# Stochastic Optimal Control & Applications.

Note Title

## §2 : Stochastic Integration.

Let's go back to Example 1.1 above, and consider the places where we waved our hands most frantically... To get (3), we derived an approximation:

$$V(X_{t+\delta t}, t+\delta t) - V(X_t, t) \approx \sigma \frac{\partial V}{\partial x} \delta B_t$$
$$+ \left( (1-u_t) \frac{\partial V}{\partial x} + \frac{\partial V}{\partial t} \right) \delta t + \frac{1}{2} \sigma^2 \frac{\partial^2 V}{\partial x^2} (\delta B)^2 + h.o.t.$$

and used this to obtain a representation for the average change in V, but how could we improve this?

Consider breaking $[0,T]$ into $0, \delta t, 2\delta t, \dots, N\delta t = T$, then:

$$V(X_T, T) - V(X_0, 0) = \sum_{n=1}^{\infty} \Big[ V(X_{i\delta t}, i\delta t) - V(X_{(i-1)\delta t}, (i-1)\delta t) \Big]$$

and we can try to sum the RHS similarly. Let

$$\phi_t(\omega) = (1 - u_t(\omega)) \frac{\partial V}{\partial x}(X^u_t(\omega), t) + \frac{\partial V}{\partial t}(X^u_t(\omega), t),$$

and observe that $X^u_t$ is a continuous function with probability 1, so if (e.g.) $\frac{\partial V}{\partial x}, \frac{\partial V}{\partial t}$ are continuous and $t \mapsto u_t(\omega)$ is measurable with probability 1, then $\int_0^T \phi_t \, dt$ exists a.s. and we should expect

$$\sum_i \phi(i\delta t)\, \delta t \longrightarrow \int_0^T \phi_t \, dt.$$

The harder question is what happens to the other terms: $\sum_{i=1}^{N} \frac{\partial V}{\partial x}(X_{i\delta t}, i\delta t)\, \delta B_{i\delta t}$, $\sum_{i=1}^{N} \frac{\partial^2 V}{\partial x^2} (\delta B_{i\delta t})^2$

Or more generally $\sum_{i=1}^{N} \phi_{i\delta t}\, \delta B_{i\delta t}$, $\sum_{i=1}^{N} \psi_{i\delta t} (\delta B_{i\delta t})^2$ for some (nice?) <u>stochastic</u> processes $\phi_t(\omega)$, $\psi_t(\omega)$.

A major issue is the following — suppose we want to compute $\int_0^T B_t \, dB_t$, a natural approximation is:

$$Y^1 := \sum_{i=0}^{N-1} B_{i\delta t}\left(B_{(i+1)\delta t} - B_{i\delta t}\right),$$

and by indep. increments, $\mathbb{E}\, Y^1 = 0$.

But an alternative approximation is

$$Y^2 := \sum_{i=0}^{N-1} B_{(i+1)\delta t}\left(B_{(i+1)\delta t} - B_{i\delta t}\right),$$

and

$$\mathbb{E}\, Y^2 = \mathbb{E}\left(Y^2 - Y^1\right) = \sum_i \mathbb{E}\left(B_{(i+1)\delta t} - B_{i\delta t}\right)^2 = \sum_i \delta t = T \quad !!$$

But both $B_{i\delta t}$ and $B_{(i+1)\delta t}$ look like $B_t$ as $\delta t \to 0$.

This suggests (in contrast to the Riemann integral) we need to choose our endpoints carefully. For the Itô integral, we will always take sums of the form $Y^1$, and it will be important that the integrand is <u>adapted</u> (i.e. $\phi_t$ is $\mathcal{F}_t$ - measurable).

<u>Setting</u>: We work on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0})$ (s.t. $\mathcal{F}_t$ satisfies the usual conditions) on which there exists a B.M. $B_t$.

