OPTIMAL CONTROL OF MARTINGALES IN A RADIALLY SYMMETRIC ENVIRONMENT

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Abstract. We study a stochastic control problem for continuous multidimensional martingales with fixed quadratic variation. In a radially symmetric environment, we are able to find an explicit solution to the control problem and find an optimal strategy. We show that it is optimal to switch between two strategies, depending only on the radius of the controlled process. The optimal strategies correspond to purely radial and purely tangential motion. It is notable that the value function exhibits smooth fit even when switching to tangential motion, where the radius of the optimal process is deterministic. Under sufficient regularity on the cost function, we prove optimality via viscosity solutions of a Hamilton-Jacobi-Bellman equation. We extend the results to cost functions that may become infinite at the origin. Extra care is required to solve the control problem in this case, since it is not clear how to define the optimal strategy with deterministic radius at the origin. Our results generalise some problems recently considered in Stochastic Portfolio Theory and Martingale Optimal Transport.

1. Introduction

In this paper we study a stochastic control problem for continuous multidimensional martingales with fixed quadratic variation. We work in a radially symmetric environment, where we are able to find optimal strategies and give the value function explicitly. We find that an optimal strategy is to switch between two behaviour regimes depending only on the current radius of the controlled martingale. Under one of the optimal strategies, which we will call *tangential motion*, the controlled martingale has a deterministically increasing radius. This property leads to two notable features of the control problem. First, we make the observation that the value function exhibits smooth fit everywhere. When it is optimal to switch into the regime of tangential motion, continuous fit is sufficient to uniquely specify the value function, since the radius is deterministic here. Therefore it is surprising that smooth fit holds. Moreover, it is not obvious how to define tangential motion at the origin. As a result, solving the control problem at the origin requires extra care and we only find approximate optimisers here. We will see that the value function can remain finite when the cost function is allowed to be infinite at the origin. Under a particular growth condition on the cost function, approximation arguments break down and it is necessary to understand how to define tangential motion at the origin.

1.1. **Problem statement.** Fix a domain $D \subseteq \mathbb{R}^d$, for some $d \geq 2$. We study the control problem of finding

$$\inf_{\mathbb{P}} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f(X_t)\, \mathrm{d}t + g(X_\tau) \right],$$

where τ is the first exit time of X from D, and the infimum is taken over a set of probability measures under which X is a continuous martingale with quadratic variation given by

$$\mathrm{d}\langle X \rangle_t = \mathrm{d}t.$$

We specialise to the *radially symmetric* case, taking D to be a d-dimensional ball, f a function of the form $f(x) = \tilde{f}(|x|)$, and g constant. This structure allows us to work with the radius of the controlled processes.

1.2. **Motivation and literature.** In one dimension, it is well known that any continuous martingale is a time-change of a standard Brownian motion. Martingales with fixed quadratic variation are a natural generalisation

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of Brownian motion to higher dimensions. Imposing this constraint will allow us to study the structure of the optimal martingales in the control problems that we consider.

Problems of stochastic control for martingales with fixed quadratic variation have appeared recently in the context of Stochastic Portfolio Theory, in the two papers [7] and [8] of Larsson and Ruf. In [7] the authors consider the problem of finding the greatest almost sure lower bound on the exit time of a martingale from some domain. Larsson and Ruf apply this control problem in [8] to find the minimal time horizon over which relative arbitrage can be achieved for a market with at least two stocks. While the control set in [7] and [8] is the same as in the problem that we study, we consider a different class of cost functions. One of the strategies that we find to be optimal here is also studied in [8].

The Hamilton-Jacobi-Bellman (HJB) equation that arises from the control problem in this paper is

$$-\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}\left(D^2u\sigma\sigma^\top\right)=f, \tag{1.1}$$

where $U := \left\{\sigma\in\mathbb{R}^{d,d}\colon \operatorname{Tr}(\sigma\sigma^\top)=1\right\}$. We can see this by considering a martingale that is adapted to the natural filtration of some Brownian motion B and has the representation $\mathrm{d}X_t=\sigma_t\,\mathrm{d}B_t$. Then, under the quadratic variation constraint $\mathrm{d}\langle X\rangle_t=\mathrm{d}t$, we have $\sigma_t\in U$, for any $t\geq 0$. This HJB equation takes a similar form to the Black-Scholes-Barenblatt (BSB) equation, as studied in [17]. Compared with the PDE (1.1), the BSB equation has an additional time derivative term, and the infimum can be taken over a more general compact control set. The BSB equation is an HJB equation corresponding to a time-inhomogeneous control problem of the type discussed in Section 3.3 of [16]. In [4], the BSB equation is applied to find a super-hedging strategy for European multi-asset derivatives.

Further motivation for studying the problems in this paper comes from a connection with Martingale Optimal Transport (MOT), motivated by the paper [15] of Tan and Touzi (see also [9]). In [15], the authors formulated the martingale optimal transport problem through penalisation. Formally, MOT is the problem of finding a martingale $(M_n)_{n=0,1}$ with marginal laws $M_0\sim\lambda$ and $M_1\sim\mu$ such that the joint distribution minimises some given quantity, for example $\mathbb{E}\left[|M_1-M_0|\right]$. In this setting, they showed that the one-dimensional MOT problem could be reformulated as an optimal stopping problem for Brownian motion, where the connection to the Brownian motion is established by requiring $B_0\sim\lambda$ and $B_\tau\sim\mu$ for some stopping time τ. The penalisation arises as a cost function of the form $H(B_\tau)$ appearing in the optimisation criterion, or equivalently, via an Itô argument, a cost function of the form $\int_0^\tau H''(B_s)\,\mathrm{d}s$. Our motivation in this paper comes from considering MOT in higher dimensions, when the optimisation problem is complicated by the fact that martingales in higher dimensions are no longer simply time-changed Brownian motion, but can incorporate spatial dependence as described above. We focus specifically on the challenge of understanding how the optimal martingales might look when the problems are radially symmetric, and we do not consider the details of how the penalised problems might arise.

1.3. **Optimal behaviour.** A key result of this paper is to show that, under sufficient regularity on the cost function, an optimal strategy can be constructed by switching between the following two behaviours. We say that a martingale X follows *radial motion* if it can be written as

$$X_t=\frac{x}{|x|}W_t,\quad t\geq 0,$$

for some $x\in D\setminus\{0\}$ and W a one-dimensional Brownian motion, so that X acts as a one-dimensional Brownian motion on the line connecting its starting point to the origin, as illustrated in Figure 3. Such a process maximises the expected time spent close to the origin and is therefore optimal for any cost function whose radial part is monotonically increasing. On the other hand, we say that X follows *tangential motion* if it solves the SDE

$$\mathrm{d}X_t=\frac{X_t^\perp}{|X_t|}\,\mathrm{d}W_t, \tag{1.2}$$

where $X_t^\perp$ is a vector orthogonal to X_t, and W is a one-dimensional Brownian motion. As we discuss below and illustrate in Figure 3, a solution of this SDE has a deterministically increasing radius. As a result, such a process minimises the expected time spent close to the origin and is optimal for cost functions with monotonically decreasing radial part. In our main results, Proposition 4.6 and Theorem 5.12, we solve the control problem explicitly and give conditions under which switching between radial and tangential motion is optimal. Our approach to proving Proposition 4.6 is first to construct a candidate value function by making the ansatz that a switching strategy of the above form is optimal, and then to verify that this is the correct value function by using the theory of viscosity solutions for the HJB equation. In Theorem 5.12 we extend Proposition 4.6 by a series of approximation arguments.

1.4. **A switching problem with smooth fit.** The most interesting optimal behaviour is so-called tangential motion, described by the SDE (1.2). This behaviour is studied by Fernholz, Karatzas and Ruf in [2], and by Larsson and Ruf in [8]. It is observed that, for a solution X of (1.2) with initial condition $x \in D$, the radius of X is the deterministically increasing process

$$t \mapsto |X_t| = \sqrt{|x| + t}.$$

This property implies optimality of tangential motion for cost functions whose radial part is monotonically decreasing. For more general cost functions, we will show that an optimal strategy can be found by switching between this behaviour and radial motion, as described above. Due to the radial symmetry of the control problem, the optimal strategy at any time depends only on the current radial position of the controlled process. Therefore we have a one-dimensional switching problem between two regimes. It is common in such switching problems to observe smooth fit criteria on the free boundaries, and indeed, we do observe such behaviour in our case. Notably, we can only justify smooth fit *heuristically* in one switching regime. To determine the optimal radius at which to switch from radial to tangential motion, we only need to impose continuous fit for the value function. However, interestingly, smooth fit also holds at such a point. We discuss this phenomenon further in Section 4.4.

1.5. **Behaviour at the origin.** Another interesting feature of the control problem is the optimal behaviour at the origin. In the case that the cost function has increasing radial part at the origin, then we can define an optimal strategy by taking a one-dimensional Brownian motion in any given direction, analogously to radial motion described above. On the other hand, in the case of a decreasing cost at the origin, it seems that tangential motion, as defined by the SDE (1.2), should be optimal, since such a process minimises the expected time spent at the origin. However, it is not immediately clear how to define tangential motion started from the origin.

In [2], the authors prove that a weak solution of (1.2) exists in dimension $d = 2$, with initial condition $X_0 = 0$. Using this result, we can solve the control problem in a weak sense, which we define in Section 2, following [1]. Under sufficient conditions on the cost, El Karoui and Tan show in [1] that this weak formulation is equivalent to a strong formulation of the control problem. However, we do not always assume that such conditions hold. In Section 5, we will recover this equivalence result for more general costs, using some approximation arguments, but we do not identify an optimal control started from the origin. Under the particular growth regime where the cost f satisfies

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty, \quad \text{for all} \quad r > 0,$$

our approximation schemes diverge, but the weak solution of (1.2) gives rise to a finite expected cost. This leads us to consider the existence of strong solutions of (1.2) started from the origin. We leave open this question of existence of strong solutions, and the question of equivalence of weak and strong control problems, and we will address these issues in a forthcoming paper.

1.6. **Outline of the paper.** In Section 2, we define the control problems precisely and prove some preliminary results on properties of the controlled processes and equivalence of the strong and weak control problems.

In Section 3, we present two motivating examples and introduce radial and tangential motion as candidates for the optimal behaviour. In the simple examples of step function costs, we prove directly that either radial or tangential motion is optimal, depending on whether the step function is increasing or decreasing.

In Section 4, we solve the control problem explicitly for radially symmetric cost functions that are sufficiently regular. We first conjecture that an optimal strategy is to switch between radial and tangential motion, and we heuristically derive a one-dimensional switching problem in Section 4.1. We then identify optimal switching points in Section 4.2 and go on to construct a candidate for the value function in Section 4.3. We observe smooth fit at the points at which the conjectured optimal radius process switches to the deterministic regime of tangential motion, and we discuss this surprising phenomenon in Section 4.4. In Section 4.5, we prove Proposition 4.6, which states that the value function is equal to the candidate function that we constructed and that switching between radial and tangential motion is indeed optimal. To prove this result, we show that the candidate function satisfies a Hamilton-Jacobi-Bellman equation in the viscosity sense, and use the facts that the value function is also a viscosity solution of this equation, and that such a solution is unique. The required results on viscosity solutions are presented without proof in Appendix A.

In Section 5, we relax the regularity conditions on the cost function, allowing for cost functions that are infinite at the origin. In this case, the theory of viscosity solutions in no longer applicable, and we do not know a priori that the weak and strong formulations of the control problem are equivalent. Under various growth conditions on the cost function, we use approximation arguments to show that all formulations of the control problem are equivalent and that the value function takes the same form as the candidate constructed in Section 4.3. We also identify the conditions under which the value function remains finite. These results are summarised in Theorem 5.12. We conclude by discussing one exceptional case under a specific growth condition in dimension $d = 2$. Here we find the weak value function, but equivalence with the strong and Markov formulations of the control problem is left open.

2. Problem formulation

We now give three formulations of the control problem, which will in many cases coincide. Fix $d \in \mathbb{N}$. We introduce the control set

$$U := \left\{\sigma \in \mathbb{R}^{d,d}\colon \ \mathrm{Tr}(\sigma\sigma^\top) = 1\right\}.$$

Let $D \subset \mathbb{R}^d$ be a domain and define the functions $f : D \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$, which we call the *running cost* and *boundary cost*, respectively. We make the following assumptions.

Assumption 2.1. Suppose that

(1) The domain D is bounded;
(2) The cost functions f and g are upper semicontinuous;
(3) The running cost f is bounded above; i.e. $f \le M$, for some $M \ge 0$;
(4) The boundary cost g is bounded above; i.e. $g \le K$ for some $K \ge 0$.

Note that, in later sections, we will also impose radial symmetry on the problem.

We introduce three variants of the control problem: a strong formulation, a weak formulation, and a Markov formulation. We will show in Proposition 2.7 that, under Assumption 2.1, the weak and strong formulations are equivalent. For the radially symmetric problem that we consider in Section 4, we will show that these formulations are also equivalent to the Markov formulation. In Section 5, we will relax our assumptions, so that Proposition 2.7 no longer applies. However, we will show that equivalence between the three formulations holds in all but one exceptional case.

2.1. **Strong formulation.**

The strong formulation of the control problem is to find the strong value function $v^S : D \to \mathbb{R}$, which we now define as in [16]. In order to define the value function, we introduce the set of controls, which will be U-valued processes, and we describe the dynamics of the controlled martingales via the stochastic integral (2.1) below.

Let $(\Omega_0, \mathcal{F}, \mathbb{P}_0)$ be a probability space on which a d-dimensional Brownian motion B is defined with natural filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$.

Control: Define the set of controls $\mathcal{U} := \{U\text{-valued } \mathbb{F}\text{-progressively measurable processes}\}$.

Dynamics: For any $x \in D$ and $\nu = (\nu_t)_{t\geq 0} \in \mathcal{U}$, define X^ν by the stochastic integral

$$X_t^\nu = x + \int_0^t \nu_s \, \mathrm{d}B_s, \quad t \geq 0. \tag{2.1}$$

Example 2.2. Let $\sigma : D \to U$ be Lipschitz. Then there is a unique strong solution X^σ of the SDE

$$\mathrm{d}X_t = \sigma(X_t) \, \mathrm{d}B_t, \quad X_0 = x.$$

Define $\nu_t = \sigma(X_t^\sigma)$, for all $t \geq 0$. Then $\nu \in \mathcal{U}$ and, for any $t \geq 0$, $X_t^\sigma = x + \int_0^t \nu_s \, \mathrm{d}B_s$.

Value function: We define the *strong value function* $v^S : D \to \mathbb{R}$ by

$$v^S(x) := \inf_{\nu \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\nu) \, \mathrm{d}s + g(X_{\tau^\nu}^\nu) \right],$$

where τ is the exit time of X^ν from the domain D, and $\mathbb{E}^x$ denotes expectation with respect to the law of X^ν, conditioned on $X_0^\nu = x$.

Remark 2.3. Note that, for any $\nu \in \mathcal{U}$, the quadratic variation of a controlled martingale X^ν is given by

$$\langle X^\nu \rangle_t = \int_0^t \mathrm{Tr}(\nu_s \nu_s^\top) \, \mathrm{d}s = t,$$

for any $t \geq 0$, by the definition of the control set U.

Definition 2.4. We say that a process X has *unit quadratic variation* if its quadratic variation is given by

$$\langle X \rangle_t = t, \quad \text{for all} \quad t \geq 0.$$

A martingale with unit quadratic variation has the property that the expected exit time of the martingale from a ball is fixed. This gives a bound on the expected exit time from the domain D as follows.

Proposition 2.5. *Let X be a continuous martingale with initial condition $X_0 = x \in D$, and suppose that X has unit quadratic variation. Fix $R > 0$ and denote by τ_R the first exit time from $B_R(0) := \{y \in \mathbb{R}^d \colon \ |y| < R\}$. Then*

$$\mathbb{E}^x[\tau_R] = R^2 - |x|^2,$$

Moreover, defining τ to be the first exit time from D, we have the bound

$$\mathbb{E}^x[\tau] \leq \mathrm{diam}(D)^2 - |x|^2 < \infty.$$

Proof. Applying Itô's formula to $|X_{\tau_R}|^2$ and taking expectations, we find that

$$\mathbb{E}^x \left[|X_{\tau_R}|^2 \right] - |x|^2 = \mathbb{E}^x \left[\langle X \rangle_{\tau_R} \right] = \mathbb{E}^x[\tau_R],$$

since X is a martingale and has unit quadratic variation. Therefore, by continuity of the paths of X, we have

$$\mathbb{E}^x[\tau_R] = R^2 - |x|^2.$$

Now set $R = \mathrm{diam}(D)$ so that $D \subseteq B_R(x)$. Then the inequality $\tau \leq \tau_R$ holds pointwise and, in particular,

$$\mathbb{E}^x[\tau] \leq \mathbb{E}^x[\tau_R] = \mathrm{diam}(D)^2 - |x|^2 < \infty,$$

as required. □

2.2. **Weak formulation.** We now introduce the weak formulation of the control problem, following El Karoui and Tan in [1]. The problem is to find the weak value function $v^W : D \to \mathbb{R}$, which we define below. In the weak formulation, the controls will take values in a set of probability measures, and the dynamics of the controlled martingales will be described as solutions of a local martingale problem.

Define the space of continuous paths $\Omega := C([0,\infty),\mathbb{R}^d)$ and denote the set of Borel measurable functions $\nu : \mathbb{R}_+ \to U$ by $\mathcal{B}(\mathbb{R}_+, U)$. Then set $\overline{\Omega} = \Omega \times \mathcal{B}(\mathbb{R}_+, U)$ and denote an element of $\overline{\Omega}$ by $\overline{\omega} = (\omega, u)$. Define the canonical process $\overline{X} = (X, \nu)$ on $\overline{\Omega}$ by $X_t(\overline{\omega}) = \omega_t$, for each $t \geq 0$, and $\nu(\overline{\omega}) = u$. We define the canonical filtration as in [1]. For $\phi \in C_b(\mathbb{R}_+ \times U)$, $s \geq 0$, define

$$M_s(\phi) := \int_0^s \phi(r, \nu_r)\,\mathrm{d}r.$$

Then define the canonical filtration $\overline{\mathbb{F}} = (\overline{\mathcal{F}}_t)_{t\geq 0}$ by

$$\overline{\mathcal{F}}_t := \sigma\left\{(X_s, M_s(\phi))\colon \phi \in C_b(\mathbb{R}_+ \times U), s \leq t\right\}, \quad t \geq 0.$$

Control: Let $\mathbb{M}$ be the set of probability measures on the set $\overline{\Omega}$. For each $x \in D$, let

$$\mathbb{M}_x = \{\mathbb{P} \in \mathbb{M}\colon \mathbb{P}(X_0 = x) = 1\}.$$

Dynamics: For $x \in D$, define

$$\mathcal{P}_x := \{\mathbb{P} \in \mathbb{M}_x : \quad t \mapsto \phi(X_t) - \phi(X_0) - \frac{1}{2}\int_0^t \mathrm{Tr}\left(D^2\phi(X_s)\nu_s\nu_s^\top\right)\mathrm{d}s \\ \text{is a } (\overline{\mathbb{F}}, \mathbb{P})\text{-local martingale for all } \phi \in C^2(\mathbb{R}^d)\},$$

and let $\tau = \inf\{t \geq 0\colon X_t \notin D\}$.

Value function: Define the *weak value function* $v^W : D \to \mathbb{R}$ by

$$v^W(x) = \inf_{\mathbb{P}\in\mathcal{P}_x} \mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s + g(X_\tau)\right],$$

where $\mathbb{E}^{\mathbb{P}}$ denotes expectation with respect to the measure $\mathbb{P}$.

Remark 2.6. A measure $\mathbb{P} \in \mathcal{P}_x$ is a solution of a local martingale problem, as defined in Definition 4.5 of [5, Chapter 5]. As shown in Problem 4.3 and Proposition 4.6 of [5, Chapter 5], there is a correspondence between solutions of a local martingale problem and weak solutions of an SDE. In our set up, a measure $\mathbb{P} \in \mathcal{P}_x$ corresponds to a weak solution of the SDE (2.1) with initial distribution δ_x.

We will now show that, under Assumption 2.1, the weak and strong value functions are equal, by referring to Theorem 4.5 of [1].

Proposition 2.7. *Suppose that Assumption 2.1 holds. Then the weak and strong formulations of the control problem are equivalent; i.e.* $v^S = v^W$ *in* D.

