2 : x^2 - y^2 = -1\}$

(iv) $\{(x, y) \in \mathbb{R}^2 : 3x + 4y = 5\}$

4 (H). Give a geometric description of the following subsets of $\mathbb{R}^3$.

(i) $\{(x, y, z) \in \mathbb{R}^3 : x + y + z = 1,\ x \geqslant 0,\ y \geqslant 0,\ z \geqslant 0\}$

(ii) $\{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leqslant 1\}$

(iii) $\{(x, y, z) \in \mathbb{R}^3 : (x - 1)^2 + (y + 1)^2 + z^2 = 1\}$

5 (W). Suppose A and B are finite sets. Suppose $|A|, |B|$ and $|A \cap B|$ are given. Does this determine $|A \cup B|$? If so, how and why?

6 (H). Suppose A, B and C are finite sets. Suppose $|A|, |B|, |C|, |A \cap B|, |B \cap C|, |C \cap A|$ and $|A \cap B \cap C|$ are given. Does this determine $|A \cup B \cup C|$? If so, how and why?

7 (W). Let X, A and B be sets.

(i) Prove that, if $X \subseteq A$ and $A \subseteq B$, then $X \subseteq B$.

(ii) Deduce that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

8 (H). Give a proof or counter-example for the following, where X, A and B are sets.

(i) $X \subseteq A \cap B$ if and only if $X \subseteq A$ and $X \subseteq B$.

(ii) $X \subseteq A \cup B$ if and only if $X \subseteq A$ or $X \subseteq B$.

What does this tell you about the relationship between $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A \cap B)$, or between $\mathcal{P}(A)$, $\mathcal{P}(B)$ and $\mathcal{P}(A \cup B)$?

9 (A). The number of subsets of size r in a set of size n (informally 'n choose r') is the binomial coefficient $\binom{n}{r}$, that is, the coefficient of t^r in the expansion of $(1+t)^n$. When r does not lie in the interval $0 \leqslant r \leqslant n$, we agree that $\binom{n}{r} = 0$.

(i) Show that $\displaystyle\sum_{r=0}^{n} \binom{n}{r} = 2^n$.

(ii) When $n > 0$, show that $\displaystyle\sum_{r \text{ odd}} \binom{n}{r} = \sum_{r \text{ even}} \binom{n}{r}$, and so both sides are equal to 2^{n-1}.

ADK 29 Sept 2020