Our set of potential integrands is:

$\mathcal{V}([0,T])$ which contains $f(t,\omega)$, $f: [0,\infty)\times\Omega \to \mathbb{R}$

(i) $f(t,\omega)$ is progressively measurable:

$\forall s \in [0,T]$, $(t,\omega) \mapsto f(t,\omega)$ is $\mathcal{B}([0,s]) \times \mathcal{F}_s$ - measurable

(this implies $f(t,\omega)$ is $\mathcal{F}_t$ - adapted, and is implied by $\mathcal{F}_t$ -adapted, r-cts or l-cts...)

(ii) $\mathbb{E}\left[\int_0^T f(t,\omega)^2\,dt\right] < \infty$.

Then $\phi \in \mathcal{V}([0,T])$ is <u>simple</u> if

$$\phi(t,\omega) = \sum_{j\geq 0} e_j(\omega)\,\mathbb{1}_{\{[t_j,t_{j+1})\}}(t),$$

some $0 \leq t_0 < t_1 < \dots < t_j \leq T$. So adapted implies $e_j(\omega)$ is $\mathcal{F}_{t_j}$ -measurable. Write $\phi \in \mathcal{S}([0,T])$.

For $\phi \in \mathcal{S}([0,T])$, we <u>define</u> the Itô integral by:

$$\int_0^T \phi(t,\omega)\,dB_t = \sum_{j\geq 0} e_j(\omega)\left(B_{t_{j+1}} - B_{t_j}\right).$$

Theorem (Itô Isometry)

For $\phi \in \mathcal{S}([0,T])$ bounded,

$$\mathbb{E}\left[\left(\int_0^T \phi(t,\omega)\, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T \phi(t,\omega)^2\, dt\right]$$

Proof. Set $\Delta B_j = B_{t_{j+1}} - B_{t_j}$. Then

$$\mathbb{E}\left[e_i e_j \Delta B_i \Delta B_j\right] = \begin{cases} 0 & i \neq j \\ \mathbb{E}[e_i^2](t_{i+1} - t_i) & i = j \end{cases}$$

since $\Delta B_j$ indep of $e_i, e_j, \Delta B_i$ if $j > i$.

So
$$\mathbb{E}\left[\left(\int_0^T \phi_s\, dB_s\right)^2\right] = \sum_{i,j} \mathbb{E}\left[e_i e_j \Delta B_i \Delta B_j\right]$$
$$= \sum_{j \geq 0} \mathbb{E}[e_i^2](t_{i+1} - t_i)$$
$$= \mathbb{E}\left[\int_0^T \phi_t^2\, dt\right] \qquad \square$$

Now we show $\phi \in \mathcal{V}([0,T])$ can be approximated by simple functions in $\mathcal{S}([0,T])$.

(1) If $t \mapsto \phi(t,\omega)$ is continuous for each $\omega \in \Omega$, and $\phi$ bounded, then $\phi_n = \phi(t_i) \mathbb{1}_{\{[t_i, t_{i+1})\}}(t)$

$$\longrightarrow \phi(t) \text{ pointwise},$$

so $\mathbb{E}\left[\int_0^T (\phi_n - \phi)^2\, dt\right] \to 0$.

(2) If $\phi_t$ bounded, there exists a continuous function

$\rho_n(t)$ such that $\rho_n(t) = 0$ if $t \notin (-1/n, 0)$,

and $\int_{-\infty}^{\infty} \rho_n(t)\, dt = 1$.

Then $g_n(t, \omega) = \int_0^T \rho_n(s-t)\, \phi(s, \omega)\, ds$

is continuous and bounded, and (non-trivial ...!) in $\mathcal{V}([0,T])$, and (non-trivial)

$$\int_0^T (g_n(s,\omega) - \phi(s,\omega))^2 \longrightarrow 0 \text{ for each } \omega \in \Omega.$$

[Again .. here is the hand-waving]

So also in $\mathbb{E}$.

(3) Finally if $\phi_t$ unbounded, since $\mathbb{E}\int \phi_t^2\, dt < \infty$,

$$\mathbb{E}\left[\int_0^T (\phi_t - ((\phi_t \wedge n) \vee (-n)))^2\, dt\right] \longrightarrow 0$$

as $n \to \infty$.