Proof. We apply Theorem 4.5 of [1], which gives conditions for equality of the weak and strong value functions. Define a function $\Phi : \Omega \to \mathbb{R}$ by

$$\Phi(\omega) = \int_0^{\tau(\omega)} f(X_s(\omega))\,\mathrm{d}s + g(X_{\tau(\omega)}(\omega)),$$

and fix $x \in D$. Then, by Theorem 4.5 of [1], it is sufficient to show that Φ is upper semicontinuous and bounded above by some random variable ξ that is uniformly integrable under the family of probability measures $\mathcal{P}_x$.

Under our assumptions, $f : D \to \mathbb{R}$ and $g : \partial D \to \mathbb{R}$ are upper semicontinuous and so Φ is also upper semicontinuous. Since we have also assumed that f and g are bounded above, we have the bound

$$\Phi(\omega) \leq M\tau(\omega) + K =: \xi(\omega).$$

Fix $\mathbb{P} \in \mathcal{P}_x$ and let (X, ν) have joint law $\mathbb{P}$. Then the process X has unit quadratic variation, and so by Proposition 2.5, $\mathbb{E}^{\mathbb{P}}[\tau] \leq \mathrm{diam}(D)^2 - |x|^2$. Hence

$$\mathbb{E}^{\mathbb{P}}[\xi] \leq M\,\mathrm{diam}(D)^2 - |x|^2 + K < \infty,$$

independently of the choice of measure $\mathbb{P}$. Therefore ξ is uniformly integrable under $\mathcal{P}_x$.

We apply Theorem 4.5 of [1] to conclude that $v^S(x) = v^W(x)$. □

With the result of Proposition 2.7 in hand, we will write $v = v^W = v^S$ and refer to v as the *value function*. We henceforth choose to work with the strong formulation of the control problem, unless we explicitly refer to the weak formulation. We additionally define the notion of a *Markov control* in the following section.

2.3. **Markov formulation.** The Markov formulation is the strongest formulation of the control problem that we will introduce in this paper. Markov controls are defined similarly in Section 3 of [3, Chapter IV] and Section 3.1 of [16].

Definition 2.8. For each $x \in D$, define the set $\mathcal{U}_x^M \subset \mathcal{U}$ of *Markov controls* as follows. A control $\nu \in \mathcal{U}$ is an element of $\mathcal{U}_x^M$ if and only if, for all $t \geq 0$, $\nu_t = \sigma(X_t^{\sigma,x})$, where $X^{\sigma,x}$ is a strong solution of the SDE

$$\mathrm{d}X_t = \sigma(X_t)\,\mathrm{d}B_t; \quad X_0 = x,$$

for some Borel function $\sigma : D \to U$. We then write $X^\nu = X^{\sigma,x}$.

The Markov formulation of the control problem is to find the *Markov value function* $v^M : D \to \mathbb{R}$, defined by

$$v^M(x) = \inf_{\nu \in \mathcal{U}_x^M} \mathbb{E}^x \left[\int_0^\tau f(X_s^\nu)\,\mathrm{d}s + g(X_\tau^\nu) \right], \quad x \in D.$$

Proposition 2.9. *For any $x \in D$, $v^M(x) \geq v^S(x)$.*

Proof. The result follows immediately from the inclusion $\mathcal{U}_x^M \subset \mathcal{U}$. □

Under mild assumptions, we will show that there is in fact equality between the strong and Markov value functions on a case by case basis, in the following sections. We also leave open a case in which the Markov and strong value functions may differ.

In the following section, we consider two simple examples of minimising and maximising the expected time spent in a ball about the origin.

3. Occupation times

From now on, we specialise to radially symmetric cost functions defined on a ball. Fix $d \geq 2$, $R > 0$ and let the domain be $D = B_R(0) \subset \mathbb{R}^d$. In this section, we will consider two examples that illustrate the optimality of the controlled processes that we call radial and tangential motion, which we later show to be optimal for more general cost functions. Note that, since the cost functions in this section have a discontinuity in $D \setminus \{0\}$, these examples are not covered the more general results in this paper. However, some of the observations made in the proofs of optimality for these examples are used in later sections.

We first consider the following example of minimising the expected time spent in a ball about the origin that is contained in D.

Example 3.1. Let $\rho \in (0, R)$, fix the boundary cost $g \equiv 0$ and define $f : D \to \mathbb{R}$ by

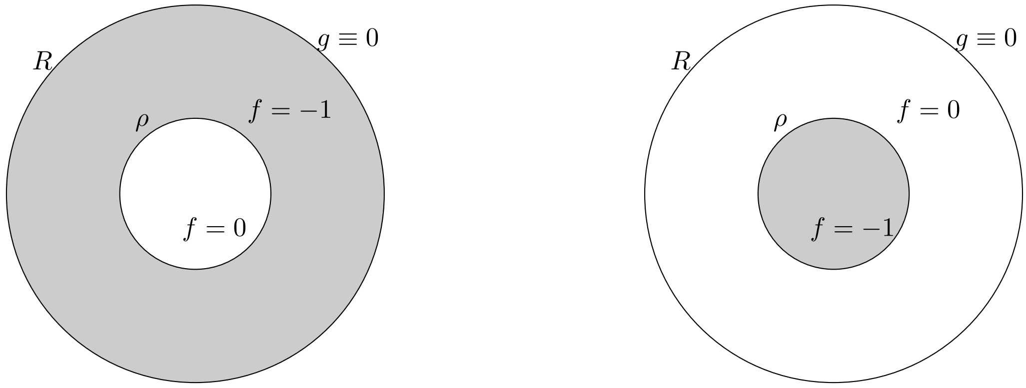
$$f(x) = \begin{cases} 0, & |x| \leq \rho, \\ -1, & |x| \in (\rho, R). \end{cases}$$

We seek the value function

$$v(x) = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s \right] = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau -\mathbb{1}_{\{|X_s^\sigma| \in (\rho, R)\}}\,\mathrm{d}s \right].$$

That is, we wish to maximise the expected time that the radius process $|X^\sigma|$ spends in the interval (ρ, R). The cost function is shown in Figure 1a.

Since the problem is radially symmetric, we expect the value function v to depend only on the radius. In fact, in this example and the example that follows, it will be convenient to work with the squared radius of any controlled process. We now derive an SDE for this squared radius process.



(A) Cost function for Example 3.1 (B) Cost function for Example 3.6

FIGURE 1

Lemma 3.2. *Let $x \in D$, $\sigma \in \mathcal{U}$, and define X^σ by the stochastic integral*

$$X_t^\sigma = x + \int_0^t \sigma_s \, \mathrm{d}B_s, \quad t \geq 0.$$

Define the squared radius process Z^σ by $Z_t^\sigma := |X_t^\sigma|^2$, $t \geq 0$. Then Z^σ satisfies the SDE

$$\mathrm{d}Z_t^\sigma = 2X_t^\top \sigma_t \, \mathrm{d}B_t + \mathrm{d}t, \quad Z_0^\sigma = |x|^2. \tag{3.1}$$

Proof. We apply Itô's formula, along with the constraint that $\sigma_t \in U$, to find

$$\begin{aligned} \mathrm{d}Z_t^\sigma &= 2X^{\sigma\top} \sigma_t \, \mathrm{d}B_t + \mathrm{Tr}(\sigma_t \sigma_t^\top) \, \mathrm{d}t \\ &= 2X^{\sigma\top} \sigma_t \, \mathrm{d}B_t + \mathrm{d}t. \end{aligned}$$ □

We conjecture that, on the boundary $\{x \in D \colon |x| = \rho\}$, any optimal control for Example 3.1 must enforce motion tangential to this internal boundary. We now define a process that exhibits this behaviour.

Definition 3.3 (Tangential motion)**.** For $x \in D \setminus \{0\}$, define

$$\sigma^0(x) := \frac{1}{|x|} \begin{bmatrix} x^\perp; & 0; & \cdots; & 0 \end{bmatrix} \in \mathbb{R}^{d,d}, \tag{3.2}$$

where $x^\perp$ denotes any $x \in \mathbb{R}^d \setminus \{0\}$ such that $x^\top x^\perp = 0$. Fix $x \in D \setminus \{0\}$ and suppose that X^{σ^0} is a strong solution of the SDE

$$\mathrm{d}X_t = \sigma^0(X_t) \, \mathrm{d}B_t, \quad X_0 = x.$$

For $t \geq 0$, define $\sigma_t^0 := \sigma^0(X_t^{\sigma^0})$, so that

$$X_t^{\sigma^0} = x + \int_0^t \sigma_s^0 \, \mathrm{d}B_s.$$

We say that the process X^{σ^0} follows *tangential motion.*

Note that $\sigma^0(0)$ is not defined. In later sections we will need to take care in cases where tangential motion is optimal close to the origin.

For σ^0 defined in Definition 3.3, we can find a formula for the squared radius process Z^{σ^0} via Lemma 3.2, as follows.

Lemma 3.4. *Suppose that X^{σ^0} follows tangential motion, as defined in Definition 3.3, with $X_0^{\sigma^0} = x \neq 0$. Then the radius process is deterministically increasing and, for any $t \geq 0$,*

$$Z_t^{\sigma^0} = \left| X_t^{\sigma^0} \right|^2 = |x| + t.$$

Proof. For $t \geq 0$, provided that $\left|X_t^{\sigma^0}\right| \neq 0$, we see that

$$\left(X_t^{\sigma^0}\right)^\top \sigma_t^0 = \frac{1}{\left|X_t^{\sigma^0}\right|}\left[\left(X_t^{\sigma^0}\right)^\top \left(X_t^{\sigma^0}\right)^\perp, \quad 0, \quad \dots, \quad 0\right] = \left[0, \quad \dots, \quad 0\right].$$

Therefore, by Lemma 3.2, Z^{σ^0} satisfies

$$\mathrm{d}Z_t^{\sigma^0} = \mathrm{d}t.$$

Let $\xi = |x|^2 \neq 0$, so that $Z_0^{\sigma^0} = \xi$. Then Z^{σ^0} is the deterministically increasing process given by

$$t \mapsto Z_t^{\sigma^0} = \xi + t. \qquad \square$$

As a consequence of the above lemma, supposing that $X_0^{\sigma^0} \neq 0$, we have that $\left|X_t^{\sigma^0}\right| > 0$ for all $t \geq 0$. Therefore the control σ^0 is well-defined when starting away from the origin. Note that, for $d \geq 3$, a control of this form is not unique, since the orthogonal vector in the definition of σ^0 can be chosen as any element of a $(d-1)$-dimensional subspace.

The observation that the process X^{σ^0} has deterministically increasing radius was made by Fernholtz, Karatzas and Ruf in Section 6.2 of [2] and again by Larsson and Ruf in Section 4.2 of [8], where they consider a problem of relative arbitrage.

In Figure 3a, we show a simulated trajectory of a process following tangential motion in dimension $d = 2$. We note that a similar simulation is produced in Figure 2 of [8].

Having defined tangential motion and proved a key property of this process, we now construct a candidate for the value function in Example 3.1.

Fix $\xi \geq \rho^2$. Then we conjecture that the control σ^0 defined in Definition 3.3 is optimal, and we compute the expected cost

$$\begin{aligned}\mathbb{E}^x\left[\int_0^\tau -\mathbb{1}_{\{|X_s^{\sigma^0}|\in(\rho,R)\}}\,\mathrm{d}s\right] &= \mathbb{E}^\xi\left[\int_0^\tau -\mathbb{1}_{\{Z_s^{\sigma^0}\in(\rho^2,R^2)\}}\,\mathrm{d}s\right] = -\int_0^\infty \mathbb{1}_{\{s\in(0,R^2-\xi)\}}\,\mathrm{d}s \\ &= -\int_0^{R^2-\xi}\mathrm{d}s = \xi - R^2.\end{aligned}$$

Now suppose that $\xi < \rho^2$. This includes the case where the process starts at the origin, where the control σ^0 is not well-defined. However, since the cost is zero in the ball $\{x \in \mathbb{R}^d \colon |x| < \rho\}$, we will see that any strategy is optimal in this region. For a fixed $r \in (\sqrt{\xi}, \rho)$ and an arbitrary $\sigma \in \mathcal{U}$, define the control $\sigma^\star$ by

$$\sigma_t^\star = \begin{cases}\sigma_t, & \left|X_t^{\sigma^\star}\right| < r,\\ \sigma_t^0, & \left|X_t^{\sigma^\star}\right| \in [r, R).\end{cases}$$

Then we compute the expected cost

$$\begin{aligned}\mathbb{E}^x\left[\int_0^\tau -\mathbb{1}_{\{|X_s^{\sigma^\star}|\in(\rho,R)\}}\,\mathrm{d}s\right] &= \mathbb{E}^\xi\left[\int_0^\tau -\mathbb{1}_{\{Z_s^{\sigma^\star}\in(\rho^2,R^2)\}}\,\mathrm{d}s\right] = \mathbb{E}^{r^2}\left[\int_0^\tau -\mathbb{1}_{\{Z_s^{\sigma^0}\in(\rho^2,R^2)\}}\,\mathrm{d}s\right] \\ &= -\int_{r^2-\xi}^\infty \mathbb{1}_{\{s\in(\rho^2-\xi,R^2-\xi)\}}\,\mathrm{d}s = -\int_{\rho^2}^{R^2}\mathrm{d}s = \rho^2 - R^2.\end{aligned}$$

This calculation gives us a conjecture for the value function in Example 3.1. Using the Itô-Tanaka formula, we will show that our candidate function satisfies a dynamic programming principle, as described in Appendix A, and we can then deduce that this function must be the value function.

Proposition 3.5. *Let $w : [0, R^2) \to \mathbb{R}$ be defined by*

$$w(\xi) = \begin{cases}\rho^2 - R^2, & \xi \leq \rho^2,\\ \xi - R^2, & \xi \in (\rho^2, R^2),\end{cases}$$

and define $\overline{v} : D \to \mathbb{R}$ by $\overline{v}(x) = w(|x|^2)$, for $x \in D$. Then the value function for Example 3.1 is given by $v = v^M = v^S = v^W = \overline{v}$.

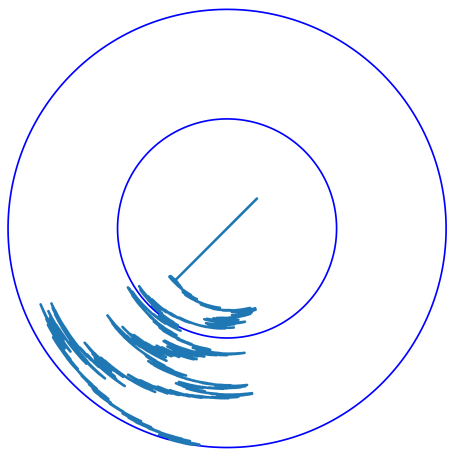


FIGURE 2. A possible trajectory for an optimal strategy in Example 3.1

Proof. We first show that $\overline{v}$ satisfies the form of the dynamic programming principle given in Remark A.3.

Define $\tilde{f} : [0, R^2) \to \mathbb{R}$ such that $f(x) = \tilde{f}(|x|)$, for $x \in D$; i.e.

$$\tilde{f}(\xi) = -\mathbb{1}_{\{\xi \in (\rho^2, R^2)\}}, \quad \xi \in [0, R^2).$$

We seek to prove that $w(Z_t^\sigma) + \int_0^t \tilde{f}(Z_s^\sigma)\,\mathrm{d}s$ is a submartingale for all $\sigma \in \mathcal{U}$, and that $w(Z_t^{\sigma^\star}) + \int_0^t \tilde{f}(Z_s^{\sigma^\star})\,\mathrm{d}s$ is a martingale for an optimal strategy $\sigma^\star \in \mathcal{U}$.

Let $\sigma \in \mathcal{U}$. We note that w is not continuously differentiable at $\xi = \rho^2$, so we apply the Itô-Tanaka formula to write down an SDE for $w(Z_t^\sigma)$. We calculate that the left derivative of w is given by

$$w'_-(\xi) = \begin{cases} 0, & \text{for} \quad \xi \le \rho^2, \\ 1, & \text{for} \quad \xi \in (\rho^2, R^2), \end{cases}$$

and the distributional derivative of w'_- is

$$w''(\mathrm{d}a) = \delta_{\rho^2}(\mathrm{d}a).$$

Hence, by the Itô-Tanaka formula,

$$\begin{aligned} w(Z_t^\sigma) - w(\xi) + \int_0^t \tilde{f}(Z_s^\sigma)\,\mathrm{d}s &= 2\int_0^t \mathbb{1}_{\{Z_s^\sigma > \rho^2\}} X_s^\top \sigma_s\,\mathrm{d}B_s + \int_0^t \mathbb{1}_{\{Z_s^\sigma > \rho^2\}}\,\mathrm{d}s + \frac{1}{2}L_t^{\sigma,\rho^2} - \int_0^t \mathbb{1}_{\{Z_s^\sigma > \rho^2\}}\,\mathrm{d}s \\ &= 2\int_0^t \mathbb{1}_{\{Z_s^\sigma > \rho^2\}} X_s^\top \sigma_s\,\mathrm{d}B_s + \frac{1}{2}L_t^{\sigma,\rho^2}. \end{aligned}$$

Since local time is always non-negative, we have shown that

$$w(Z_t^\sigma) + \int_0^t \tilde{f}(Z_s^\sigma)\,\mathrm{d}s$$

is a submartingale for any $\sigma \in \mathcal{U}$.

Now we note that, for any $\sigma \in \mathcal{U}$,

$$\overline{v}(X_\tau^\sigma) = w(Z_\tau^\sigma) = w(R^2) = 0,$$

by continuity of the paths of X^σ. Therefore, we can use the submartingale property and the optional sampling theorem to find that

$$\begin{aligned} \mathbb{E}^x\left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s\right] &= \mathbb{E}^x\left[\int_0^\tau f(X_s^\sigma)\,\mathrm{d}s + \overline{v}(X_\tau^\sigma)\right] \\ &= \mathbb{E}^\xi\left[\int_0^\tau \tilde{f}(Z_s^\sigma)\,\mathrm{d}s + w(Z_\tau^\sigma)\right] \ge w(\xi) = \overline{v}(x). \end{aligned}$$

Now, supposing that $\xi \neq 0$, consider the control $\sigma^\star = \sigma^0$, so that $Z_t^{\sigma^\star} = \xi + t$, for any $t \geq 0$. Then

$$\mathbb{E}^\xi \left[-\int_0^\tau \mathbb{1}_{\left\{Z_s^{\sigma^\star} \in (\rho^2, R^2)\right\}} \,\mathrm{d}s \right] = -\int_0^\infty \mathbb{1}_{\{s \in (\rho^2 - \xi \wedge \rho^2, R^2 - \xi)\}} \,\mathrm{d}s = \begin{cases} \rho^2 - R^2, & \xi \in (0, \rho^2], \\ \xi - R^2, & \xi \in (\rho^2, R^2). \end{cases}$$

In the case that $\xi = 0$, fix $r \in (0, \rho)$ and $\sigma \in \mathcal{U}_0^M$, and take

$$\sigma_t^\star = \begin{cases} \sigma_t, & \left|X_t^{\sigma^\star}\right| < r, \\ \sigma_t^0, & \left|X_t^{\sigma^\star}\right| \in [r, R). \end{cases}$$

Then

$$\mathbb{E}^0 \left[-\int_0^\tau \mathbb{1}_{\left\{Z_s^{\sigma^\star} \in (\rho^2, R^2)\right\}} \,\mathrm{d}s \right] = \mathbb{E}^{r^2} \left[-\int_0^\tau \mathbb{1}_{\left\{Z_s^{\sigma^\star} \in (\rho^2, R^2)\right\}} \,\mathrm{d}s \right] = \rho^2 - R^2.$$

We conclude that, for any $x \in D$,

$$\overline{v}(x) = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma) \,\mathrm{d}s \right] = v^S(x) = v^W(x).$$

Hence the conjectured function $\overline{v}$ is indeed the value function. Note also that, for any $x \in D$, $\sigma^\star \in \mathcal{U}_x^M$, and so $v^M = v^S = v^W = \overline{v}$. $\square$

We now turn to a second example of maximising the expected time spent in a ball around the origin.

Example 3.6. Fix $\rho \in (0, R)$, define the cost $f : D \to \mathbb{R}$ by

$$f(x) = \begin{cases} -1, & |x| < \rho \\ 0, & |x| \in [\rho, R), \end{cases}$$

and fix the boundary cost $g \equiv 0$. We seek the value function

$$v(x) = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X_s^\sigma) \,\mathrm{d}s \right] = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau -\mathbb{1}_{\{|X_s^\sigma| < \rho\}} \,\mathrm{d}s \right].$$

That is, we wish to maximise the expected time that the martingale spends in the ball $B_\rho(0)$. The cost function is shown in Figure 1b.

We propose that an optimal strategy is to run as a Brownian motion on the radius of the domain. We now define a process that follows this strategy.