So, given $\phi \in \mathcal{V}([0,T])$, we can find $\phi_n \in \mathcal{S}([0,T])$ such that $E\left[\int_0^T |\phi_n - \phi|^2\, dt\right] \to 0$.

But $\int_0^T \phi_n(t)\, dB_t \in L^2(\mathbb{P})$ for all $n \in \mathbb{N}$. But it follows that $\left\| \int \phi_n\, dB_t - \int \phi_m\, dB_t \right\|_2^2 = \mathbb{E}\left(\int_0^T (\phi_m - \phi_n)^2\, dt\right)$

$\longrightarrow 0$

ie. $\int \phi_n\, dB_t$ are a Cauchy sequence in $L^2(\mathbb{P})$, which is a complete space $\Rightarrow \exists!$ Limit!

Hence, for $\phi \in \mathcal{V}([0,T])$, we can <u>define</u> the Itô integral as the limit of the Itô integrals of approximating simple functions.

Then also:

Corrollary (Itô Isometry):

$$\mathbb{E}\left[\left(\int_0^T \phi_t \, dB_t\right)^2\right] = \mathbb{E}\left[\int_0^T \phi_t^2 \, dt\right] \qquad \forall \phi \in \mathcal{V}([0,T]).$$

Example: Assume (as usual) $B_0 = 0$. Then:

$$\int_0^t B_s \, dB_s = \frac{1}{2} B_t^2 - \frac{1}{2} t.$$

Proof: Take $\phi_n(t) := \sum B_{t_j} \mathbb{1}\{[t_j, t_{j+1})\}(t)$

then

$$\mathbb{E}\left[\int_0^T (\phi_n(t) - B_t)^2 dt\right] = \mathbb{E}\left[\sum \int_{t_j}^{t_{j+1}} (B_{t_j} - B_s)^2 ds\right]$$

$$= \sum \int_{t_j}^{t_{j+1}} (s - t_j) \, ds = \frac{1}{2} \sum (t_{j+1} - t_j)^2 \to 0.$$

$$\int_0^T \phi_n(t) \, dB_t = \sum B_{t_j}(B_{t_{j+1}} - B_{t_j})$$

$$= \frac{1}{2} \sum \left[B_{t_{j+1}}^2 - B_{t_j}^2 - (B_{t_{j+1}} - B_{t_j})^2\right]$$

$$= \frac{1}{2} \left[B_{t_N}^2 - B_{t_{N-1}}^2 + B_{t_{N-1}}^2 - B_{t_{N-2}}^2 + \dots - B_0^2\right]$$

$$- \frac{1}{2} \sum (B_{t_{j+1}} - B_{t_j})^2$$

But $\sum (B_{t_{j+1}} - B_{t_j})^2 \longrightarrow T$ in $L^2(\mathbb{P})$,

So $\int_0^T B_t \, dB_t = \frac{1}{2}(B_T^2 - T)$ !

Properties of the stochastic / Itô Integral:

(i) $\int_s^T f_t \, dB_t = \int_s^u f_t \, dB_t + \int_u^T f_t \, dB_t \qquad (s \leq u \leq T)$

(ii) Linear $\left( \int (\alpha f + \beta g)\, dB_t = \alpha \int f \, dB_t + \beta \int g \, dB_t, \; \alpha, \beta \in \mathbb{R} \right)$.

(iii) $\int_0^T f \, dB_t$ is $\mathcal{F}_T$-measurable

$X_t$ a version of $Y_t$ if $\mathbb{P}(X_t = Y_t) = 1 \;\forall t$.

(iv) There exists a continuous version of $t \mapsto \int_0^t f_s \, dB_s$ which is a martingale. (In future, we will work with this version).

More generally, a stochastic integral or an Itô process is a stochastic process $X_t$ such that:

$$X_t = X_0 + \int_0^t u(s,\omega)\, ds + \int_0^t v(s,\omega)\, dB_s ,$$

where $v$ is progressively meas. & $\mathbb{P}\left( \int_0^t v(s,\omega)^2 \, ds < \infty \;\forall t \geq 0 \right) = 1$.