Definition 3.7 (Radial motion)**.** Define a function $\sigma^1 : D \to \mathbb{R}$ by

$$\sigma^1(x) = \begin{cases} \frac{1}{|x|} \begin{bmatrix} x; & 0; & \cdots; & 0 \end{bmatrix}, & x \neq 0, \\ \begin{bmatrix} e_1; & 0; & \cdots; & 0 \end{bmatrix}, & x = 0, \end{cases} \tag{3.3}$$

where e_1 is the unit vector in the first coordinate direction. Fix $x \in D$ and define σ^1 to be the constant control given by $\sigma_t^1 = \sigma(x)$, for all $t \geq 0$. Define X^{σ^1} by

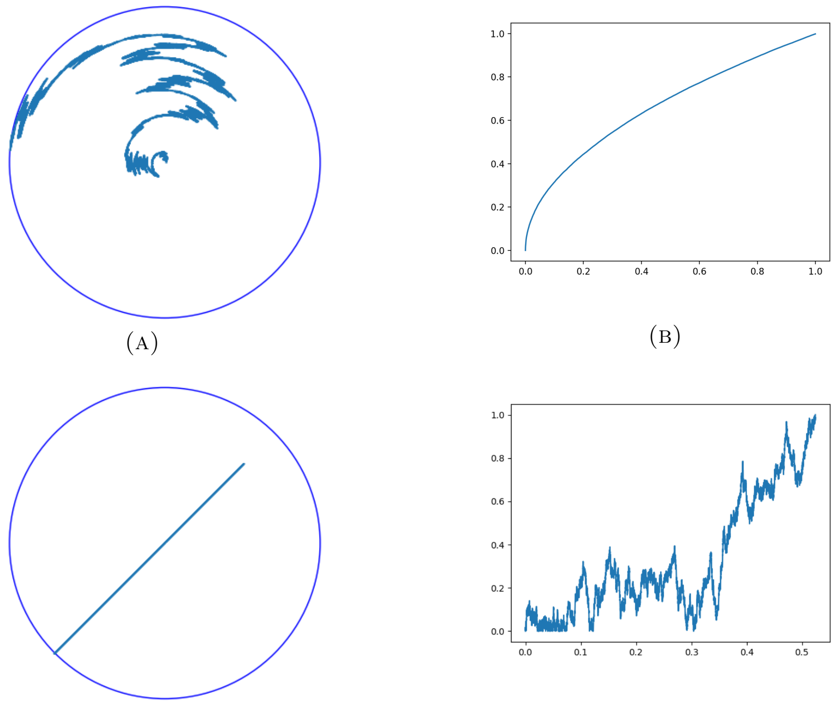
$$X_t^{\sigma^1} = x + \int_0^t \sigma_s^1 \,\mathrm{d}B_s = x + \sigma^1(x) B_t, \quad t \geq 0.$$

We say that the process X^{σ^1} follows *radial motion.*

A simulated trajectory of a process following radial motion is shown in Figure 3c, along with the sample path of its radius in Figure 3d.

Let W be the first component of B, and note that W is a one-dimensional Brownian motion. Then, defining σ^1 as in Definition 3.7, we see that

$$X_t^{\sigma^1} = \begin{cases} x + \int_0^t \sigma_s^1 \,\mathrm{d}B_s = x + W_t \frac{x}{|x|}, & x \neq 0, \\ W_t e_1, & x = 0. \end{cases}$$



(A) (B)

(C) (D)

FIGURE 3. Sample paths of processes $(X_t^{\sigma^0})_{t\geq 0}$ following tangential motion and $(X_t^{\sigma^1})_{t\geq 0}$ following radial motion are shown in (A) and (C), respectively. The corresponding radius processes are shown in (B) and (D) respectively.

Hence $\left|X_t^{\sigma^1}\right| = \left||x| + W_t\right|$, and so

$$\mathbb{E}^x\left[\int_0^\tau -\mathbb{1}_{\{|X_t^{\sigma^1}|<\rho\}}\right] = \mathbb{E}^{|x|}\left[-\int_0^{\tau_R} \mathbb{1}_{\{W_t\in(-\rho,\rho)\}}\right],$$

where $\tau_R = \inf\{t \geq 0\colon\ |W_t| = R\}$.

We can compute this expected cost by using the Green's function and speed measure for one-dimensional Brownian motion, as defined in Section 3 of [11, Chapter VII]. The speed measure m is given by $\int m(\mathrm{d}y) = 2\int \mathrm{d}y$, and the Green's function G on the interval $[-\rho,\rho]$ is given by

$$G(r,y) = \begin{cases} \frac{(y+R)(R-r)}{2R}, & y \leq r, \\ \frac{(r+R)(R-y)}{2R}, & y \geq r, \end{cases}$$

for $r, y \in [-\rho,\rho]$. By Corollary 3.8 of [11, Chapter VII], we have

$$\mathbb{E}^{|x|}\left[-\int_0^{\tau_R} \mathbb{1}_{\{W_t\in(-\rho,\rho)\}}\right] = -\int_{-\rho}^{\rho} G(|x|,y)m(\mathrm{d}y) = \begin{cases} |x|^2 + \rho^2 - 2\rho R, & |x| < \rho, \\ 2\rho\,|x| - 2\rho R, & |x| \geq \rho. \end{cases}$$

This gives us a candidate for the value function in Example 3.6. Again, we present this function in terms of the radius squared. We can then apply Itô's formula, using the SDE for the squared radius process that we derived in Lemma 3.2.

Proposition 3.8. *Let $w : [0, R^2) \to \mathbb{R}$ be defined by*

$$w(\xi) = \begin{cases} \xi + \rho^2 - 2\rho R, & \xi \leq \rho^2, \\ 2\rho\xi^{\frac{1}{2}} - 2\rho R, & \xi \in (\rho^2, R^2), \end{cases}$$

and define $\overline{v} : D \to \mathbb{R}$ by $\overline{v}(x) = w(|x|^2)$, for $x \in D$. Then the value function for Example 3.6 is given by $v = v^M = v^S = v^W = \overline{v}$.

Proof of Proposition 3.8. Again we will show that $\overline{v}$ satisfies the form of the dynamic programming principle given in Remark A.3.

Note first that w is continuously differentiable and twice piecewise continuously differentiable, with

$$w'(\xi) = \begin{cases} 1, & \xi \le \rho^2, \\ \rho\xi^{-\frac{1}{2}}, & \xi \in (\rho^2, R^2), \end{cases} \qquad \text{and} \qquad w''(\xi) = \begin{cases} 0, & \xi \le \rho^2, \\ -\frac{1}{2}\rho\xi^{-\frac{3}{2}}, & \xi \in (\rho^2, R^2). \end{cases}$$

Hence we can apply Itô's formula to $w(Z^\sigma_t)$, for any $\sigma \in \mathcal{U}$, recalling that $Z^\sigma_t = |X^\sigma_t|^2$. Let $Z^\sigma_0 = \xi \in [0, R^2)$. Then, for $t > 0$,

$$\begin{aligned} w(Z^\sigma_t) - w(\xi) = &\int_0^t \mathbb{1}_{\{Z^\sigma_s \le \rho^2\}} \, \mathrm{d}Z^\sigma_s + \rho \int_0^t \mathbb{1}_{\{Z^\sigma_s \in (\rho^2, R^2)\}} (Z^\sigma_s)^{-\frac{1}{2}} \, \mathrm{d}Z^\sigma_s \\ &- \frac{\rho}{4} \int_0^t \mathbb{1}_{\{Z^\sigma_s \in (\rho^2, R^2)\}} (Z^\sigma_s)^{-\frac{3}{2}} \, \mathrm{d}\langle Z^\sigma \rangle_s. \end{aligned} \tag{3.4}$$

Substituting in the SDE (3.1) for Z^σ, we find that there is a square-integrable martingale M^σ such that

$$\begin{aligned} w(Z^\sigma_t) - w(\xi) &= \int_0^t \mathrm{d}M^\sigma_s + \int_0^t \mathbb{1}_{\{Z^\sigma_s \le \rho^2\}} \, \mathrm{d}s + \rho \int_0^t \mathbb{1}_{\{Z^\sigma_s \in (\rho^2, R^2)\}} (Z^\sigma_s)^{-\frac{1}{2}} \, \mathrm{d}s \\ &\quad - \rho \int_0^t \mathbb{1}_{\{Z^\sigma_s \in (\rho^2, R^2)\}} (Z^\sigma_s)^{-\frac{3}{2}} \operatorname{Tr}\left(X^\sigma_s X^{\sigma\top}_s \sigma_s \sigma_s^\top \right) \mathrm{d}s \\ &= \int_0^t \mathrm{d}M^\sigma s - \int_0^t f(X^\sigma_s) \, \mathrm{d}s + \rho \int_0^t \mathbb{1}_{\{|X^\sigma_s| \in (\rho, R)\}} \, |X^\sigma_s|^{-3} \operatorname{Tr}\left(\left[|X^\sigma_s|^2 I - X^\sigma_s X^{\sigma\top}_s \right] \sigma_s \sigma_s^\top \right) \mathrm{d}s, \end{aligned}$$

where I denotes the identity matrix. Noting that the matrix $|x|^2 I - xx^\top$ is positive semi-definite for any $x \in \mathbb{R}^d$, we see that the final integral in the above equation is always non-negative, and so

$$\overline{v}(X^\sigma_t) + \int_0^t f(X^\sigma_s) \, \mathrm{d}s$$

is a submartingale for any $\sigma \in \mathcal{U}$.

Now take $\sigma = \sigma^1$ and let W be the first component of the Brownian motion B. Then, from the SDE (3.1) for the squared radius process, we see that $Z := Z^{\sigma^1}$ is a one-dimensional squared Bessel process satisfying

$$\mathrm{d}Z_t = 2\sqrt{Z_t} \, \mathrm{d}W_t + \mathrm{d}t.$$

Substituting this SDE for Z into our calculation (3.4), and defining $X := X^{\sigma^1}$, we find that there is a square-integrable martingale M such that, for any $t > 0$,

$$\begin{aligned} w(Z_t) - w(\xi) &= \int_0^t \mathrm{d}M_s + \int_0^t \mathbb{1}_{\{Z_s \le \rho^2\}} \, \mathrm{d}s + \rho \int_0^t \mathbb{1}_{\{Z_s \in (\rho^2, R^2)\}} Z_s^{-\frac{1}{2}} \, \mathrm{d}s - \frac{\rho}{4} \int_0^t \mathbb{1}_{\{Z_s \in (\rho^2, R^2)\}} Z_s^{-\frac{3}{2}} \cdot 4 Z_s \, \mathrm{d}s \\ &= \int_0^t \mathrm{d}M_s - \int_0^t f(X_s) \, \mathrm{d}s. \end{aligned}$$

Hence $\overline{v}(X_t) + \int_0^t f(X_s) \, \mathrm{d}s$ is a martingale.

From the above submartingale and martingale properties, we can conclude in a similar manner as in the proof of Proposition 3.5 that

$$\overline{v}(x) = \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f(X^\sigma_s) \, \mathrm{d}s \right] = v^S(x) = v^W(x).$$

For each $x \in D$, the constant control σ^1 is a Markov control, and so we also have $v^M(x) = v^S(x) = \overline{v}(x)$. □

For the step cost functions considered above, optimal controls involve tangential and radial motion, as defined in Definition 3.3 and Definition 3.7, respectively. A consequence of Proposition 4.6 in the next section will be that either tangential or radial motion is optimal for any continuous monotone cost function. More generally, we will show that, for a continuous radially symmetric cost function with sufficient regularity, an optimal control is to switch between radial and tangential motion.

4. Explicit solution for radially symmetric costs

In this section, we consider the control problem for more general radially symmetric cost functions. We make the ansatz that the optimal strategy is to switch between two extreme behaviours in the control set, namely the strategies of tangential and radial motion defined in Definition 3.3 and Definition 3.7, respectively. In this way, we reduce the control problem to a one-dimensional optimal switching problem for the radius process. We use the principles of smooth and continuous fit to identify the optimal switching points, and we provide an algorithm to construct a candidate for the value function. We are able to write this function explicitly in Definition 4.5. We refer to the theory of viscosity solutions, which we summarise in Appendix A, in order to verify that the candidate function is indeed equal to the value function.

We make the following assumptions.

Assumption 4.1. Suppose that

(1) The domain is $D = B_R(0) \subset \mathbb{R}^d$, for some $R > 0$ and $d \geq 2$;
(2) The cost function f is radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$, for some function $\tilde{f} : [0, R) \to \mathbb{R}$;
(3) The boundary cost g is constant;
(4) The cost function f is continuous;
(5) There exists $\eta > 0$ such that the cost function $\tilde{f}$ is monotone on the interval $(0, \eta)$;
(6) The one-sided derivative $\tilde{f}'_+(r)$ exists for all $r > 0$ and changes sign only finitely many times;
(7) There exists $\delta > 0$ such that $\tilde{f}$ is continuously differentiable on $(0, \delta)$ and $\lim_{r\to 0} r\tilde{f}'(r) = 0$.

Remark 4.2. In Section 5, we will relax the fourth condition on continuity and the seventh condition on differentiability.

We rule out the case that the cost function oscillates at the origin by imposing the fifth condition on monotonicity. We will see in the following sections that the fifth and sixth conditions allow us to find an optimal strategy that switches between two regimes finitely many times. We believe that we would still be able to solve the control problem explicitly if we relax the fifth and sixth conditions, but in this case an optimal strategy may not exist. To simplify our exposition, we do not treat this case here.

Recall the definitions of the functions σ^0 in (3.2) and σ^1 in (3.3), which are associated to tangential and radial motion, respectively. We conjecture that, in the case that $\tilde{f}$ is increasing at the origin, there exists a sequence of points $0 = s_0 < r_1 < s_1 < \ldots < R$ such that an optimal control is of the form

$$\sigma^\star_t = \begin{cases} \sigma^1\big(X^{\sigma^\star}_0\big), & \big|X^{\sigma^\star}_t\big| \in [0, r_1), \\ \sigma^0\big(X^{\sigma^\star}_t\big), & \big|X^{\sigma^\star}_t\big| \in [r_i, s_i], \quad i \geq 1, \\ \sigma^1\big(X^{\sigma^\star}_{\tau_{s_i}}\big), & \big|X^{\sigma^\star}_t\big| \in (s_i, r_{i+1}), \quad i \geq 1, \end{cases} \tag{4.1}$$

where, for each $i \geq 1$, we define the hitting time $\tau_{s_i} := \inf\big\{t \geq 0\colon \big|X^{\sigma^\star}_t\big| = s_i\big\}$. Note that $t \mapsto \big|X^{\sigma^\star}_t\big|$ is deterministically increasing when $\big|X^{\sigma^\star}_t\big| \in [r_i, s_i]$, for any $i \geq 1$, by Lemma 3.2. Therefore, if $\big|X^{\sigma^\star}_0\big| \geq r_1$, $\big|X^{\sigma^\star}_t\big| \geq r_1$ for all $t \geq 0$.

Similarly, if $\tilde{f}$ is decreasing at the origin, we conjecture that there is a sequence of points $0 = r_0 < s_0 < r_1 < \ldots < R$ such that an optimal control is of the form

$$\sigma^\star_t = \begin{cases} \sigma^0\left(X^{\sigma^\star}_t\right), & \big|X^{\sigma^\star}_t\big| \in (0, s_0], \\ \sigma^1\big(X^{\sigma^\star}_{\tau_{s_i}}\big), & \big|X^{\sigma^\star}_t\big| \in (s_{i-1}, r_i), \quad i \geq 1, \\ \sigma^0\left(X^{\sigma^\star}_t\right), & \big|X^{\sigma^\star}_t\big| \in [r_i, s_i], \quad i \geq 1. \end{cases} \tag{4.2}$$

Note that, in this second case, we do not make any claim about the optimal behaviour at the origin. Since $\sigma^0(0)$ is not defined, we will require some approximation at the origin in this case. We explore this further in Section 5 where we relax Assumption 4.1.

In either case, we conjecture that, at any time, an optimally controlled process should follow either radial motion or tangential motion, depending only on the current radial position of the process. We present a simulated trajectory of such a controlled process for an example with two switching points in Figure 4. In Proposition 4.6, we will prove that the control $\sigma^\star$ is optimal.

Note that $\sigma^\star$ does not satisfy the definition of a Markov control given in Definition 2.8. We now demonsrate how to recover the Markov property.

Remark 4.3. Let the control $\sigma^\star \in \mathcal{U}$ be as defined in (4.1). First note that, if $|x| \in [0, r_1)$, then $X^{\sigma^\star}$ is a strong solution of the SDE

$$\mathrm{d}X_t = \sigma^1(x)\,\mathrm{d}B_t, \quad X_0 = x,$$

up to the first hitting time of radius r_1, where the coefficient $\sigma^1(x)$ is a constant. When $\left|X_t^{\sigma^\star}\right| \in [r_i, s_i]$, for some $i \geq 1$, we have $\sigma_t^\star = \sigma^0(X_t^{\sigma^\star})$. When $\left|X_t^{\sigma^\star}\right| \in (s_i, r_{i+1})$, for some $i \geq 1$, we have $\sigma_t^\star = \sigma^1(X_{\tau_{s_i}}^{\sigma^\star})$, and we can calculate that $X^{\sigma^\star}$ solves the SDE

$$\mathrm{d}X_t = \sigma^1(X_t)\,\mathrm{d}B_t$$

in this region. Therefore, fixing $x \in D$, there is a Markov control $\tilde{\sigma}^\star \in \mathcal{U}_x^M$ such that

$$\mathbb{E}^x\left[\int_0^\tau f(X_s^{\tilde{\sigma}^\star})\,\mathrm{d}s\right] = \mathbb{E}^x\left[\int_0^\tau f(X_s^{\sigma^\star})\,\mathrm{d}s\right].$$

Now fix $x \in D \setminus \{0\}$ and define the control $\sigma^\star$ as in (4.2). By similar reasoning as above, there exists a Markov control $\tilde{\sigma}^\star \in \mathcal{U}_x^M$ such that

$$\mathbb{E}^x\left[\int_0^\tau f(X_s^{\tilde{\sigma}^\star})\,\mathrm{d}s\right] = \mathbb{E}^x\left[\int_0^\tau f(X_s^{\sigma^\star})\,\mathrm{d}s\right].$$

We have shown that it is possible to replace the control $\sigma^\star$ with a Markov control $\tilde{\sigma}^\star$ under which we have the same expected cost. We will make use of this Markov control in later sections in order to show that the Markov value function coincides with the strong value function under additional assumptions.

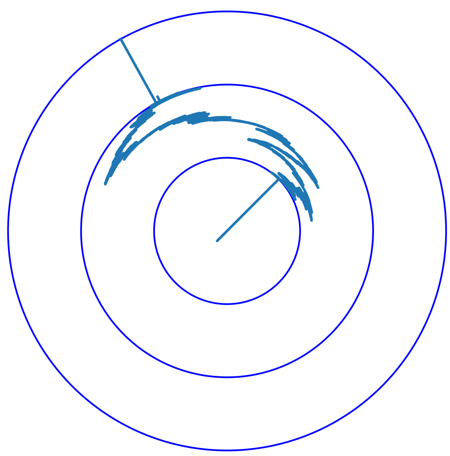


FIGURE 4. A sample path of an optimal controlled process in a case with two switching points

4.1. **Reduction to a switching problem.** Before beginning to construct a candidate for the value function, we give the following justification for our conjecture that switching between radial motion and tangential motion is optimal. We will work with the radius of the controlled process in this section. We now derive an SDE for the radius process under some simplifying assumptions.

Proposition 4.4. *Let $\sigma \in \mathcal{U}$ be of the form $\sigma_t = \left[\overline{\sigma}_t; \quad 0; \quad \ldots; \quad 0\right]$, where $\overline{\sigma}_t \in \mathbb{R}^d$ with $|\overline{\sigma}_t| = 1$, for $t \geq 0$. Let $x \in D \setminus \{0\}$ and suppose that X^σ solves the SDE*

$$\mathrm{d}X_t^\sigma = \sigma_t\,\mathrm{d}B_t = \overline{\sigma}_t\,\mathrm{d}W_t; \quad X_0^\sigma = 0,$$

where W is the first component of B. Set $r_0 := |x|$, let $\varepsilon \in (0, r)$, and define $\tau_\varepsilon := \inf\left\{t > 0\colon \left|R^\lambda_t - r_0\right| = \varepsilon\right\}$. Then there exists a $[0,1]$-valued process λ such that $|X^\sigma_t| = R^\lambda_t$, where R^λ solves the SDE

$$\mathrm{d}R^\lambda_t = \lambda_t \,\mathrm{d}W_t + \frac{1-\lambda_t^2}{2R^\lambda_t}\,\mathrm{d}t, \quad t \in [0, \tau_\varepsilon]; \quad R^\lambda_0 = r_0.$$

Proof. Let $t \in [0, \tau_\varepsilon]$, so that $X^\sigma_t \neq 0$. Then we can apply Itô's formula to find that the radius of X^σ satisfies the SDE

$$\mathrm{d}|X^\sigma_t| = |X^\sigma_t|^{-1}\,(X^\sigma_t)^\top \overline{\sigma}_t\,\mathrm{d}W_t + \frac{1}{2}\,|X^\sigma_t|^{-3}\,\mathrm{Tr}\left(\left[|X^\sigma_t|^2\, I - X^\sigma_t (X^\sigma_t)^\top\right]\overline{\sigma}_t\overline{\sigma}_t^\top\right)\mathrm{d}t. \tag{4.3}$$

Now let $(X^\sigma_t)^\perp$ denote the vector with norm $\left|(X^\sigma_t)^\perp\right| = |X^\sigma_t|$ that is orthogonal to the vector X^σ_t and satisfies

$$\overline{\sigma}_t = |X^\sigma_t|^{-1}\left(\lambda_t X^\sigma_t + \mu_t (X^\sigma_t)^\perp\right), \tag{4.4}$$

for some $\lambda_t, \mu_t \in \mathbb{R}$. Using the condition $|\overline{\sigma}_t| = 1$, we see that

$$1 = \lambda_t^2 + \mu_t^2,$$

and so $\lambda_t \in [0,1]$ and $\mu_t = \sqrt{1-\lambda_t^2}$.

Substituting the expression (4.4) for $\overline{\sigma}$ back into the SDE (4.3) for $|X^\sigma|$, and repeatedly using the identities $(X^\sigma_t)^\top X^\sigma_t = |X^\sigma_t|^2$ and $(X^\sigma_t)^\top (X^\sigma_t)^\perp = 0$, we have

$$\mathrm{d}|X^\sigma_t| = \lambda_t\,\mathrm{d}W_t + \frac{1}{2}\,|X^\sigma_t|^{-1}\left(1-\lambda_t^2\right)\mathrm{d}t.$$

Therefore, writing $R^\lambda_t = |X^\sigma_t|$, where λ_t is defined via (4.4), we arrive at the desired form of the SDE. $\square$

Now, suppose further that the process λ in the proof of Proposition 4.4 takes the form

$$\lambda_t = \lambda(R^\lambda_t), \quad t \geq 0.$$

Then we can write down the infinitesimal generator $\mathcal{L}^\lambda$ for the process R^λ as

$$\mathcal{L}^\lambda u(r) = -\frac{1}{2}\lambda^2(r) u''(r) - \frac{1-\lambda^2(r)}{2r} u'(r),$$

for $r \in (r_0 - \varepsilon, r_0 + \varepsilon)$ and any smooth function $u \in C^2((r_0-\varepsilon, r_0+\varepsilon), \mathbb{R})$.

Consider the following simplification of the control problem. Restrict the control set to contain only those controls that give rise to a process λ of the form specified above. Let $v^R : D \to \mathbb{R}$ be the value function of this simplified problem. By radial symmetry, we can write

$$v^R(x) = \tilde{v}^R(|x|),$$

for some $\tilde{v}^R : [0, R) \to \mathbb{R}$. Supposing that $\tilde{v}^R$ is twice continuously differentiable, we expect $\tilde{v}^R$ to be a classical solution of a Hamilton-Jacobi-Bellman equation. By the results of Section 3.3 of [16], $\tilde{v}^R$ solves

$$\inf_\lambda \mathcal{L}^\lambda \tilde{v}^R = \tilde{f},$$

in the interval $(r_0-\varepsilon, r_0+\varepsilon)$, where the infimum is taken over functions $\lambda : (r_0-\varepsilon, r_0+\varepsilon) \to [0,1]$.

Note that we can rewrite the generator as

$$\mathcal{L}^\lambda \tilde{v}^R(r) = -\frac{1}{2r}(\tilde{v}^R)'(r) - \frac{r}{2}\lambda^2(r)\left[\frac{1}{r}(\tilde{v}^R)'(r)\right]'.$$

Hence, at points r such that $\left[\frac{1}{r}(\tilde{v}^R)'(r)\right]' > 0$, the infimum is attained for $\lambda(r) = 1$, while at points r such that $\left[\frac{1}{r}(\tilde{v}^R)'(r)\right]' < 0$, the infimum is attained for $\lambda(r) = 0$. At a point r such that $\left[\frac{1}{r}(\tilde{v}^R)'(r)\right]' = 0$, the infimum is attained for any value $\lambda(r) \in [0,1]$.

Returning to the expression (4.4) for $\overline{\sigma}$ in terms of λ, we see that setting $\lambda_t = 1$ gives $\overline{\sigma}_t = \frac{X^\sigma_t}{|X^\sigma_t|}$, with generator $\mathcal{L}^1$ given by

$$\mathcal{L}^1 u(r) = -\frac{1}{2}u''(r). \tag{4.5}$$

Note that, away from the origin, a controlled process following this control has the same behaviour as radial motion, as defined in Definition 3.7. On the other hand, $\lambda = 0$ corresponds to tangential motion, as defined in Definition 3.3, with generator

$$\mathcal{L}^0 u(r) = -\frac{1}{2r} u'(r). \tag{4.6}$$

Therefore the above calculations support our claim that the optimal strategy should be to switch between these two behaviour regimes.

We note that, in the above discussion, we restricted the control set and made the strong assumption that the value function is twice continuously differentiable. In order to prove that the behaviour described above is optimal without these restrictions, we will need to refer to the theory of viscosity solutions for HJB equations that we summarise in Appendix A.

We now identify the conjectured optimal switching points and construct a candidate for the value function, before proving optimality in Proposition 4.6.

4.2. **Optimal switching points.** With the justification of the previous section, we make the ansatz that there exists an optimal strategy of the form described in either (4.1) or (4.2). We now seek the optimal switching points r_i and s_i.

We will find that we require continuous fit and a condition on the first derivative to fix the points r_i, and we will need to impose smooth fit and a condition on the second derivative to fix the points s_i. It is interesting to note that smooth fit also holds at the points r_i, although we do not enforce it.

Under the conjectured optimal behaviour, the value function is of the form

$$v(x) = \tilde{v}(|x|), \quad x \in D,$$

for some $\tilde{v} : [0, R) \to \mathbb{R}$. To identify the optimal switching points, we will assume that $\tilde{v}$ is differentiable in the interval $(0, R)$ and satisfies the boundary condition $\tilde{v}(R) = g$. Then, for any $r \in (0, R)$, we have

$$\tilde{v}(r) = g - \int_r^R \tilde{v}'(s)\,\mathrm{d}s.$$

When we verify our candidate for the value function in Proposition 4.6, we will show that v is in fact continuously differentiable in D and attains the boundary condition $v = g$ on ∂D.

By definition of the value function, the expected cost associated to any admissible control at some radius $r \in (0, R)$ is greater than the value $\tilde{v}(r)$. Therefore the derivative of such an expected cost at some $r \in (0, R)$ must be less than the derivative of the value function $\tilde{v}'(r)$. We will use this observation to determine the optimal switching points.

Let $\tilde{V} : [0, R] \to \mathbb{R}$ and define a candidate value function $V : D \to \mathbb{R}$ by $V(x) = \tilde{V}(|x|)$ for $x \in D$. The first step in constructing this function V is to find the optimal switching points, as follows.

Suppose that there exists some $i \geq 1$ such that $0 < s_{i-1} < r_i < R$. Then we expect that the optimal control switches from tangential motion to radial motion at the point s_{i-1}. In some interval (s, s_{i-1}), we set $\tilde{V} = w_{i-1}$, where w_{i-1} solves the ODE

$$\mathcal{L}^0 w_{i-1}(r) = -2r\tilde{f}(r),$$

and $\mathcal{L}^0$ is the generator associated to tangential motion that is defined in (4.6). This ODE is equivalent to the first order ODE

$$w'_{i-1}(r) = -2r\tilde{f}(r).$$

In the interval (s_{i-1}, r_i), we set $\tilde{V} = u_i$, where u_i solves the ODE

$$\mathcal{L}^1 u_i(r) = \tilde{f}(r),$$

and $\mathcal{L}^1$ is the generator associated to radial motion that is defined in (4.5). We can write this ODE as

$$u''_i(r) = -2\tilde{f}(r).$$

We fix the boundary conditions

$$u_i(s_{i-1}) = w_{i-1}(s_{i-1}), \quad \text{and} \quad u'_{i+}(s_{i-1}) = w'_{i-1}(s_{i-1}) = -2s_{i-1}\tilde{f}(s_{i-1}),$$

to define u_i uniquely.

Now, in the interval $(r_i, s_i \wedge R)$, we suppose that tangential motion is optimal and set $\tilde{V} = w_i$, where w_i solves the first order ODE

$$w'_i(r) = -2r\tilde{f}(r).$$

We then have the following free boundary problem:

$$\begin{cases} \tilde{V}''(r) = -2\tilde{f}(r), & r \in (s_{i-1}, r_i), \\ \tilde{V}'(r) = -2r\tilde{f}(r), & r \in (r_i, s_i \wedge R), \\ \tilde{V}(r_i+) = \tilde{V}(r_i-), \end{cases} \tag{4.7}$$

where the point r_i is to be found. Note that we require the continuous fit condition at r_i in order to solve the first order ODE in $(r_i, s_i \wedge R)$.

As noted above, we determine the switching point by comparing the derivatives of u_i and w_i. The point r_i should be the first point at which $w'_i(r) = -2r\tilde{f}(r)$ is greater than the first derivative of u_i. Therefore we define r_i by

$$r_i := \inf\left\{ r > s_{i-1} \colon s_{i-1}\tilde{f}(s_{i-1}) + \int_{s_{i-1}}^{r} \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r) \right\}.$$

That is the first point after s_{i-1} at which the running average of the cost function becomes greater than its current value. Note that this point cannot be in a region where $\tilde{f}$ is increasing and so r_i is greater than or equal to the first point of decrease of $\tilde{f}$ after s_{i-1}.

In Figure 5b, we show an example of choosing the switching point r_1 by comparing derivatives. We see in Figure 5a that, for this example, the switching point r_1 is strictly greater than the turning point at which the cost function starts to decrease. Also note that, although we have only imposed continuous fit at the point r_1, we can see in Figure 5b that the derivatives are equal at r_1. For any continuous cost function, this smooth fit property arises in the same way; we will discuss this in detail in Section 4.4.

Let us now suppose that $s_i < R$. We suppose that, in the interval $(s_i, r_{i+1} \wedge R)$, radial motion is once again optimal, and we set $\tilde{V} = u_{i+1}$, where u_{i+1} solves the second order ODE

$$u''_{i+1}(r) = -2\tilde{f}(r).$$

Then we have a second free boundary problem

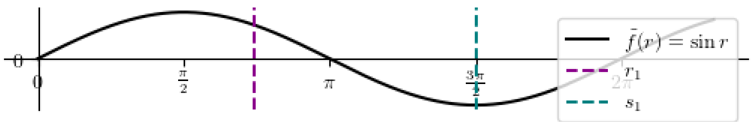
$$\begin{cases} \tilde{V}'(r) = -2r\tilde{f}(r), & r \in (r_i, s_i), \\ \tilde{V}''(r) = -2\tilde{f}(r), & r \in (s_i, r_{i+1} \wedge R), \\ \tilde{V}(s_i+) = \tilde{V}(s_i-), \\ \tilde{V}'_+(s_i) = \tilde{V}'_-(s_i), \end{cases} \tag{4.8}$$

where the point s_i is to be found. Here we require both smooth fit and continuous fit at the point s_i in order to solve the second order ODE in the interval $(s_i, r_{i+1} \wedge R)$.

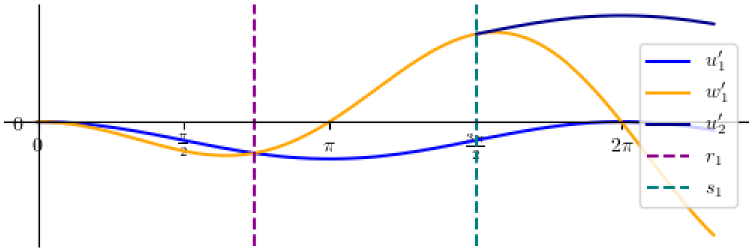
Having imposed the smooth fit condition $\tilde{V}'_+(s_i) = \tilde{V}'_-(s_i)$, the first derivatives of solutions of $w'_i(r) = -2r\tilde{f}(r)$ and $u''_{i+1}(r) = -2\tilde{f}(r)$ are equal for any choice of s_i. In order to fix the point s_i, we require a second order condition. Recall from Assumption 4.1 that we assume that the right derivative of $\tilde{f}$ exists everywhere. This allows us to define s_i to be the first point at which $u''_{i+1}(r) = -2\tilde{f}(r)$ is greater than the one-sided second derivative from the right of the solution of $w'_i(r) = -2r\tilde{f}(r)$. Thus there is an interval of positive length on which the first derivatives are in this same order.

We can calculate the one-sided second derivative from the right of w_i as

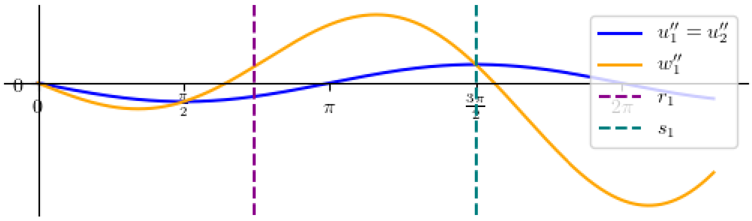
$$w''_{i\,+}(r) = -2\tilde{f}(r) - 2r\tilde{f}'_+(r).$$



(A) Radial part of the cost function $\tilde{f}(r) = \sin r$



(B) First derivatives of the expected costs u_1, w_1, and u_2



(C) Second derivatives of the expected costs u_1, w_1, and u_2

FIGURE 5. The first two switching points r_1, s_1 are shown for the cost function $f(x) = \sin|x|$. The switching point r_1 is the first point at which $w_1'(r) = -2r\tilde{f}(r)$ exceeds u_1', where u_1 solves $u_1''(r) = -2\tilde{f}(r)$, with ${u_1}'_+(0) = 0$, as shown in (B). The switching point s_1 is the first point after r_1 at which $u_2'' = -2\tilde{f}$ exceeds w_1'', as shown in (C). Fixing $u_2'(s_1) = w_1'(s_1)$, we see in (B) that s_1 is chosen such that u_2' remains greater than w_1' over an interval of positive length.

This leads us to define s_i by

$$s_i := \inf\left\{ s > r_i \colon \tilde{f}_+'(s) > 0 \right\}.$$

In this case, the switching point is exactly the turning point at which $\tilde{f}$ starts to increase. For the example in Figure 5, we can see that the switching point s_1 does indeed coincide with this turning point. Figure 5c shows how this switching point is chosen by comparing second derivatives, and Figure 5b shows that the first derivatives at this point have the desired properties.

Note that the sixth condition of Assumption 4.1 implies that there are finitely many switching points s_i and thus finitely many points r_i. Taking the above definitions of r_i and s_j for all values of i, j such that $r_i, s_j < R$, we now solve the ODEs in (4.7) and (4.8) to construct a candidate for the value function.

4.3. **Construction of the value function.** In this section we construct the candidate function V, which we will go on to prove is equal to the value function. We break the construction down into two cases depending on the behaviour of the cost function at the origin, and then into two further sub-cases depending on the behaviour of the cost function at the boundary of the domain.

4.3.1. *Case I: Increasing cost at the origin.* Suppose first that $\tilde{f}$ is increasing on the interval $(0, \eta)$. We summarise the construction of the candidate value function in this case in Algorithm 1.

Fix $s_0 = 0$. Since we expect the optimal control to enforce radial motion in the ball $B_\eta(0)$, we solve the second order ODE

$$u_1''(r) = -2\tilde{f}(r), \quad r \in (0, R).$$

We require two boundary conditions in order to uniquely define the solution u_1. We impose the boundary condition ${u_1}'_+(0) = 0$ for the following reasons.

Algorithm 1 Construction of the value function in Case I

Define $s_0 = 0$.
Solve $u_1''(r) = -2\tilde{f}(r)$, with ${u_1}'_+(0) = 0$, $u_1(0) = \alpha$, for some $\alpha \in \mathbb{R}$.
Define $r_1 := \inf\left\{r > 0\colon \int_0^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\}$.
Set $\tilde{V} = u_1$ on $(0, r_1 \wedge R]$.
if $r_1 < R$ **then**
  for $i \geq 1$ **do**
    Solve $w_i'(r) = -2r\tilde{f}(r)$, with $w_i(r_i) = u_i(r_i)$.
    Define $s_i := \inf\left\{r > r_i\colon \tilde{f}'_+(s) > 0\right\}$.
    Set $\tilde{V} = w_i$ on $(r_i, s_i \wedge R]$.
    if $s_i \geq R$ **then**
      break
    end if
    Solve $u_{i+1}''(r) = -2\tilde{f}(r)$, with $u_{i+1}'(s_i+) = -2s_i\tilde{f}(s_i)$ and $u_{i+1}(s_i) = w_i(s_i)$.
    Define $r_{i+1} := \inf\left\{r > s_i\colon s_i\tilde{f}(s_i) + \int_{s_i}^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\}$.
    Set $\tilde{V} = u_{i+1}$ on $(s_i, r_{i+1} \wedge R]$.
    if $r_{i+1} \geq R$ **then**
      break
    end if
  end for
end if
Fix α such that $\tilde{V}(R) = g$.

First, from the discussion in the previous section, we recall that we will define the first switching point to be

$$r_1 = \inf\left\{r > 0\colon u_1'(r) < -2r\tilde{f}(r)\right\},$$

since we are seeking to maximise the derivative of the candidate value function. Therefore, for $r \in (0, r_1)$, we must have $u_1'(r) \geq -2r\tilde{f}(r)$ and, in particular

$${u_1}'_+(0) = \lim_{r\downarrow 0} u_1'(r) \geq -2\lim_{r\downarrow 0} r\tilde{f}(r) = 0.$$

To get the opposite inequality, fix $\delta \in (0, \eta)$ and $r \in (0, \delta)$ and apply Itô's formula to $u_1(\delta) = u_1\big(\big|X^{\sigma^1}_{\tau_\delta}\big|\big)$ to see that

$$u_1(\delta) - u_1(r) = \frac{1}{2}\mathbb{E}^r\left[\int_0^{\tau_\delta} u_1''\big(\big|X^{\sigma^1}_{\tau_\delta}\big|\big)\,\mathrm{d}s\right] = -\mathbb{E}^r\left[\int_0^{\tau_\delta} \tilde{f}\big(\big|X^{\sigma^1}_{\tau_\delta}\big|\big)\,\mathrm{d}s\right].$$

Then, applying dominated convergence to take the limit as $r \downarrow 0$, and using the fact that $\tilde{f}$ is increasing, we have that

$$\begin{aligned}\lim_{r\downarrow 0}\frac{1}{\delta}\left(u_1(\delta) - u_1(r)\right) &= -\frac{1}{\delta}\mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}\big(\big|X^{\sigma^1}_{\tau_\delta}\big|\big)\,\mathrm{d}s\right]\\ &\leq -\frac{1}{\delta}\tilde{f}(0)\mathbb{E}^0[\tau_\delta] = -\delta\tilde{f}(0).\end{aligned}$$

Hence

$$0 \leq {u_1}'_+(0) \leq -\lim_{\delta\downarrow 0}\delta\tilde{f}(0) = 0.$$

As well as imposing the above condition on the first derivative, we also fix an arbitrary value $u_1(0) = \alpha \in \mathbb{R}$. Having constructed the candidate value function, up to this arbitrary constant, on the whole domain, we will use the external boundary condition $\tilde{V}(R) = g$ to determine the value of α. We now have

$$u_1(r) = \alpha - 2\int_0^r\int_0^s \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s.$$

Define

$$r_1 := \inf\left\{r > 0\colon \int_0^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\},$$

and set $\tilde{V}(r) = u_1(r)$ for $r \in (0, r_1 \wedge R]$.