We will often write this as

$$dX_t = u_t \, dt + v_t \, dB_t .$$

So $\int_0^T B_t \, dB_t = \frac{1}{2}(B_T^2 - T)$ is equivalently

$$d\left(\tfrac{1}{2} B_T^2\right) = \tfrac{1}{2} dt + B_t \, dB_t .$$

<u>Theorem</u> (Itô's formula). Let $X_t$ be an Itô process,

$$dX_t = u_t dt + v_t \, dB_t$$

Let $g(t,x) \in C^2([0,\infty) \times \mathbb{R})$. Then

$$Y_t = g(X_t, t)$$

is again an Itô process, and

$$dY_t = \left( \frac{\partial g}{\partial t}(t, X_t) + u_t \frac{\partial g}{\partial x}(t, X_t) + \frac{1}{2} v_t^2 \frac{\partial^2 g}{\partial x^2}(t, X_t) \right) dt + v_t \frac{\partial g}{\partial x}(t, X_t)\, dB_t .$$

which can also be thought of as:

$$dY_t = \frac{\partial g}{\partial t}(t, X_t)\, dt + \frac{\partial g}{\partial x}(t, X_t)\, dX_t + \frac{1}{2}\frac{\partial^2 g}{\partial x^2}(t, X_t)\cdot (dX_t)^2$$

where $(dX_t)^2$ is computed by $dt \cdot dt = dt \cdot dB_t = dB_t \cdot dt = 0$,
$dB_t \cdot dB_t = dt$.

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Sketch Proof: By approximation if necessary, we can assume $f$ and all its derivatives are bounded. Applying Taylor's theorem, we get:

$$g(t, X_t) - g(0, X_0) = \sum_j \frac{\partial g}{\partial t}\Delta t_j + \sum_j \frac{\partial g}{\partial x}\Delta X_j + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial t^2}(\Delta t_j)^2$$
$$+ \sum_j \frac{\partial^2 g}{\partial x \partial t}(\Delta t_j)(\Delta X_j) + \frac{1}{2}\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2$$
$$+ \sum_j R_j,$$

where all partial derivatives are evaluated at $(t_j, X_j)$; and $R_j = o((\Delta t)^2 + (\Delta X_j)^2)$. Since the functions are assumed bounded, we get

$$\sum_j \frac{\partial g}{\partial t}\Delta t_j \to \int \frac{\partial g}{\partial t}\, dt, \quad \sum_j \frac{\partial g}{\partial x}\Delta X_j \to \int_0^T \frac{\partial g}{\partial x}(s, X_s)\, dX_s$$

If also $u, v$ are simple:

$$\sum_j \frac{\partial^2 g}{\partial x^2}(\Delta X_j)^2 = \sum_j \frac{\partial^2 g}{\partial x^2} u_j^2 (\Delta t_j)^2 + 2\sum_j \frac{\partial^2 g}{\partial x^2} u_j v_j (\Delta t_j)(\Delta B_j)$$
$$+ \sum_j \frac{\partial^2 g}{\partial x^2} v_j^2 (\Delta B_j)^2,$$

and e.g.

$$\mathbb{E}\Big[\Big(\sum_j \frac{\partial^2 g}{\partial x^2} u_i v_j (\Delta t_i)(\Delta B_j)\Big)^2\Big]$$
$$= \sum_j \mathbb{E}\Big[\Big(\frac{\partial^2 g}{\partial x^2} u_i v_j\Big)^2\Big](\Delta t_i)^3 \to 0$$

as $(\Delta t) \to 0$

So the term we need to analyse is the last one.

Claim: $$\sum_j \frac{\partial^2 g}{\partial x^2} v_j^2 (\Delta B_j)^2 \to \int_0^t v_s^2 \frac{\partial^2 g}{\partial x^2}\, ds.$$

in $L^2(\mathbb{P})$.