If $r_1 < R$, we then expect the optimal control to switch to enforcing tangential motion. Therefore we solve the first order ODE

$$w_1'(r) = -2r\tilde{f}(r), \quad r \in (r_1, R).$$

In order to uniquely define the solution w_1, we impose the continuous fit condition $w_1(r_1) = \tilde{V}(r_1)$. Then we have

$$\begin{aligned} w_1(r) &= \tilde{V}(r_1) - 2\int_{r_1}^r s\tilde{f}(s)\,\mathrm{d}s \\ &= \alpha - 2\int_{r_1}^r s\tilde{f}(s)\,\mathrm{d}s - 2\int_0^{r_1}\int_0^s \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s. \end{aligned}$$

Now define

$$s_1 := \inf\left\{r > r_1\colon \tilde{f}_+'(r) > 0\right\},$$

and set $\tilde{V}(r) = w_1(r)$ for $r \in (r_1, s_1 \wedge R]$.

If $s_1 < R$, then we expect the optimal control to switch back to enforcing radial motion, and so we solve the second order ODE

$$u_2''(r) = -2\tilde{f}(r), \quad r \in (s_1, R).$$

At this point, we impose both the continuous fit condition $u_2(s_1) = \tilde{V}(s_1)$ and the smooth fit condition ${u_2}_+'(s_1) = \tilde{V}'(s_1)$ in order to uniquely define u_2. We then find that

$$u_2'(r) = \tilde{V}'(s_1) - 2\int_{s_1}^r \tilde{f}(s)\,\mathrm{d}s,$$

and so

$$\begin{aligned} u_2(r) &= \tilde{V}(s_1) + (r - s_1)\tilde{V}'(s_1) - 2\int_{s_1}^r\int_{s_1}^s \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &= \alpha - 2\int_{s_1}^r\int_{s_1}^s \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s - 2\int_0^{r_1}\int_0^s \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s - 2\int_{r_1}^{s_1} s\tilde{f}(s)\,\mathrm{d}s - 2(r_1 - s_1)r_1\tilde{f}(r_1). \end{aligned}$$

Defining

$$r_2 := \inf\left\{r > s_1\colon s_1\tilde{f}(s_1) + \int_{s_1}^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\},$$

we set $\tilde{V}(r) = u_2(r)$ for $r \in (s_1, r_2 \wedge R]$.

We continue in this way until reaching the boundary of the domain, setting

$$\tilde{V}(r) = \begin{cases} u_i(r), & r \in (s_{i-1}, r_i \wedge R], \\ w_i(r), & r \in (r_i, s_i \wedge R], \end{cases}$$

for each $i \geq 1$.

In order to determine the value of $\tilde{V}(0) = \alpha$, we use the boundary condition on ∂D. Let $K \in \mathbb{N}$ be such that $R \in (s_{K-1}, s_K]$. Suppose first that $R \in (s_{K-1}, r_K]$. Then we expect radial motion to be optimal close to the boundary of the domain, and we have $\tilde{V}(r) = u_{K-1}(r)$ for $r \in (s_K, R]$. Imposing the boundary condition $V(x) = g$ for $x \in \partial D$, we have $u_K(R) = g$. Now suppose that $R \in (r_K, s_K]$, so that we expect tangential motion to be optimal close to the boundary of the domain. Then we have $\tilde{V}(r) = w_K(r)$ for $r \in (r_K, R]$. Now imposing the boundary condition $V(x) = g$ for $x \in \partial D$ gives us $w_K(R) = g$. In either case, the value of α is then specified uniquely.

We state the candidate value function explicitly in Definition 4.5 below.

4.3.2. *Case II: Decreasing cost at the origin.* We now turn to the second case where $\tilde{f}$ is decreasing on the interval $(0, \eta)$. We summarise the construction of the candidate value function in this case in Algorithm 2.

Algorithm 2 Construction of the value function in Case II

Define $r_0 = 0$.
Solve $w_0'(r) = -2r\tilde{f}(r)$, with $w_0(r) = \alpha$, for some $\alpha \in \mathbb{R}$.
Define $s_0 := \inf\left\{r > 0 \colon \tilde{f}_+'(r) > 0\right\}$.
Set $\tilde{V} = w_0$ on $(0, s_0 \wedge R]$.
if $s_0 < R$ **then**
  for $i \geq 0$ **do**
    Solve $u_{i+1}''(r) = -2\tilde{f}(r)$, with $u_{i+1}'(s_i+) = -2s_i\tilde{f}(s_i)$ and $u_{i+1}(s_i) = w_i(s_i)$.
    Define $r_{i+1} := \inf\left\{r > s_i \colon s_i\tilde{f}(s_i) + \int_{s_i}^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\}$.
    Set $\tilde{V} = u_{i+1}$ on $(s_i, r_{i+1} \wedge R]$.
    if $r_{i+1} \geq R$ **then**
      break
    end if
    Solve $w_{i+1}'(r) = -2r\tilde{f}(r)$, with $w_{i+1}(r_{i+1}) = u_{i+1}(r_{i+1})$.
    Define $s_{i+1} := \inf\left\{r > r_{i+1} \colon \tilde{f}_+'(r) > 0\right\}$.
    Set $\tilde{V}(R) = g$ on $(r_{i+1}, s_{i+1} \wedge R]$.
    if $s_{i+1} \geq R$ **then**
      break
    end if
  end for
end if
Fix α such that $\tilde{V}(R) = g$.

We expect the optimal control to enforce tangential motion in $B_\eta(0) \setminus B_\varepsilon(0)$, for any $\varepsilon \in (0, \eta)$. As we will see in Section 5, it will be possible to define a control at the origin whose cost approximates the cost associated to tangential motion. Without further justification here, we fix $r_0 = 0$ and seek the solution w_0 to the first order ODE

$$w_0'(r) = -2r\tilde{f}(r), \quad r \in (0, R).$$

Note that this ODE fixes the first derivative and, in particular, ${w_1}'_+(0) = 0$. In order to uniquely define w_1, we need to impose one boundary condition. As in the previous section, we will fix an arbitrary value $w_1(0) = \alpha \in \mathbb{R}$, and we will determine the value of α from the external boundary condition $\tilde{V}(R) = g$, once we have constructed the candidate value function on the whole domain.

The construction of the value function proceeds in the same way as in Case I, and we omit the details here. We state the candidate value function in both cases in the following Definition 4.5.

Definition 4.5 (Candidate value function)**.** Let the cost functions f and g be as in Assumption 4.1. For $k \in \mathbb{N}$ and $i = 0, \dots, k$, define the constant

$$\mathfrak{F}_i^k := 2\sum_{j=i+1}^{k}\left[(r_j - s_{j-1})s_{j-1}\tilde{f}(s_{j-1}) + \int_{s_{j-1}}^{r_j}\int_{s_{j-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \int_{r_j}^{s_j} s\tilde{f}(s)\,\mathrm{d}s\right].$$

Then we define the candidate value function $V : D \to \mathbb{R}$ as follows.
Case I. If $\tilde{f}$ is increasing in $(0, \eta)$, then set $s_0 = 0$ and let $K \in \mathbb{N}$ be such that $R \in (s_{K-1}, s_K]$. For $x \in D$, define

$$\begin{aligned} V(x) = g &- 2\int_{R\vee r_K}^{s_K} s\tilde{f}(s)\,\mathrm{d}s - 2(r_K - R\wedge r_K)s_{K-1}\tilde{f}(s_{K-1}) - 2\int_{R\wedge r_K}^{r_K}\int_{s_{K-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s \\ &+ 2\sum_{i=1}^{K}\mathbb{1}_{\{(s_{i-1},s_i]\}}(|x|)\left[(r_i - |x|\wedge r_i)s_{i-1}\tilde{f}(s_{1-1}) + \int_{|x|\wedge r_i}^{r_i}\int_{s_{i-1}}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \int_{|x|\vee r_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s + \mathfrak{F}_i^K\right]. \end{aligned}$$

Case II. If $\tilde{f}$ is decreasing in $(0,\eta)$, then set $r_0 = 0$ and let $L \in \mathbb{N}$ be such that $R \in (r_L, r_{L+1}]$. For $x \in D$, define

$$V(x) = g - 2\int_{R\wedge s_L}^{s_L} s\tilde{f}(s)\,\mathrm{d}s + 2(R\vee s_L - s_L)s_L\tilde{f}(s_L) + 2\int_{s_L}^{R\vee s_L}\int_{s_L}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s$$
$$+ 2\sum_{i=0}^{L}\mathbb{1}_{\{(r_i,r_{i+1}]\}}(|x|)\left[\int_{|x|\wedge s_i}^{s_i} s\tilde{f}(s)\,\mathrm{d}s - (|x|\vee s_i - s_i)s_i\tilde{f}(s_i) - \int_{s_i}^{|x|\vee s_i}\int_{s_i}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + \mathfrak{F}_i^L\right].$$

Before turning to the rigorous proof of optimality in Section 4.5, we make a digression to discuss the smooth fit property that the candidate value function exhibits.

4.4. **The principle of smooth fit.** In the preceding construction, the smooth fit condition is required to fix the switching points s_i. It is notable, however, that we do not need to impose smooth fit to uniquely identify the points r_i, and it is thus surprising to us that the smooth fit condition is nevertheless satisfied at these switching points.

A heuristic argument for the smooth fit condition in diffusion problems typically comes from two competing criteria. The heuristic we look to exploit is that we expect $\tilde{v}(|X_t^\nu|) + \int_0^t \tilde{f}(|X_s^\nu|)\,\mathrm{d}s$ to be a submartingale for all admissible strategies, and a martingale for the optimal strategy. Now suppose that the function $\tilde{v}$ displays a discontinuity in its first derivative at the boundary of two types of behaviour, at $|X| = r$ say. Then, applying the Itô-Tanaka formula, we expect a local time term of the form $(\tilde{v}'_+(r) - \tilde{v}'_-(r))\,\mathrm{d}L_t^r$ to appear in $\mathrm{d}\tilde{v}(|X_t|)$. Since there is no cancelling term in the time integral component, it follows immediately that if $\tilde{v}'_+(r) < \tilde{v}'_-(r)$, then the process will be a supermartingale for any strategy which has positive local time at r. Consequently, we expect $\tilde{v}'_+(r) \geq \tilde{v}'_-(r)$ for all $r \in (0,R)$. Moreover, *if* the optimal strategy accrues local time at r, then a similar argument forces $\tilde{v}'_+(r) = \tilde{v}'_+(r)$, and we deduce the smooth fit condition.

However, this justification breaks down at the switching points r_i described above. Under the conjectured optimal strategy, there is no local time accrued at such switching points, and therefore the heuristic justification for smooth fit fails. However, from our construction of the value function, we see that the smooth fit condition still holds!

In general, we have no heuristic justification for such a condition. We note that, in optimal stopping problems for Lévy processes, or more generally jump diffusions, it is common to observe continuous fit conditions where there is no diffusive boundary behaviour [6]. This is comparable to the behaviour that we observe under the optimal strategy at points r_i, which arises from the notable fact that our control process can produce both diffusive and non-diffusive behaviour at boundary points. We are unaware of similar behaviour occurring in a diffusive setting, and we leave further study of this behaviour for future work.

4.5. **Proof of optimality.** We now turn to the proof that the candidate function that we have constructed is indeed the value function.

Proposition 4.6. *Under Assumption 4.1, the value function v is continuously differentiable and takes the form $v = V$, where V is defined in Definition 4.5.*

Moreover, there exists an optimal control $\sigma^\star \in \mathcal{U}$ in the following cases. If $\tilde{f}$ is increasing in $(0,\eta)$, then the control $\sigma^\star$ defined in (4.1) *is optimal. If $\tilde{f}$ is decreasing in $(0,\eta)$ and the initial condition is $x \in D\setminus\{0\}$, then the control $\sigma^\star$ defined in* (4.2) *is optimal. In each of these cases, the Markov value function v^M also coincides with the value function v.*

In order to prove this result, we refer to the theory of viscosity solutions for HJB equations that we summarise in Appendix A. The main result that we require is the following theorem, which we restate here for reference.

Theorem A.5. *Suppose that Assumption A.1 holds, and suppose further that the domain D is uniformly convex, the running cost f is continuous in D, and the boundary cost g is uniformly continuous on ∂D.*

Then the value function $v : D \to \mathbb{R}$ defined in Section 2 extends continuously to $\overline{D}$ and is the unique viscosity solution of the HJB equation

$$-\frac{1}{2}\inf_{\sigma\in U} \operatorname{Tr}\left(D^2 v \sigma\sigma^\top\right) - f = 0$$

in D, with boundary condition

$$v = g \quad \text{on} \quad \partial D.$$

In this section, we will prove that the candidate function V is a viscosity solution of the HJB equation

$$-\frac{1}{2}\inf_{\sigma\in U} \operatorname{Tr}\left(D^2 V(x)\sigma\sigma^\top\right) = f(x), \quad x \in D, \tag{4.9}$$

with boundary condition $V = g$ on ∂D. We then appeal to Theorem A.5, as stated above, to see that the value function v is a viscosity solution of the same boundary value problem and, moreover, such a solution is unique. From this, we conclude that the function V is equal to the value function v.

We first show that V is a classical solution of (4.9) in the regions where we expect radial motion to be optimal.

Lemma 4.7. *For each $i \geq 1$, define $u_i : (s_{i-1}, r_i \wedge R] \to \mathbb{R}$ by*

$$u_i(r) = 2\int_r^{r_i}\int_{s_{i-1}}^{s} \tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + 2(r_i - r)s_{i-1}\tilde{f}(s_{i-1}) + C_i^u, \quad r \in (s_{i-1}, r_i \wedge R],$$

for an arbitrary constant C_i^u, and define the set

$$D_i := \{x \in D \colon\ |x| \in (s_{i-1}, r_i \wedge R)\}.$$

Then $U_i : D_i \to \mathbb{R}$, defined by $U_i(x) = u_i(|x|)$, is a classical solution of the PDE (4.9) *in the region D_i.*

Proof. Fix $i \geq 1$ and let $x \in D_i$. Observe that, by definition of r_i,

$$u_i'(|x|) \geq -2\,|x|\,\tilde{f}(|x|). \tag{4.10}$$

We have that U_i is twice continuously differentiable at x and

$$D^2 U_i(x) = |x|^{-3}\left[|x|\,u_i''(|x|) - u_i'(|x|)\right] x x^\top + |x|^{-1} u_i'(|x|) I.$$

Substituting in $u_i''(|x|) = -2\tilde{f}(|x|)$ and rearranging gives

$$\begin{aligned} D^2 U_i(x) &= -|x|^{-3}\left[2\,|x|\,\tilde{f}(|x|) + u_i'(|x|)\right] x x^\top + |x|^{-1} u_i'(|x|) I \\ &= -2\tilde{f}(|x|) I + |x|^{-3}\left[2\,|x|\,\tilde{f}(|x|) + u_i'(|x|)\right]\left[|x|^2 I - x x^\top\right]. \end{aligned}$$

Hence, for any $\sigma \in U$,

$$\operatorname{Tr}\left(D^2 U_i(x)\sigma\sigma^\top\right) = -2\tilde{f}(|x|)\operatorname{Tr}(\sigma\sigma^\top) + |x|^{-3}\left[2\,|x|\,\tilde{f}(|x|) + u_i'(|x|)\right]\operatorname{Tr}\left(\left[|x|^2 I - x x^\top\right]\sigma\sigma^\top\right).$$

Noting that $|x|^2 I - x x^\top$ is positive semi-definite, and using (4.10), we have

$$\operatorname{Tr}\left(D^2 U_i(x)\sigma\sigma^\top\right) \geq -2\tilde{f}(|x|)\operatorname{Tr}(\sigma\sigma^\top) = -2f(x),$$

for any $\sigma \in U$.

Taking $\sigma = \sigma^1(x)$, where $\sigma^1 : D \to \mathbb{R}$ is the function defined in Definition 3.7, we see that

$$\operatorname{Tr}\left(\left[|x|^2 I - x x^\top\right]\sigma^1(x)\sigma^1(x)^\top\right) = 0,$$

and so

$$\operatorname{Tr}\left(D^2 U_i(x)\sigma^1(x)\sigma^1(x)^\top\right) = -2f(x).$$

Hence U_i is a classical solution of the PDE (4.9) in the the region D_i. □

We next show that V is a viscosity solution of (4.9) in the regions where we expect tangential motion to be optimal.

Lemma 4.8. *For each $i \geq 0$, define $w_i : (r_i, s_i \wedge R] \to \mathbb{R}$ by*

$$w_i(r) = 2\int_r^{s_i} s\tilde{f}(s)\,\mathrm{d}s + C_i^w, \quad r \in (r_i, s_i \wedge R],$$

for an arbitrary constant C_i^w, and define the set

$$\overline{D}_i := \{x \in D\colon\ |x| \in (r_i, s_i \wedge R)\}.$$

Then $W_i : \overline{D}_i \to \mathbb{R}$, defined by $W_i(x) = w_i(|x|)$, is a viscosity solution of the PDE (4.9) *in the region $\overline{D}_i$.*

Note that w_i is twice continuously differentiable if and only if $\tilde{f}$ is continuously differentiable. We first suppose that this is the case and prove the following lemma.

Lemma 4.9. *Fix $i \geq 0$ and suppose that $\tilde{f}$ is continuously differentiable in the interval $(r_i, s_i \wedge R)$. Then W_i defined in Lemma 4.8 is a classical solution of the PDE* (4.9) *in the region $\overline{D}_i$.*

Proof. Let $x \in \overline{D}_i$ and observe that, by definition of s_i,

$$w''_{i+1}(|x|) \geq -2\tilde{f}(|x|). \tag{4.11}$$

Since $\tilde{f}$ is assumed to be continuously differentiable, we have that w_i and W_i are both twice continuously differentiable, and

$$D^2W_i(x) = |x|^{-3}\left[|x|\,w''_i(|x|) - w'_i(|x|)\right]xx^\top + |x|^{-1}\,w'_i(|x|)I.$$

Substituting in $w'_i(|x|) = -2\,|x|\,\tilde{f}(|x|)$, we have

$$D^2W_i(x) = |x|^{-2}\left[w''_i(|x|) + 2\tilde{f}(|x|)\right]xx^\top - 2\tilde{f}(|x|)I.$$

Hence, for any $\sigma \in U$,

$$\begin{aligned}\operatorname{Tr}\left(D^2W_i(x)\sigma\sigma^\top\right) &= |x|^{-2}\left[w''_i(|x|) + 2\tilde{f}(|x|)\right]\operatorname{Tr}(xx^\top\sigma\sigma^\top) - 2\tilde{f}(|x|)\operatorname{Tr}(\sigma\sigma^\top)\\ &\geq -2\tilde{f}(|x|)\operatorname{Tr}(\sigma\sigma^\top) = -2f(x),\end{aligned}$$

using the inequality (4.11).

Taking $\sigma = \sigma^0(x)$, where $\sigma^0 : D \to \mathbb{R}$ is the function defined in Definition 3.3, we see that

$$\operatorname{Tr}\left(xx^\top\sigma^0(x)\sigma^0(x)^\top\right) = 0,$$

and so

$$\operatorname{Tr}\left(D^2W_i(x)\sigma^0(x)\sigma^0(x)^\top\right) = -2f(x).$$

Hence W_i is a classical solution of the PDE (4.9) in the region $\overline{D}_i$. □

We can now prove Lemma 4.8, by using smooth approximations to the continuous function $\tilde{f}$ and applying a standard stability result for viscosity solutions, which can be found, for example, in Lemma 6.2 of [3, Chapter II].

Proof of Lemma 4.8. Fix $i \geq 1$. Since $\tilde{f}$ is continuous on $[r_i, s_i \wedge R]$, we can approximate $\tilde{f}$ uniformly by polynomials $(\tilde{f}^k)_{k\in\mathbb{N}}$ (see e.g. Theorem 7.26 of [13]). Let $k \in \mathbb{N}$ and define $W_i^k : \overline{D}_i \to \mathbb{R}$ by

$$W_i^k(x) := -2\int_{r_i}^{|x|} \tilde{f}^k(s)s\,\mathrm{d}s + C_i^w.$$

Define $f^k : \overline{D}_i \to \mathbb{R}$ by $f^k(x) = \tilde{f}^k(|x|)$, and define $F^k : \overline{D}_i \times \mathbb{R}^{d,d} \to \mathbb{R}$ by

$$F^k(x, X) = -\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}(X\sigma\sigma^\top) - f^k(x).$$

Then, since $\tilde{f}^k$ is continuously differentiable, we can apply Lemma 4.9 to see that W_i^k is a classical solution, and therefore a viscosity solution, of

$$F^k(x, D^2W_i^k(x)) = 0 \quad \text{for} \quad x \in \overline{D}_i.$$

We now show that F^k converges uniformly to $F : \overline{D}_i \times \mathbb{R}^{d,d} \to \mathbb{R}$, defined by

$$F(x, X) = -\frac{1}{2} \inf_{\sigma \in U} \operatorname{Tr}(X\sigma\sigma^\top) - f(x),$$

and that W_i^k converges uniformly to W_i.

Let $\varepsilon > 0$. Then, by uniform convergence of $(\tilde{f}^k)_{k\in\mathbb{N}}$, there exists $N \in \mathbb{N}$ such that

$$\left|\tilde{f}(r) - \tilde{f}^k(r)\right| < \varepsilon, \quad \text{for all} \quad r \in [r_0, R] \quad \text{and} \quad k \geq N.$$

Let $k \geq N$, $x \in \overline{D}_i$ and $X \in \mathbb{R}^{d,d}$. Then $|x| \in [r_i, s_i \wedge R]$, and so

$$\left|F(x, X) - F^k(x, X)\right| = \left|f(x) - f^k(x)\right| = \left|\tilde{f}(|x|) - \tilde{f}^k(|x|)\right| < \varepsilon.$$

Therefore $F^k \to F$ uniformly on $\overline{D}_i \times \mathbb{R}^{d,d}$.

Now choose $M \in \mathbb{N}$ such that

$$\left|\tilde{f}(r) - \tilde{f}^k(r)\right| < \frac{\varepsilon}{2s_i(s_i - r_i)}, \quad \text{for all} \quad r \in [r_i, s_i] \quad \text{and} \quad k \geq M.$$

Let $k \geq M$ and $x \in \overline{D}_i$. Then $|x| \in [r_i, s_i \wedge R]$, and so

$$\begin{aligned}\left|W_i(x) - W_i^k(x)\right| = 2\left|\int_{r_i}^{|x|} \left(\tilde{f}(s) - \tilde{f}^k(s)\right) s \,\mathrm{d}s\right| &\leq 2\int_{r_i}^{s_i} \left|\tilde{f}(s) - \tilde{f}^k(s)\right| |s| \,\mathrm{d}s \\ &\leq 2(s_i - r_i)\frac{\varepsilon}{2s_i(s_i - r_i)} s_i = \varepsilon.\end{aligned}$$

Hence $W_i^k \to W_i$ uniformly on $\overline{D}_i$.

We can now apply the stability result given in Lemma 6.2 of [3, Chapter II], to conclude that W_i is a viscosity solution of

$$F(x, D^2 W_i(x)) = 0 \quad \text{for} \quad x \in \overline{D}_i;$$

i.e. W_i is a viscosity solution of the PDE (4.9) in the region $\overline{D}_i$. □

We now combine the above lemmas to prove that V is the value function.

Proof of Proposition 4.6. We divide the domain D into disjoint regions and prove first that V is a viscosity solution of (4.9) in the interior of each region.

Step 1: Fix $i \geq 1$ such that $s_{i-1} \leq R$, if such a point exists. In the region $D_i = \{x \in D\colon\ |x| \in (s_{i-1}, r_i \wedge R)\}$, we have $V = U_i$, for a particular choice of constant C_i^u. So by Lemma 4.7, V is a viscosity solution of (4.9) in this region.

Now fix $i \geq 0$ such that $r_i \leq R$, if such a point exists. In the region $\overline{D}_i = \{x \in D\colon\ |x| \in (r_i, s_i \wedge R)\}$, we have $V = W_i$ for a particular choice of constant C_i^w, and so V is a viscosity solution of (4.9) in this region, by Lemma 4.8.

Step 2: We next prove that V is a viscosity solution of (4.9) on each of the internal boundaries between the regions.

Let $i \geq 0$ be such that $r_i < R$, if such a point exists. Consider $x_i \in D$ such that $|x_i| = r_i$. Note that

$$\begin{aligned}\lim_{|x|\to r_i-} D^2 V(x) &= \lim_{|x|\to r_i-} D^2 U_i(x) \\ &= -\lim_{|x|\to r_i-} \left[2\tilde{f}(|x|)I + |x|^{-3}\left(2|x|\tilde{f}(|x|) + u_i'(|x|)\right)\left[|x|^2 I - xx^\top\right]\right] = -2\tilde{f}(r_i)I,\end{aligned} \tag{4.12}$$

since $2r_i\tilde{f}(r_i) + u_i{}'_-(r_i) = 0$, by definition of r_i and continuity of $\tilde{f}$.

To show that V is a viscosity subsolution at x_i, let $x_i \in \arg\min(\phi - V)$, for some $\phi \in C^\infty(D)$. Since $V \in C^1(D)$, it must be the case that $D\phi(x_i) = DV(x_i)$, and that the Hessian of ϕ satisfies

$$D^2\phi(x_i) \geq \lim_{|x|\to r_i-} D^2 V(x) = -2\tilde{f}(r_i)I,$$

as calculated in (4.12). Hence, for any $\sigma \in U$,

$$\operatorname{Tr}\left(D^2\phi(x_i)\sigma\sigma^\top\right) \geq -2\tilde{f}(r_i)\operatorname{Tr}(\sigma\sigma^\top) = -2f(x_i),$$

and so

$$-\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}\left(D^2\phi(x_i)\sigma\sigma^\top\right) \leq f(x_i),$$

as required.

To show the supersolution property, let $x_i \in \arg\max(\psi - V)$, for some $\psi \in C^\infty(D)$. Then by a similar argument to the one above, we have

$$D^2\psi(x_i) \leq -2\tilde{f}(r_i)I,$$

and so

$$\operatorname{Tr}\left(D^2\psi(x_i)\sigma\sigma^\top\right) \leq -2f(x_i),$$

for any $\sigma \in U$, which implies that

$$-\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}\left(D^2\psi(x_i)\sigma\sigma^\top\right) \geq f(x_i).$$

Now let $i \geq 0$ be such that $s_i < R$, if such a point exists, and consider $x_i \in D$ such that $|x_i| = s_i$. Here, note that

$$\begin{aligned}\lim_{|x|\to s_i+} D^2V(x) &= \lim_{|x|\to s_i+} D^2U_{i+1}(x)\\ &= -\lim_{|x|\to s_i+}\left[2\tilde{f}(|x|)I + |x|^{-3}\left(2\,|x|\,\tilde{f}(|x|) + u_{i+1}'(|x|)\right)\left[|x|^2\,I - xx^\top\right]\right] = -2\tilde{f}(s_i)I,\end{aligned} \tag{4.13}$$

using the fact that $2s_i\tilde{f}(s_i) + {u_{i+1}}'_+(s_i) = 0$, by definition of s_i and the smooth fit property.

To show that V is a viscosity solution at points of radius s_i, we follow the same reasoning as we did for points of radius r_i. For $x_i \in \arg\min(\phi - V)$ and $\phi \in C^\infty(D)$, we have that

$$D^2\phi(x_i) \geq \lim_{|x|\to s_i+} D^2V(x) = -2\tilde{f}(s_i)I,$$

using (4.13). So, for any $\sigma \in U$,

$$\operatorname{Tr}\left(D^2\phi(x_i)\sigma\sigma^\top\right) \geq -2f(x_i),$$

which implies that the subsolution property holds.

Similarly, for $x_i \in \arg\max(\psi - V)$ and $\psi \in C^\infty(D)$, we have

$$D^2\psi(x_i) \leq -2\tilde{f}(s_i)I,$$

and so, for any $\sigma \in U$,

$$\operatorname{Tr}\left(D^2\psi(x_i)\sigma\sigma^\top\right) \leq -2f(x_i),$$

which implies the supersolution property.

Step 3: We have shown that V is a viscosity solution of (4.9) in $D \setminus \{0\}$. We now consider the behaviour at the origin. Recall from Assumption 4.1 that we have assumed that $\tilde{f}$ is monotone on some interval $(0, \eta)$.

Case I:. Suppose that $\tilde{f}$ is strictly increasing on $(0, \eta)$. Then $V = U_1$ in some neighbourhood of the origin. We see that $r_1 \geq \eta$, and so $V = U_1$ in $B_\eta(0)(0)$. Let $x \in B_\eta(0)$ and consider

$$D^2V(x) = -2\tilde{f}(|x|)I + |x|^{-3}\left(2\,|x|\,\tilde{f}(|x|) + u_1'(|x|)\right)\left[|x|^2\,I - xx^\top\right].$$

Since $|x| < r_1$, we have

$$2\,|x|\,\tilde{f}(|x|) + u_1'(|x|) > 0.$$

Substituting in the value of u_1' and considering a first order Taylor expansion around 0, we find that there exists $C > 0$ such that

$$\begin{aligned} 2\,|x|\,\tilde{f}(|x|) + u_1'(|x|) = -2\int_0^{|x|} \tilde{f}(s)\,\mathrm{d}s &= 2\,|x|\left(\tilde{f}(|x|) - \tilde{f}(0)\right) + o(|x|) \\ &\le 2\,|x|\left(\tilde{f}(|x|) - \tilde{f}(0)\right) + C\,|x|^2 . \end{aligned}$$

Hence, for $j, k \in \{1, \ldots, d\}$,

$$\begin{aligned} 0 \le |x|^{-3}\left(2\,|x|\,\tilde{f}(|x|) + u_i'(|x|)\right)\left|\left[|x|^2\,I - xx^\top\right]_{jk}\right| &\le |x|^{-1}\left(2\,|x|\,\tilde{f}(|x|) + u_i'(|x|)\right) \\ &\le 2\left(\tilde{f}(|x|) - \tilde{f}(0)\right) C\,|x| . \end{aligned}$$

Taking the limit as $|x| \to 0+$, by continuity of $\tilde{f}$, we have

$$\lim_{x\to 0} D^2V(x) = -2\tilde{f}(0)I.$$

It is then easy to see that V is a viscosity solution of (4.9) at the origin.
Case II:. On the other hand, if $\tilde{f}$ is decreasing in $(0, \eta)$, we have that $V = W_1$ in $B_\eta(0)$. Recall from Assumption 4.1 that $\tilde{f}$ is continuously differentiable on some interval $(0, \delta)$, and consider $x \in D$ such that $|x| < \delta \wedge \eta$. Then

$$\begin{aligned} D^2V(x) &= |x|^{-2}\left[w_1''(|x|) + 2\tilde{f}(|x|)\right] xx^\top - 2\tilde{f}(|x|)I \\ &= 2\,|x|^{-2}\left[-\,|x|\,\tilde{f}'(|x|) - \tilde{f}(|x|) + \tilde{f}(|x|)\right] xx^\top - 2\tilde{f}(|x|)I \\ &= -2\,|x|^{-1}\,\tilde{f}'(|x|)xx^\top - 2\tilde{f}(|x|)I. \end{aligned}$$

Since $\tilde{f}'(|x|) \le 0$, we get the following bound. For $j, k \in \{1, \ldots, d\}$,

$$0 \le -2\,|x|^{-1}\,\tilde{f}'(|x|)\,|x_jx_k| \le -2\,|x|\,\tilde{f}'(|x|) \to 0, \quad \text{as} \quad |x| \to 0+,$$

where the limit is given by the fifth statement of Assumption 4.1.

Therefore $\lim_{x\to 0} D^2V(x) = -2\tilde{f}(0)I$, as in Case I, and so V is a viscosity solution of (4.9) at the origin.

Step 4: By construction of the function V, the boundary condition $V = g$ on ∂D is satisfied. We conclude, by Theorem A.5, that the function V is equal to the value function v. Also, by the construction of V, we have that the value function v is continuously differentiable in D.

Step 5: Finally, we turn to the proof that the control $\sigma^\star$ is optimal and that there also exists a Markov control $\tilde{\sigma}^\star$ that attains the value function. For the proof of optimality of $\sigma^\star$, it is sufficient to show that

$$t \mapsto V(X_t^{\sigma^\star}) + \int_0^t f(X_s^{\sigma^\star})\,\mathrm{d}s$$

is a martingale. We will work with the squared radius of the process $X^{\sigma^\star}$, writing $Z_t^{\sigma^\star} = \left|X_t^{\sigma^\star}\right|^2$, for $t \ge 0$. We also let $\overline{V} : [0, R^2) \to \mathbb{R}$ be such that $V(x) = \overline{V}(|x|^2)$ for all $x \in D$.

Suppose that $\tilde{f}$ is increasing on the interval $(0, \eta)$. Then $\sigma^\star$ is given by (4.1). Letting W be the first component of the Brownian motion B, Lemma 3.2 tells us that $Z^{\sigma^\star}$ satisfies the SDE

$$\mathrm{d}Z_t^{\sigma^\star} = \mathrm{d}t + 2\left(\sum_i \mathbb{1}_{\left\{Z_t^{\sigma^\star} \in (s_i^2, r_{i+1}^2 \wedge R^2)\right\}} + \mathbb{1}_{\left\{Z_t^{\sigma^\star} \in [0, r_1^2 \wedge R^2)\right\}}\right)\sqrt{Z_t^{\sigma^\star}}\,\mathrm{d}W_t,$$

where the index i runs from 1 to the first i such that $r_{i+1} \ge R$.

In each interval $[r_i^2, s_i^2]$, there is a constant C such that

$$\overline{V}(z) = 2\int_{\sqrt{z}}^{s_i} s\tilde{f}(s)\,\mathrm{d}s + C.$$

Therefore, since $\mathrm{d}Z_t^{\sigma^\star} = \mathrm{d}t$ when $Z_t^{\sigma^\star} \in [r_i^2, s_i^2]$, we can make a change of variables to find that

$$\mathbb{1}_{\{Z_t^{\sigma^\star}\in[r_i^2,s_i^2]\}}\,\mathrm{d}\overline{V}(Z_t^{\sigma^\star}) = -\mathbb{1}_{\{Z_t^{\sigma^\star}\in[r_i^2,s_i^2]\}}\tilde{f}\big(\sqrt{Z_t^{\sigma^\star}}\big)\,\mathrm{d}t. \tag{4.14}$$

Now, in each interval (s_i^2, r_{i+1}^2), there is a constant C such that

$$\overline{V}(z) = 2\int_{\sqrt{z}}^{r_{i+1}}\int_{s_i}^{s}\tilde{f}(t)\,\mathrm{d}t\,\mathrm{d}s + 2(r_{i+1}-\sqrt{z})s_i\tilde{f}(s_i) + C.$$

We see that V is twice continuously differentiable in such an interval, and so we can apply Itô's formula to $\overline{V}(Z^{\sigma^\star})$. We calculate the derivatives

$$\overline{V}'(z) = -z^{-\frac{1}{2}}\int_{s_i}^{\sqrt{z}}\tilde{f}(s)\,\mathrm{d}s - z^{-\frac{1}{2}}s_i\tilde{f}(s_i),$$

and

$$\overline{V}''(z) = \frac{1}{2}z^{-\frac{3}{2}}\int_{s_i}^{\sqrt{z}}\tilde{f}(s)\,\mathrm{d}s - \frac{1}{2}Z^{-1}\tilde{f}(\sqrt{z}) + \frac{1}{2}z^{-\frac{3}{2}}s_i\tilde{f}(s_i).$$

Then, by Itô's formula, we find that

$$\mathbb{1}_{\{Z_t^{\sigma^\star}\in(s_i^2,r_{i+1}^2)\}}\,\mathrm{d}\overline{V}(Z_t^{\sigma^\star}) = -\mathbb{1}_{\{Z_t^{\sigma^\star}\in(s_i^2,r_{i+1}^2)\}}\tilde{f}\big(\sqrt{Z_t^{\sigma^\star}}\big)\,\mathrm{d}t + 2\mathbb{1}_{\{Z_t^{\sigma^\star}\in(s_i^2,r_{i+1}^2)\}}\overline{V}'(Z_t^{\sigma^\star})\sqrt{Z_t^{\sigma^\star}}\,\mathrm{d}W_t.$$

We have a similar expression for the interval $[0, r_1^2)$, and so combining this with (4.14), we have

$$V(X_t^{\sigma^\star}) - V(X_0^{\sigma^\star}) = -\int_0^t f(X_s^{\sigma^\star})\,\mathrm{d}s + 2\int_0^t\left(\sum_i \mathbb{1}_{\{Z_s^{\sigma^\star}\in(s_i^2,r_{i+1}^2)\}} + \mathbb{1}_{\{Z_s^{\sigma^\star}\in[0,r_1^2)\}}\right)\sqrt{Z_s^{\sigma^\star}}\,\mathrm{d}W_s,$$

for any $t \geq 0$. This shows that the required martingale property holds, and so $\sigma^\star$ is an optimal control. Setting $X_0^{\sigma^\star} = x$, for any $x \in D$, we see by Remark 4.3 that there exists a Markov control $\tilde{\sigma}^\star \in \mathcal{U}_x^M$, such that

$$\mathbb{E}^x\left[\int_0^\tau f(X_s^{\tilde{\sigma}^\star})\,\mathrm{d}s\right] + g = \mathbb{E}^x\left[\int_0^\tau f(X_s^{\sigma^\star})\,\mathrm{d}s\right] + g = v(x).$$

Since $v^M(x) \geq v(x)$ by Proposition 2.9, we conclude that $v^M(x) = v(x)$.

Now suppose that $\tilde{f}$ is decreasing on the interval $(0, \eta)$, and let $X_0^{\sigma^\star} = x$, for some $x \in D \setminus \{0\}$. In this case $\sigma^\star$ is given by (4.2), and $Z^{\sigma^\star}$ satisfies

$$\mathrm{d}Z_t^{\sigma^\star} = \mathrm{d}t + 2\sum_i \mathbb{1}_{\{Z_t^{\sigma^\star}\in(s_i^2,r_{i+1}^2\wedge R^2)\}}\sqrt{Z_t^{\sigma^\star}}\,\mathrm{d}W_t,$$

where now the index i runs from 0 to the first i such that $r_{i+1} \geq R$. We see that $Z^{\sigma^\star}$ never hits the origin.

We can make the same calculations as above to find that, for any $t \geq 0$,

$$V(X_t^{\sigma^\star}) - V(X_0^{\sigma^\star}) = -\int_0^t f(X_s^{\sigma^\star})\,\mathrm{d}s + 2\int_0^t\sum_i \mathbb{1}_{\{Z_s^{\sigma^\star}\in(s_i^2,r_{i+1}^2)\}}\sqrt{Z_s^{\sigma^\star}}\,\mathrm{d}W_s,$$

and so the required martingale property holds once again. We conclude that $\sigma^\star$ is optimal.

Once again, by Remark 4.3, there exists a control $\tilde{\sigma}^\star \in \mathcal{U}_x^M$, such that

$$\mathbb{E}^x\left[\int_0^\tau f(X_s^{\tilde{\sigma}^\star})\,\mathrm{d}s\right] = \mathbb{E}^x\left[\int_0^\tau f(X_s^{\sigma^\star})\,\mathrm{d}s\right],$$

and so $v^M(x) = v(x)$, as required. □

We required the smoothness conditions on the running cost f in Assumption 4.1 in order to show that the candidate value function is a viscosity solution at the origin. In Section 5, we will relax these assumptions and extend the above result to include cost functions that have an infinite discontinuity at the origin. In this case, we cannot define a viscosity solution of the HJB equation (4.9) at the origin, and so Theorem A.5 will no longer be applicable.

5. Infinite cost at the origin

We now extend Proposition 4.6 by considering the case where the cost function is continuous on the whole domain, except at the origin where it may become infinite. We will show that the value function takes the same form as we saw in Proposition 4.6. We will also find growth conditions on the cost function under which the value function becomes infinite. We note that, in allowing the cost function to become infinite at the origin, we must take care to check that we still have equality between the strong value function v^S and the weak value function v^W, as we showed in Proposition 2.7 for the case of continuous cost functions. In a particular growth regime, we do not prove that $v^S(0) = v^W(0)$ in dimension $d = 2$. This result will be treated in a forthcoming paper, using the theory of Brownian filtrations.

We relax the regularity conditions on the cost function f from Assumption 4.1, as follows.

Assumption 5.1. We assume that

(1) The domain is $D = B_R(0) \subset \mathbb{R}^d$, for some $R > 0$ and $d \geq 2$;
(2) The cost function f is radially symmetric; i.e. $f(x) = \tilde{f}(|x|)$, for some function $\tilde{f} : [0, R) \to \mathbb{R}$;
(3) The boundary cost g is constant;
(4) The cost function f is continuous on $D \setminus \{0\}$;
(5) There exists $\eta > 0$ such that the cost function $\tilde{f}$ is monotone on the interval $(0, \eta)$;
(6) The one-sided derivative $\tilde{f}'_+(r)$ exists for all $r > 0$ and changes sign only finitely many times.

Note that we retain the fifth statement in this assumption to ensure that the cost function does not oscillate as it approaches the origin, and we retain the sixth statement so that there are finitely many switching points and these are well-defined.

Having relaxed the conditions on the cost function f, we can no longer use the theory of viscosity solutions. To prove the following results, we once again treat the cases of increasing and decreasing costs separately, and we distinguish between regimes of slow and fast growth at the origin. The different growth regimes will be determined by the convergence of the integrals

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s.$$

In each of the proofs in this section, we make the simplifying assumption that the boundary cost is $g = 0$. However, the results still hold for any constant boundary cost g.