Writing $a_t = v_t \frac{\partial^2 g}{\partial x^2}$, then:

$$\mathbb{E}\left[\left(\sum_j a_j (\Delta B_t)^2 - \sum_j a_j \Delta t_j\right)^2\right]$$

$$= \sum_{i,j} \mathbb{E}\left[a_i a_j ((\Delta B_i)^2 - \Delta t_i)((\Delta B_j)^2 - \Delta t_j)\right]$$

But for $i \neq j$, the terms vanish, and we get:

$$\sum_j \mathbb{E}\left[a_j^2 ((\Delta B_t)^2 - \Delta t)^2\right] = \sum_j \mathbb{E}[a_j^2] \cdot \mathbb{E}\underbrace{[\cdots]}_{=2(\Delta t)^2}$$

$$\longrightarrow 0 \quad \text{as } \Delta t \to 0.$$

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Martingale Representation Theorem:

Let $B_t$ be a BM, and $\mathcal{F}_t^B$ its natural filtration, and suppose $M_t$ an $\mathcal{F}_t^{(n)}$-martingale, and $M_t \in L^2(\mathbb{P})$ for all $t \geq 0$. Then there exists $g(s,\omega)$ such that:

$$M_t(\omega) = \mathbb{E}[M_0] + \int_0^t g(s,\omega)\, dB_s \quad \text{a.s. } \forall t \geq 0.$$

(Sketch proof)

Can show linear span of $\exp\left\{\int_0^T h_t\, dB_t - \frac{1}{2}\int_0^T h_t^2\, dt\right\}$, $h \in L^2([0,T])$ (deterministic), is dense in $L^2(\mathcal{F}_T^B, \mathbb{P})$.

But if $F = \exp\left\{\int_0^T h_s\, dB_s - \frac{1}{2}\int_0^T h_s^2\, ds\right\}$

$$\Rightarrow \quad F = Y_T, \quad Y_t = \exp\left\{ \int_0^t h_s \, dB_s - \tfrac{1}{2}\int_0^t h_s^2 \, ds \right\}$$

$$\text{Itô} \Rightarrow dY_t = Y_t h_t \, dB_t + \tfrac{1}{2} Y_t h_t^2 \, dt - \tfrac{1}{2} Y_t h_t^2 \, dt$$

$$= Y_t h_t \, dB_t,$$

So can approximate a dense set of $F \in L^2(P)$ by martingales. Use Itô isometry to extend from dense set to all $F$.

Finally, given $M$, using $F = M_T$, we get

$$M_t = \mathbb{E}[M_T \mid \mathcal{F}_t] = \mathbb{E}\left[ \mathbb{E}[M_T] + \int_0^T \phi_s \, dB_s \mid \mathcal{F}_t \right]$$

$$= M_0 + \int_0^t \phi_s \, dB_s + \mathbb{E}\left[\int_t^T \phi_s \, dB_s \mid \mathcal{F}_t\right]$$

$$= M_0 + \int_0^t \phi_s \, dB_s,$$

So $M_t = M_0 + \int_0^t \phi_s \, dB_s$. $\square$.

## §3. Stochastic Differential Equations:

An SDE is an equation of the form:

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t, \quad X_0 \text{ given.}$$

**Theorem**: Fix $T > 0$, sse $b(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$, $\sigma(\cdot,\cdot) : [0,T] \times \mathbb{R} \to \mathbb{R}$ measurable s.t

$$|b(t,x)| + |\sigma(t,x)| \leq C(1 + |x|)$$

and

$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq D|x - y|$$

for some constants, $C, D$. Let $Z$ be an r.v.

independent of $\mathcal{F}_\infty^B$, $E|Z|^2 < \infty$.

Then the SDE

$$\begin{cases} dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dB_t \\ 0 \le t \le T, \; X_0 = Z \end{cases}$$

has a unique $t$-continuous solution $X_t(\omega)$ adapted to $\mathcal{F}_t^Z$, gen by $Z, B$, and

$$E\left[\int_0^T |X_t|^2\, dt\right] < \infty.$$

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Note that the bounded & Lipschitz conditions are natural: •) $dX_t = X_t^2\, dt \Rightarrow X_t = \frac{1}{1-t}$, explodes...