5.1. **Cost functions increasing at the origin.**

We first consider cost functions that are increasing in some neighbourhood around the origin. In this case, we will find that radial motion, as defined in Definition 3.7, is optimal close to the origin.

Proposition 5.2. *Suppose that Assumption 5.1 holds and there exists $\eta > 0$ such that $\tilde{f}$ is negative and increasing on the interval $(0, \eta)$. Then the strong, weak and Markov value functions defined in Section 2 are equal, and we can write $v = v^S = v^W = v^M$. Moreover, for the candidate value V defined in Definition 4.5,*

$$v = \begin{cases} V \in (-\infty, \infty), & \text{if} \quad \int_0^r \tilde{f}(s)\,\mathrm{d}s > -\infty, \quad \text{for any} \quad r > 0, \\ -\infty, & \text{if} \quad \int_0^r \tilde{f}(s)\,\mathrm{d}s = -\infty, \quad \text{for any} \quad r > 0. \end{cases}$$

Remark 5.3. Note that, since $\tilde{f}$ is increasing on $(0, \eta)$, the function V is defined in Case I of Definition 4.5, with $s_0 = 0$ and $r_1 = \inf\left\{r > 0\colon \int_0^r \tilde{f}(s)\,\mathrm{d}s > r\tilde{f}(r)\right\}$. When $\int_0^r \tilde{f}(s)\,\mathrm{d}s > -\infty$ for any $r > 0$, the switching point r_1 is well-defined.

Proof of Proposition 5.2. First suppose that, for any $r > 0$,

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s > -\infty.$$

For $N \in \mathbb{N}$, define an approximating sequence of functions $\tilde{f}_N : [0, R) \to \mathbb{R}$ by

$$\tilde{f}_N(r) = \begin{cases} \tilde{f}(\frac{1}{N}), & r \leq \frac{1}{N}, \\ \tilde{f}(r), & r > \frac{1}{N}, \end{cases}$$

and define $f_N : D \to \mathbb{R}$ by $f_N(x) = \tilde{f}_N(|x|)$ for $x \in D$. Then f_N is continuous and bounded. Moreover, fixing $N > \frac{1}{\eta}$, we have the bound $f_N \geq f$. Now define $v^S_N : D \to \mathbb{R}$ by

$$v^S_N(x) := \inf_{\sigma \in \mathcal{U}} \mathbb{E}^x \left[\int_0^\tau f_N(X^\sigma_s) \, \mathrm{d}s \right], \quad x \in D,$$

using the same notation as in the definition of the strong value function v^S in Section 2. Note that $v^S_N \geq v^S$.

Let V_N denote the candidate value function defined in Case I of Definition 4.5 with the function $\tilde{f}$ replaced by $\tilde{f}_N$. Since Assumption 4.1 is satisfied for the value function v^S_N, we can apply Proposition 4.6 to see that $v^S_N = V_N$. We can also see that, for any $x \in D$, $\lim_{N\to\infty} V_N(x) = V(x)$ and $V(x)$ is finite, since $\int_0^r \tilde{f}(s) \, \mathrm{d}s > -\infty$ for any $r > 0$. We will show that $\lim_{N\to\infty} v^S_N(x) = v^S(x)$ and conclude that $v^S(x) = V(x)$.

Fix $\sigma \in \mathcal{U}$ and $x \in D$. We have

$$\mathbb{E}^x \left[\int_0^\tau \tilde{f}_N(|X^\sigma_s|) \, \mathrm{d}s \right] = \mathbb{E}^x \left[\int_0^\tau \tilde{f}(|X^\sigma_s|) \mathbb{1}_{\left\{|X^\sigma_s| \in (\frac{1}{N}, R)\right\}} \, \mathrm{d}s \right] + \tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^x \left[\int_0^\tau \mathbb{1}_{\left\{|X^\sigma_s| \leq \frac{1}{N}\right\}} \right]. \tag{5.1}$$

Define $K := \sup\{f(x) \colon x \in D\}$ and note that $K < \infty$ by continuity of f in $D \setminus \{0\}$. Then the sequence

$$\left(\int_0^\tau \tilde{f}(|X^\sigma_s|) \mathbb{1}_{\left\{|X^\sigma_s| \in (\frac{1}{N}, R)\right\}} \, \mathrm{d}s \right)_{N \in \mathbb{N}}$$

is decreasing for $N > \frac{1}{\eta}$ and bounded above by τK. Since τ has finite expectation by Proposition 2.5, we can apply monotone convergence (see e.g. Theorem 1 of [14, Chapter II, §6]) to show that

$$\lim_{N\to\infty} \mathbb{E}^x \left[\int_0^\tau \tilde{f}(|X^\sigma_s|) \mathbb{1}_{\left\{|X^\sigma_s| \in (\frac{1}{N}, R)\right\}} \, \mathrm{d}s \right] = \mathbb{E}^x \left[\lim_{N\to\infty} \int_0^\tau \tilde{f}(|X^\sigma_s|) \mathbb{1}_{\left\{|X^\sigma_s| \in (\frac{1}{N}, R)\right\}} \, \mathrm{d}s \right] = \mathbb{E}^x \left[\int_0^\tau \tilde{f}(|X^\sigma_s|) \, \mathrm{d}s \right].$$

We will show that the second term of (5.1) vanishes as $N \to \infty$ by referring to Proposition 3.8 on the control problem for a step cost function. Note that $\tilde{f}(\frac{1}{N}) < 0$. For $x \neq 0$, we can choose $N > \frac{1}{|x|}$, so that, by Proposition 3.8,

$$\begin{aligned} 0 > \tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^x \left[\int_0^\tau \mathbb{1}_{\left\{|X^\sigma_s| \leq \frac{1}{N}\right\}} \right] &= -\tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^x \left[\int_0^\tau -\mathbb{1}_{\left\{|X^\sigma_s| \leq \frac{1}{N}\right\}} \right] \\ &\geq -\frac{2}{N} \tilde{f}\left(\frac{1}{N}\right) (R - |x|) \xrightarrow{N\to\infty} 0, \end{aligned}$$

using the condition that $\int_0^r \tilde{f}(s) \, \mathrm{d}s > -\infty$ to find the limit.

For $x = 0$, Proposition 3.8 gives us

$$\begin{aligned} 0 > \tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^0 \left[\int_0^\tau \mathbb{1}_{\left\{|X^\sigma| \leq \frac{1}{N}\right\}} \, \mathrm{d}s \right] &= -\tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^0 \left[-\int_0^\tau \mathbb{1}_{\left\{|X^\sigma| \leq \frac{1}{N}\right\}} \, \mathrm{d}s \right] \\ &\geq \tilde{f}\left(\frac{1}{N}\right) \left(\frac{2R}{N} - \frac{1}{N^2} \right) \xrightarrow{N\to\infty} 0. \end{aligned}$$

Hence

$$\lim_{N\to\infty} \mathbb{E}^x \left[\int_0^\tau \tilde{f}_N(|X^\sigma_s|) \, \mathrm{d}s \right] = \mathbb{E}^x \left[\int_0^\tau \tilde{f}(|X^\sigma_s|) \, \mathrm{d}s \right],$$

for any $\sigma \in \mathcal{U}$, $x \in D$.

Now fix $x \in D$ and $\varepsilon > 0$ and choose σ^ε to be an ε-optimal strategy for the cost function f; i.e.

$$\mathbb{E}^x \left[\int_0^\tau f(X^{\sigma^\varepsilon}_s) \, \mathrm{d}s \right] \leq v^S(x) + \varepsilon.$$

Then

$$v^S(x) + \varepsilon \geq \mathbb{E}^x \left[\int_0^\tau f(X^{\sigma^\varepsilon}_s) \, \mathrm{d}s \right] = \lim_{N\to\infty} \mathbb{E}^x \left[\int_0^\tau f_N(X^{\sigma^\varepsilon}_s) \, \mathrm{d}s \right] \geq \lim_{N\to\infty} v^S_N(x) \geq v^S(x).$$

Taking the limit as $\varepsilon \downarrow 0$, we see that

$$v^S(x) = \lim_{N\to\infty} v_N^S(x),$$

and by uniqueness of the limit, we have that $v^S(x) = V(x)$.

As in Proposition 2.7, we can apply Theorem 4.5 of [1] to see that $v^S = v^W$. Since f is continuous in $D \setminus \{0\}$, upper semicontinuous at 0, and bounded above by a constant, we can deduce that the conditions of Theorem 4.5 of [1] are met in the same way as in the proof of Proposition 2.7. Moreover, for any $x \in D$, we can repeat the above argument, replacing the set of strong controls $\mathcal{U}$ with the Markov controls $\mathcal{U}_x^M$, to find that $v^M(x) = V(x)$. Hence $v^W = v^S = v^M = V$.

Now suppose that, for any $r > 0$,

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = -\infty.$$

We will show that radial motion is an optimal strategy and that this strategy gives a negative infinite cost. Let the control σ^1 be as defined in Definition 3.7, and define X^{σ^1} by

$$X_t^{\sigma^1} = x + \int_0^t \sigma_s^1\,\mathrm{d}B_s, \quad t \geq 0.$$

Let W be the first component of the Brownian motion B. First suppose that $x \neq 0$. Then, for any $t \geq 0$,

$$X_t^{\sigma^1} = x + \frac{x}{|x|} W_t,$$

and so

$$\mathbb{E}^x\left[\int_0^\tau \tilde{f}(|X_s^{\sigma^1}|)\,\mathrm{d}s\right] = \mathbb{E}^{|x|}\left[\int_0^\tau \tilde{f}(W_s)\mathbb{1}_{\{W_s\geq 0\}}\,\mathrm{d}s\right] + \mathbb{E}^{|x|}\left[\int_0^\tau \tilde{f}(-W_s)\mathbb{1}_{\{W_s<0\}}\,\mathrm{d}s\right].$$

We can now use the Green's function G for the one-dimensional Brownian motion W on the interval $(-R, R)$, as calculated in Example 3.6. By Corollary 3.8 of [11, Chapter VII], we see that

$$\begin{aligned}\mathbb{E}^x\left[\int_0^\tau \tilde{f}(X_s^{\sigma^1})\,\mathrm{d}s\right] &= 2\int_0^R G(|x|, y)\tilde{f}(y)\,\mathrm{d}y + 2\int_{-R}^0 G(|x|, y)\tilde{f}(-y)\,\mathrm{d}y \\ &= \frac{|x|+R}{R}\int_{|x|}^R (R-y)\tilde{f}(y)\,\mathrm{d}y + \frac{R-|x|}{R}\int_0^{|x|}(y+R)\tilde{f}(y)\,\mathrm{d}y + \frac{R-|x|}{R}\int_{-R}^0 (y+R)\tilde{f}(-y)\,\mathrm{d}y.\end{aligned}$$

Making a change of variables $y \mapsto -y$ in the last integral gives

$$\mathbb{E}^x\left[\int_0^\tau \tilde{f}(X_s^{\sigma^1})\,\mathrm{d}s\right] = 2\int_{|x|}^R (R-y)\tilde{f}(y)\,\mathrm{d}y + 2(R-|x|)\int_0^{|x|}\tilde{f}(y)\,\mathrm{d}y.$$

Since f is bounded above and $\int_0^{|x|}\tilde{f}(y)\,\mathrm{d}y = -\infty$, we have

$$\mathbb{E}^x\left[\int_0^\tau \tilde{f}(X_s^{\sigma^1})\,\mathrm{d}s\right] = -\infty.$$

Now let $x = 0$. Then, $X_t^{\sigma^1} = e_1 W_t$, for $t \geq 0$. Using the symmetry of the Green's function G for W about zero, together with the growth condition on f, we have

$$\begin{aligned}\mathbb{E}^0\left[\int_0^\tau \tilde{f}(|X_s^{\sigma^1}|)\,\mathrm{d}s\right] &= 2\mathbb{E}^0\left[\int_0^\tau \tilde{f}(W_s)\mathbb{1}_{\{W_s\geq 0\}}\,\mathrm{d}s\right] \\ &= 4\int_0^R G(0,y)\tilde{f}(y)\,\mathrm{d}y = 2\int_0^R (R-y)\tilde{f}(y)\,\mathrm{d}y = -\infty.\end{aligned} \tag{5.2}$$

For any $x \in D$, note that $\sigma^1 \in \mathcal{U}_x^M$. Therefore we can conclude that

$$v^W(x) \leq v^S(x) \leq v^M(x) \leq \mathbb{E}^x\left[\int_0^\tau \tilde{f}(X_s^{\sigma^1})\,\mathrm{d}s\right] = -\infty.$$

□

We have shown that, for cost functions increasing at the origin, there is a dichotomy depending on the convergence of $\int_0^r \tilde{f}(s)\,\mathrm{d}s$. When $\int_0^r \tilde{f}(s)\,\mathrm{d}s > -\infty$ for any $r > 0$, the value function is finite and equal to V, and when $\int_0^r \tilde{f}(s)\,\mathrm{d}s = -\infty$ for any $r > 0$, the value is identically equal to negative infinity.

5.2. **Cost functions decreasing at the origin.** We now consider cost functions that are decreasing in some neighbourhood around the origin. Excluding the origin from this neighbourhood, an optimal strategy is tangential motion, as defined in Definition 3.3. We will first show that, away from the origin, the form of the value function is unchanged from the value function in Proposition 4.6.

Proposition 5.4. *Suppose that Assumption 5.1 holds and there exists $\eta > 0$ such that $\tilde{f}$ is positive and decreasing on the interval $(0, \eta)$. Then, for $x \in D \setminus \{0\}$, $v(x) = v^S(x) = v^W(x) = v^M(x) = V(x) \in (-\infty, \infty)$, where V is the candidate value function defined in Definition 4.5.*

Remark 5.5. In this case, since $\tilde{f}$ is decreasing on $(0, \eta)$, V is defined in Case II of Definition 4.5.

Proof of Proposition 5.4. For $N \in \mathbb{N}$, define $\tilde{f}_N$, f_N and v_N^S as in the proof of Proposition 5.2. Now, for $N > \frac{1}{\eta}$, we have $\tilde{f}_N \leq \tilde{f}$, $f_N \leq f$, and $v_N^S \leq v_N$. Recall that, by Proposition 4.6, $v_N^S = V_N$, where V_N is the candidate value function defined in Case II of Definition 4.5 with the cost function $\tilde{f}$ replaced by $\tilde{f}_N$. Fix $x \in D \setminus \{0\}$ and $N > \frac{1}{|x|} \vee \frac{1}{\eta}$. Then we can see that $V_N(x) = V(x) \in (-\infty, \infty)$. We will show that $v_N^S(x) = v^S(x)$ and conclude that $v^S(x) = V(x)$.

Let $\sigma^\star$ be the control defined in (4.2). Since $\tilde{f}^N$ is decreasing in the interval $(0, \eta)$, Proposition 4.6 shows that $\sigma^\star$ is optimal, and so

$$\begin{aligned} v_N^S(x) &= \mathbb{E}^x \left[\int_0^\tau \tilde{f}_N\left(\left|X_s^{\sigma^\star}\right|\right) \mathrm{d}s \right] \\ &= \tilde{f}\left(\frac{1}{N}\right) \mathbb{E}^x \left[\int_0^\tau \mathbb{1}_{\left\{\left|X_s^{\sigma^\star}\right| \leq \frac{1}{N}\right\}} \mathrm{d}s \right] + \mathbb{E}^x \left[\int_0^\tau \tilde{f}\left(\left|X_s^{\sigma^\star}\right|\right) \mathbb{1}_{\left\{\left|X_s^{\sigma^\star}\right| \in (\frac{1}{N}, R)\right\}} \mathrm{d}s \right]. \end{aligned} \tag{5.3}$$

When $\left|X_t^{\sigma^\star}\right| \in (0, \eta)$, the radius process $t \mapsto \left|X_t^{\sigma^\star}\right|$ is deterministically increasing, by Lemma 3.2. Therefore, $\mathbb{1}_{\left\{\left|X_t^{\sigma^\star}\right| \leq \frac{1}{N}\right\}} = 0$, for all $t \geq 0$, since $|x| > \frac{1}{N}$. Hence, by (5.3) and the definition of v^S, we have

$$v^S(x) \geq v_N^S(x) = \mathbb{E}^x \left[\int_0^\tau \tilde{f}\left(\left|X_s^{\sigma^\star}\right|\right) \mathrm{d}s \right] \geq v^S(x),$$

and so $v^S(x) = v_N^S(x) = V_N(x) = V(x)$.

We can repeat the above argument, replacing the strong value functions with Markov value functions, and replacing the optimal control $\sigma^\star$ with the optimal Markov control $\tilde{\sigma}^\star$ defined in Remark 4.3. Note that the process $t \mapsto \left|X^{\tilde{\sigma}^\star}\right|$ is still deterministically increasing on the interval $(0, \eta)$, and so the above argument holds for the Markov value function. Therefore $v^M(x) = V(x)$, for any $x \in D \setminus \{0\}$.

Finally, for $N \in \mathbb{N}$, define $v_N^W : D \to \mathbb{R}$ by

$$v_N^W(x) := \inf_{\mathbb{P} \in \mathcal{P}_x} \mathbb{E}^{\mathbb{P}} \left[\int_0^\tau f_N(X_s) \, \mathrm{d}s \right], \quad x \in D,$$

using the same notation as in the definition of the weak value function v^W in Section 2. By Proposition 2.7, $v_N^S = v_N^W$. Once again, fix $x \in D \setminus \{0\}$ and $N > \frac{1}{|x|} \vee \frac{1}{\eta}$, so that $v_N^W \leq v^W$. Then we have

$$v_N^S(x) = v_N^W(x) \leq v^W(x) \leq v^S(x) = V(x) = v_N^S(x),$$

and we conclude that $v^W(x) = v^S(x) = v^M(x) = V(x)$. ☐

At the origin, we have not shown that there exists an optimal control. The function σ^0 introduced in Definition 3.3 is not defined at the origin, and so we require an approximation to tangential motion. We consider different growth rates separately, as we did for increasing costs.

Proposition 5.6. *Suppose that Assumption 5.1 holds and there exists $\eta > 0$ such that $\tilde{f}$ is positive and decreasing on the interval $(0, \eta)$. Suppose further that, for any $r > 0$,*

$$\int_0^r \tilde{f}(s) \, \mathrm{d}s < \infty.$$

Then $v(0) = v^S(0) = v^W(0) = v^M(0) = V(0) \in (-\infty, \infty)$, where V is the candidate value defined in Definition 4.5.

Proof. For $N \in \mathbb{N}$, define $\tilde{f}_N$, f_N and v_N^S as in the proof of Proposition 5.2. Letting V_N be the candidate value function in Case II of Definition 4.5 with $\tilde{f}$ replaced by $\tilde{f}_N$, we have $v_N^S(0) = V_N(0)$, by Proposition 4.6. We also see that $\lim_{N\to\infty} V_N(0) = V(0)$, and the value $V(0)$ is finite due to the growth condition on $\tilde{f}$. We will show that $v^S(0) = \lim_{N\to\infty} v_N^S(0)$ and conclude that $v^S(0) = V(0)$.

Fix $\delta \in (0,\eta)$ and $N > \frac{1}{\delta}$. Denote by e_1 the unit vector in the first coordinate direction. Let $\sigma^N \in \mathcal{U}$ be an optimal control for the cost f_N, so that

$$v_N^S(0) = \mathbb{E}^0\left[\int_0^\tau \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right].$$

Since $\tilde{f}_N$ is constant on $(0,\frac{1}{N})$ and decreasing on $(\frac{1}{N},\eta)$, by Proposition 4.6 we can choose σ^N such that

$$\sigma_t^N = \begin{cases} \begin{bmatrix} e_1; & 0; & \dots; & 0 \end{bmatrix}, & \text{for} \quad |X_t| < \frac{1}{N}, \\ \sigma^0(X_t), & \text{for} \quad |X_t| \in \big[\frac{1}{N},\eta\big). \end{cases}$$

Also define a control σ^δ that coincides with σ^N except that we set $\sigma_t^\delta = \begin{bmatrix} e_1; & 0; & \dots; & 0 \end{bmatrix}$, for $|X_t| \in [\frac{1}{N},\delta)$.

Under either control σ^N or σ^δ, the process $t \mapsto |X_t|$ is deterministically increasing on the interval (δ,η), by Lemma 3.4. Therefore, writing τ^δ for the exit time from the ball $B_\delta(0)$, the error between the value $v_N^S(0)$ and the expected cost of choosing the control σ^δ with the cost f is

$$\begin{aligned} 0 \le E_N(\delta) :=& \mathbb{E}^0\left[\int_0^\tau \tilde{f}\big(|X_s^{\sigma^\delta}|\big)\,\mathrm{d}s\right] - v_N^S(0) \\ =& \mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}\big(|X_s^{\sigma^\delta}|\big)\,\mathrm{d}s\right] - \mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right]. \end{aligned}$$

In the ball $B_\delta(0)$, the process X^{σ^δ} is equal to a one-dimensional Brownian motion in the direction e_1 and so, making a calculation with the Green's function similar to (5.2) in the proof of Proposition 5.2, we find that

$$\mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}\big(|X_s^{\sigma^\delta}|\big)\,\mathrm{d}s\right] = 2\int_0^\delta (\delta - y)\,\tilde{f}(y)\,\mathrm{d}y.$$

We now compute the expected cost under the control σ^N. When $\big|X_t^{\sigma^N}\big| \in (\frac{1}{N},\delta)$, the process X^{σ^N} follows tangential motion, and so we can calculate

$$\mathbb{E}^{\frac{1}{N}}\left[\int_0^{\tau_\delta} \tilde{f}_N\big(|X_s^{\sigma^{N,\varepsilon}}|\big)\,\mathrm{d}s\right] = \int_0^{\delta^2 - N^{-2}} \tilde{f}\big(\sqrt{N^{-2}+s}\big)\,\mathrm{d}s = 2\int_{\frac{1}{N}}^\delta s\tilde{f}(s)\,\mathrm{d}s.$$

In the ball $B_{\frac{1}{N}}(0)$, the process X^{σ^N} is a one-dimensional Brownian motion and so, making another calculation with the Green's function, we can write

$$\begin{aligned} \mathbb{E}^0\left[\int_0^{\tau_\delta} \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right] &= \mathbb{E}^0\left[\int_0^{\tau_{\frac{1}{N}}} \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right] + \mathbb{E}^{\frac{1}{N}}\left[\int_0^{\tau_\delta} \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right] \\ &= \frac{1}{N^2}\tilde{f}\left(\frac{1}{N}\right) + 2\int_{\frac{1}{N}}^\delta y\tilde{f}(y)\,\mathrm{d}y. \end{aligned}$$

Therefore the error is

$$E_N(\delta) = 2\int_0^\delta (\delta - y)\tilde{f}(y)\,\mathrm{d}y - 2\int_{\frac{1}{N}}^\delta y\tilde{f}(y)\,\mathrm{d}y - \frac{1}{N^2}\tilde{f}\left(\frac{1}{N}\right).$$

Since $\int_0^r \tilde{f}(s)\,\mathrm{d}s < \infty$, for any $r > 0$, we can take the limit as $N \to \infty$ to get

$$E(\delta) := \lim_{N\to\infty} E_N(\delta) = 2\int_0^\delta (\delta - 2y)\,\tilde{f}(y)\,\mathrm{d}y,$$

and then taking the limit as $\delta \to 0$ gives

$$0 \le E(\delta) = 2\int_0^\delta (\delta - 2y)\,\tilde{f}(y)\,\mathrm{d}y \xrightarrow{\delta\to 0} 0. \tag{5.4}$$

Returning to the definition of $E_N(\delta)$, for fixed $\delta \in (0,\eta)$ and $N > \frac{1}{\delta}$, we recall that

$$v_N^S(0) + E_N(\delta) = \mathbb{E}^0\left[\int_0^\tau \tilde{f}\big(|X_s^{\sigma^\delta}|\big)\,\mathrm{d}s\right]. \tag{5.5}$$

Since $\tilde{f}_N \le \tilde{f}$, we have

$$v^S(0) + E_N(\delta) \ge v_N^S(0) + E_N(\delta),$$

and, by the definition of v^S,

$$\mathbb{E}^0\left[\int_0^\tau \tilde{f}\big(|X_s^{\sigma^\delta}|\big)\,\mathrm{d}s\right] \ge v^S(0).$$

Combining these inequalities with (5.5), we see that

$$v^S(0) + E_N(\delta) \ge v_N^S(0) + E_N(\delta) \ge v^S(0).$$

Since the sequence $\big(v_N^S(0)\big)_{N\in\mathbb{N}}$ is monotone, we can take the limit as $N \to \infty$ and find that

$$v^S(0) + E(\delta) \ge \lim_{N\to\infty} v_N^S(0) + E(\delta) \ge v^S(0).$$

Having calculated that $\lim_{\delta\to 0} E(\delta) = 0$ in (5.4), we have

$$v^S(0) = \lim_{N\to\infty} v_N^S(0) = V(0).$$

Defining v_N^W as in the proof of Proposition 5.4, we have $v_N^S(0) = v_N^W(0) \le v^W(0)$ for any $N \in \mathbb{N}$, and so $v^S(0) = v^W(0) = V(0)$. We can also replace σ^N with an optimal Markov control $\tilde{\sigma}^N \in \mathcal{U}_0^M$ that satisfies the same properties as σ^N. Following the same arguments as above we can then conclude that $v^M(0) = V(0)$. □

Remark 5.7. Note that, if the growth rate of $\tilde{f}$ is such that, for any $r > 0$, $\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty$, then the error $E(\delta)$ in the proof of Proposition 5.6 is infinite for all δ. Therefore the above argument does not generalise to costs with faster growth at the origin.

We now consider decreasing costs with faster growth at the origin.

Proposition 5.8. *Suppose that Assumption 5.1 holds and that there exists $\eta > 0$ such that $\tilde{f}$ is positive and decreasing on the interval $(0,\eta)$. If, for any $r > 0$,*

$$\int_0^r s\tilde{f}(s)\,\mathrm{d}s = \infty,$$

then $v^S(0) = v^W(0) = v^M(0) = +\infty$.

Proof. Once again define $\tilde{f}_N$, f_N and v_N^S as in the proof of Proposition 5.2. Let $N > \frac{1}{\eta}$ and define the control σ^N as in the proof of Proposition 5.6, so that σ^N is optimal for the cost f_N. Using the calculations of the expected cost under the control σ^N from the proof of Proposition 5.6, we find that

$$\begin{aligned} v_N^S(0) = \mathbb{E}^0\left[\int_0^\tau \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right] &= \mathbb{E}^0\left[\int_0^{\tau_\eta} \tilde{f}_N\big(|X_s^{\sigma^N}|\big)\,\mathrm{d}s\right] + \mathbb{E}^0\left[\int_0^\tau \tilde{f}\big(|X_s^{\sigma^N}|\big)\mathbb{1}_{\{|X^{\sigma^N}|\in(\eta,R)\}}\,\mathrm{d}s\right] \\ &\ge \frac{1}{N^2}\tilde{f}\left(\frac{1}{N}\right) + 2\int_{\frac{1}{N}}^{\eta} y\tilde{f}(y)\,\mathrm{d}y + (R^2-\eta^2)\min\left\{\tilde{f}(r)\colon r\in(\eta,R)\right\}. \end{aligned}$$

By the growth condition on $\tilde{f}$, we have

$$\lim_{N\to\infty}\int_{\frac{1}{N}}^{\eta} y\tilde{f}(y)\,\mathrm{d}y = +\infty.$$

Also, since $\tilde{f}$ is continuous on $(0,R)$, we have $\min\left\{\tilde{f}(r)\colon r\in(\eta,R)\right\} > -\infty$, and so

$$\lim_{N\to\infty} v_N^S(0) = +\infty.$$

Defining v_N^W as in the proof of Proposition 5.4 for any $N \in \mathbb{N}$, we conclude that

$$v^M(0) \ge v^S(0) \ge v^W(0) \ge \lim_{N\to\infty} v_N^W(0) = \lim_{N\to\infty} v_N^S(0) = +\infty.$$

□

We have now fully characterised the value function for any radially symmetric cost, except for the value at the origin when the cost function is decreasing at the origin and grows at such a rate that, for any $r > 0$,

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty.$$

This remaining growth regime has many interesting features, which we will study in detail in a forthcoming paper and can be found in Chapter 3 of the thesis [12]. Here we present the following partial result.

Proposition 5.9. *Suppose that Assumption 5.1 holds and that there exists $\eta > 0$ such that $\tilde{f}$ is positive and decreasing on the interval $(0, \eta)$. If, for any $r > 0$,*

$$\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty \quad \textit{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty,$$

then, for V equal to the candidate value function defined in Definition 4.5,

$$v^W(0) = V(0) \in (-\infty, \infty).$$

Moreover, if $d \geq 3$, then $v(0) = v^S(0) = v^W(0) = v^M(0) = V(0)$.

Note that we do not make any claim about the strong or Markov value functions in dimension $d = 2$.

We split the proof of Proposition 5.9 into the two lemmas, first proving the result for dimensions $d \geq 3$.

Lemma 5.10. *Under the conditions of Proposition 5.9 with $d \geq 3$, we have*

$$v(0) = v^S(0) = v^W(0) = v^M(0) = V(0) \in (-\infty, \infty).$$

Proof. In this case, we can follow the same argument as in the proof of Proposition 5.6 except that we replace the constant control $\begin{bmatrix} e_1; & 0; & \dots; & 0 \end{bmatrix}$ with $\frac{1}{d}I$, where I is the d-dimensional identity matrix. Instead of following a one-dimensional Brownian motion at the origin, the controlled processes under σ^N and σ^δ follow a scaled d-dimensional Brownian motion, as do the processes under the Markov controls $\tilde{\sigma}^N$ and $\tilde{\sigma}^\delta$. We now verify that the approximation arguments in Proposition 5.6 hold with this change.

We will use the Green's function for the d-dimensional Brownian motion B, as defined in Section 3.3 of the book [10] of Mörters and Peres. By Theorem 3.32 and 3.33 of [10] and the radial symmetry of f, there are constants $C, C' > 0$ such that, for any $\delta \in (0, \eta)$,

$$\mathbb{E}^0\left[\int_0^{\tau_\delta} f(B_s)\,\mathrm{d}s\right] = C\int_{B_\delta} |y|^{2-d} f(y)\,\mathrm{d}y = C'\int_0^\delta r\tilde{f}(r)\,\mathrm{d}r \xrightarrow{\delta \to 0} 0,$$

where the limit follows from the assumption that $\int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty$ for any $r > 0$. Hence, following the same arguments as in the proof of Proposition 5.6, we deduce the desired result. $\square$

Now suppose that $d = 2$. Note that, from the form of the Green's function for 2-dimensional Brownian motion given in Theorem 3.34 of [10], we can see that the argument used for $d \geq 3$ is no longer valid. In the following lemma, we treat the weak control problem in dimension $d = 2$.

Lemma 5.11. *Under the conditions of Proposition 5.9 with $d = 2$, the weak value function is given by*

$$v^W(0) = V(0) \in (-\infty, \infty).$$

Proof. Retaining the notation of the proof of Proposition 5.8, we have that, for any $y \in D$ with $|y| = \eta$,

$$v^W(0) \geq \lim_{N\to\infty} v_N^S(0) = V(0) = 2\int_0^\eta \xi\tilde{f}(y)\,\mathrm{d}\xi + V(y), \tag{5.6}$$

by Proposition 4.6 and the definition of V in Definition 4.5.

In Theorem 4.3 of [8], Larsson and Ruf prove that, for $d = 2$, there exists a weak solution X^{σ^0} of the SDE

$$\mathrm{d}X_t = \sigma^0(X_t)\,\mathrm{d}B_t; \quad X_0 = 0. \tag{5.7}$$

The process X^{σ^0} follows tangential motion starting from the origin, as defined in Definition 3.3. By Lemma 3.4, we have $\left|X_t^{\sigma^0}\right| = \sqrt{t}$, for any $t \geq 0$, and so

$$\mathbb{E}^0\left[\int_0^{\tau_\eta} f(X_s^{\sigma^0})\,\mathrm{d}s\right] = \int_0^{\eta^2} \tilde{f}(\sqrt{s})\,\mathrm{d}s = 2\int_0^{\eta} \xi\tilde{f}(\xi)\,\mathrm{d}\xi.$$

Note that Assumption A.1 holds, and so we can apply the dynamic programming principle from Proposition A.2 to see that, for any $y \in D$ with $|y| = \eta$,

$$v^W(0) \leq v^S(0) \leq \mathbb{E}^0\left[\int_0^{\tau_\eta} f(X_s^{\sigma^0})\,\mathrm{d}s + v^S(X_{\tau_\eta}^{\sigma^0})\right] = 2\int_0^{\eta} \xi\tilde{f}(\xi)\,\mathrm{d}\xi + V(y),$$

using the result of Proposition 5.4 that $v^S = V$ away from the origin.

Combining the above inequality with (5.6), we have $v^W(0) = V(0)$, as required. □

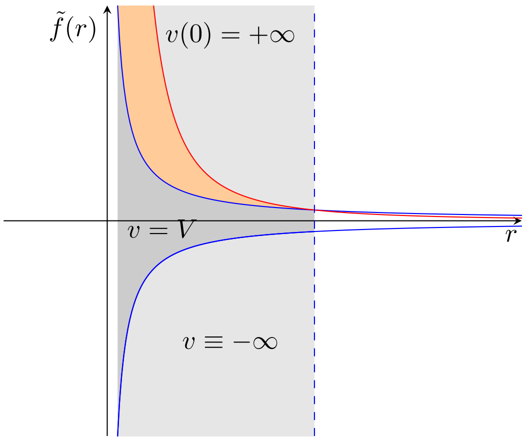


FIGURE 6. Figure showing the distinct growth regimes for the cost function in Theorem 5.12, highlighting the case where, for any $r > 0$, $\int_0^r \tilde{f}(s)\,\mathrm{d}s = \infty$ and $\int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty$, as in Proposition 5.9.

We summarise the preceding results in the following extension of Proposition 4.6.

Theorem 5.12. *Suppose that Assumption 5.1 is satisfied, and let $V : D \to \mathbb{R}$ be the candidate value function defined in Definition 4.5. Let $x \in D$ and suppose, moreover, that one of the following conditions holds:*

(i) $\tilde{f}$ is increasing on the interval $(0, \eta)$;
(ii) $\tilde{f}$ is decreasing on the interval $(0, \eta)$ and $x \in D \setminus \{0\}$;
(iii) $\tilde{f}$ is decreasing on the interval $(0, \eta)$, $x = 0$ and, for any $r > 0$, $\int_0^r \tilde{f}(s) < \infty$;
(iv) $\tilde{f}$ is decreasing on the interval $(0, \eta)$, $x = 0$ and, for any $r > 0$, $\int_0^r s\tilde{f}(s) = \infty$;
(v) $\tilde{f}$ is decreasing on the interval $(0, \eta)$ and $d \geq 3$.

Then the value function is given by

$$v(x) = v^S(x) = v^W(x) = v^M(x) = V(x).$$

Furthermore, we can determine when the value function is finite. If $\tilde{f}$ is increasing on the interval $(0, \eta)$, then

$$\begin{cases} v > -\infty, & \text{if} \quad \int_0^r \tilde{f}(s)\,\mathrm{d}s > -\infty \quad \text{for any} \quad r > 0, \\ v \equiv -\infty, & \text{if} \quad \int_0^r \tilde{f}(s)\,\mathrm{d}s = -\infty \quad \text{for any} \quad r > 0. \end{cases}$$

If $\tilde{f}$ is decreasing on the interval $(0, \eta)$, then $v(x) < \infty$ for $x \in D \setminus \{0\}$, and

$$\begin{cases} v(0) = \infty, & \text{if} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s = \infty \quad \text{for any} \quad r > 0, \\ v(0) < \infty, & \text{if} \quad d \geq 3 \quad \text{and} \quad \int_0^r s\tilde{f}(s)\,\mathrm{d}s < \infty \quad \text{for any} \quad r > 0, \\ v(0) < \infty, & \text{if} \quad d = 2 \quad \text{and} \quad \int_0^r \tilde{f}(s)\,\mathrm{d}s < \infty \quad \text{for any} \quad r > 0. \end{cases}$$

We now discuss what remains to find the strong and Markov value functions under the assumptions of Proposition 5.9 in the case $d = 2$.

Remark 5.13. Recall that, in Proposition 2.7, we appealed to Theorem 4.5 of El Karoui and Tan's paper [1] to show equality between weak and strong value functions, under the assumption that the cost function f is upper semicontinuous and bounded above by a constant.

Under the assumptions of Proposition 5.9, we cannot apply Theorem 4.5 of [1], since one of the conditions of that theorem is no longer satisfied. Namely, in our setup, Theorem 4.5 of [1] is only applicable if the random variable $F_\tau := \int_0^\tau f(X_s)\,\mathrm{d}s$ is bounded above by some random variable that is uniformly integrable under the family of probability measures $\mathcal{P}_0$ defined in Section 2. We show that this condition is not satisfied as follows.

Let e_1 be the unit vector in the first coordinate direction and define X^1 by $X^1_t = e_1 B_t$, for $t \geq 0$. Then let $\mathbb{P}^{X^1}$ be the law of the process X^1 and define the product measure $\mathbb{P} := \mathbb{P}^{X^1} \times \delta_{e_1} \in \mathcal{P}_0$. Following the same Green's function calculation as in (5.2) in the proof of Proposition 5.2, we compute that

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^\tau f(X_s)\,\mathrm{d}s\right] = \int_0^R (R-r)\tilde{f}(r)\,\mathrm{d}r = +\infty,$$

due to the growth condition on $\tilde{f}$ at the origin. Hence there does not exist any uniformly integrable upper bound on F_τ and Theorem 4.5 of [1] does not apply.

In Lemma 5.11, we found the weak value function at the origin by using the fact that there exists a weak solution of the SDE (5.7) describing tangential motion started from the origin. However, the SDE (5.7) has no strong solution. The proof of this fact will appear in a forthcoming paper and can also be found in Chapter 3 of the thesis [12]. Since there exists no strong solution, we cannot follow the same argument as in the proof of Lemma 5.11 to find the strong value function. Nevertheless, we will show in our forthcoming paper that the strong and weak value functions are in fact equal, using the theory of Brownian filtrations. Again the details can be found in Chapter 3 of the thesis [12].

Another consequence of the lack of strong solutions of the SDE (5.7) is that the optimal strong control that we will construct cannot depend only on the current position of the controlled process. This leads us to conjecture there is a gap between the Markov value function and the strong and weak value functions at the origin.

Appendix A. Dynamic programming and comparison principles

In this appendix, we state the main results from the theory of dynamic programming and viscosity solutions that we use in the paper. The proofs of these results are fairly standard, but we have been unable to find versions of these results in the literature that completely cover the conditions required here. Full details can be found in the doctoral thesis [12]. We require the following strengthening of Assumption 2.1.

Assumption A.1. Suppose that Assumption 2.1 holds and, moreover, the domain D is strictly convex and the value function v satisfies $v(x) > -\infty$, for any $x \in D$.

Proposition A.2. *Suppose that Assumption A.1 is satisfied. Then v is continuous and the following dynamic programming principle holds. For any $x \in D$ and any stopping time θ with $\theta \in [0, \tau]$ almost surely, v satisfies*

$$v(x) = \inf_{\nu \in \mathcal{U}} \mathbb{E}^x\left[\int_0^\theta f(X^\nu_s)\,\mathrm{d}s + v(X^\nu_\theta)\right]. \tag{A.1}$$

Remark A.3. If there exists an optimal control $\sigma^\star \in \mathcal{U}$, then (A.1) is equivalent to stating that

$$v(X^\sigma_t) + \int_0^t f(X^\sigma_s)\,\mathrm{d}s \quad \text{is} \quad \begin{cases} \text{a submartingale,} & \text{for all} \quad \sigma \in \mathcal{U}, \\ \text{a martingale,} & \text{for} \quad \sigma = \sigma^\star. \end{cases}$$

Proposition A.4. *Suppose that Assumption 2.1 holds and that $f : D \to \mathbb{R}$ is continuous. Then we have the following comparison principle for the HJB equation*

$$-\frac{1}{2}\inf_{\sigma\in U}\operatorname{Tr}\left(D^2 v\sigma\sigma^\top\right) - f = 0. \tag{A.2}$$

Suppose that $u \in \mathrm{USC}(\overline{D})$ is a viscosity subsolution of (A.2)*, $v \in \mathrm{LSC}(\overline{D})$ is a viscosity supersolution of* (A.2)*, and $u \le v$ on ∂D. Then $u \le v$ on $\overline{D}$.*

Theorem A.5. *Suppose that Assumption A.1 holds, and suppose further that the domain D is uniformly convex, the running cost f is continuous in D, and the boundary cost g is uniformly continuous on ∂D.*

Then the value function $v : D \to \mathbb{R}$ defined in Section 2 extends continuously to $\overline{D}$ and is the unique viscosity solution of the HJB equation (A.2) *in D, with boundary condition $v = g$ on ∂D.*

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