•) $dX_t = 3 X_t^{2/3}\, dt$, $X_0 = 0$ has solutions:

$$X_t = \begin{cases} 0 & t \le a \\ (t-a)^3 & t > a \end{cases}$$

for any $a > 0$.

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Proof

(Uniqueness:) Let $X^1, X^2$ be solutions:

$$E|X_T^1 - X_T^2|^2 = E\Big[\Big(\int_0^T \underbrace{[b(s, X_s^1) - b(s, X_s^2)]}_{:= a_s} ds + \int_0^T \underbrace{(\sigma(s, X_s^1) - \sigma(s, X_s^2))}_{:= \gamma_s} dB_s\Big)^2\Big].$$

$$\le 2\, E\Big[\Big(\int_0^T a\, ds\Big)^2\Big] + 2\, E\Big[\Big(\int_0^T \gamma_s\, dB_s\Big)^2\Big]$$

$$\le 2T\, E\Big[\int_0^T a^2\, ds\Big] + 2\, E\Big[\int_0^T \gamma_s^2\, ds\Big]$$

$$\leq 2(T+1)D^2 \mathbb{E}\left[\int_0^T |X_s^1 - X_s^2|^2 \, ds\right]$$

$$\leq 2(T+1)\, D^2 \int_0^T \mathbb{E}\left[|X_s^1 - X_s^2|^2\right] ds.$$

i.e. if $v(s) = \mathbb{E}|X_s^1 - X_s^2|^2$,

$$V(t) \leq C\int_0^t v(s)\, ds,$$

and by Gronwall's inequality: $v(t) = 0 \quad \forall t \geq 0$.

Path continuity $\Rightarrow X^1 = X^2$ for all $t \in [0,T]$ a.s..

(Existence) Define $Y_t^{(0)} = X_0$,

$$Y_t^{(k+1)} = X_0 + \int_0^t b(s, Y_s^k)\, ds + \int_0^t \sigma(t, Y_t^k)\, dB_s.$$

Then $\mathbb{E}|Y_t^{k+1} - Y_t^k|^2 \leq (1+T)\, 2D^2 \int_0^t \mathbb{E}|Y_s^{k+1} - Y_s^k|^2 ds$

for $k \geq 1$ and

$$\mathbb{E}|Y_t^1 - Y_t^0|^2 \leq 4C^2t^2(1 + \mathbb{E}|X_0|^2)$$

$$\leq A_1 t$$

By induction on $k$, get

$$\mathbb{E}|Y_t^{k+1} - Y_t^k|^2 \leq \frac{A_2^{k+1}\; t^{k+1}}{(k+1)!}, \quad k \geq 0,\ t \in [0,T].$$

So if $\|r_s\| = \mathbb{E}\left[\int_0^T r_s^2 \, ds\right]^{1/2}$,

$$\|Y_t^{(m)} - Y_t^n\| = \left\|\sum_{k=n}^{m-1}(Y_t^{k+1} - Y_t^k)\right\|$$

$$\leq \sum_{k=n}^{m-1} \| Y_t^{k+1} - Y_t^k \|$$

$$\leq \sum \left( \int_0^T \frac{A_2^{k+1} \, t^{k+1}}{(k+1)!} \, dt \right)^{1/2}$$

$$= \sum \frac{A_2^{k+1} \, T^{k+2}}{(k+2)!} \longrightarrow 0$$

Therefore $Y_t^n$ a Cauchy sequence & convergent. By Hölder inequality & Itô isometry, integrals converge, so $X_t = \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s$ a.s.

(also, $X_t$ is $\mathcal{F}_t^Z$-meas. & can be chosen to be continuous.)

□

Weak + Strong solutions:

Above, we've been given a BM & a filtration. Such a solution is called a strong solution. In some circumstances, we may not be able to do this, but, given $\sigma$, $b$, we may be able to construct a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(B_t, X_t, \mathcal{F}_t)$ such that $X_t$ solves the SDE.

E.g. Tanaka equation: $dX_t = \mathrm{sign}(X_t)\, dB_t$.

## §4 Diffusions

A (time-homogeneous) Itô diffusion is a solution to the SDE:

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dB_t$$

for $t \geq s$, $X_s = x$, $X_t \in \mathbb{R}^n$, $b : \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : \mathbb{R}^n \to \mathbb{R}^{n \times m}$, $B_t$ an $m$-dim. B.M., and $\sigma, b$ are Lipschitz.

NB: often: $X_t = X_t^{s,x}$, $X_t^x = X_t^{0,x}$.

Now $\{X^{s,x}_{s+h}\}_{h \geq 0}$ and $\{X^{0,x}_h\}_{h\geq 0}$ have same distributions.

(Strong)

**Thm** (Markov Property) $f$ bounded Borel, $\mathbb{R}^n \to \mathbb{R}$. Then, for $t, h \geq 0$ $E^x[f(X_{t+h}) \mid \mathcal{F}_t^{(m)}] = E^{X_t(\omega)}[f(X_h)]$ where $\mathcal{F}_t^{(m)}$ is the filtration generated by the driving BM & $\tau$ a $\mathcal{F}_t^{(m)}$-stopping time.

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Let $\{X_t\}$ be a time-homog. diffusion on $\mathbb{R}^n$. Then the (infinitesimal) generator $A$ of $X_t$ given by

$$Af(x) = \lim_{t \downarrow 0} \frac{E^x[f(X_t)] - f(x)}{t} \qquad x \in \mathbb{R}^n$$

The set of functions $f : \mathbb{R}^n \to \mathbb{R}$ such that the limit exists for all $x \in \mathbb{R}^n$ is $\mathcal{D}_A$.

Lemma: Let $X_t$ be an Itô process as above, and $f \in C_0^2(\mathbb{R}^n)$ (twice diff. & compact support), and $\tau$ a stopping time s.t. $\mathbb{E}^x[\tau] < \infty$, then

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau \left(\sum_i b_i(X_s)\frac{\partial f}{\partial x_i}(X_s) + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}\frac{\partial^2 f}{\partial x_i \partial x_j}(X_s)\,ds\right)\right]$$

Sketch proof: (in 1-D)

Applying Itô to $f(X_s)$, and taking expectations, we get the above formula plus the term:

$$\mathbb{E}^x\left[\int_0^\tau \sigma(X_s)\frac{\partial f}{\partial x}\,dB_s\right]$$

By the martingale property, this is 0 if $\tau$ bounded. In general,

$$\mathbb{E}^x\left[\left(\left(\int_0^\tau - \int_0^{\tau\wedge k}\right)\sigma(X_s)\frac{\partial f}{\partial x_s}\,dB_s\right)^2\right]$$

$$\leq \mathbb{E}^x\int_{\tau\wedge k}^\tau \sigma(X_s)\frac{\partial f}{\partial x_s}\,ds$$

$$\leq M\mathbb{E}(\tau - \tau\wedge k) \to 0$$

as $k \to \infty$.

$\square$

Corrolorary:

If $f \in C_0^2(\mathbb{R}^n)$, $f \in D_A$ and

$$Af(x) = \sum_i b_i(x)\frac{\partial f}{\partial x_i} + \frac{1}{2}\sum_{i,j}(\sigma\sigma^T)_{i,j}(x)\frac{\partial^2 f}{\partial x_i \partial x_j}.$$

Corrollorary II: Dynkin's Formula

Let $f \in C_0^2(\mathbb{R}^n)$, and $\tau$ a stopping time s.t. $\mathbb{E}^x[\tau] < \infty$. Then:

$$\mathbb{E}^x[f(X_\tau)] = f(x) + \mathbb{E}^x\left[\int_0^\tau Af(X_s)\,ds\right].$$

Kolmogorov's Backward Equation:

Let $f \in C_0^2(\mathbb{R}^n)$, $X_t$ as above. Then

$$u(t,x) := \mathbb{E}^x[f(X_t)]$$

satisfies $u(t,\cdot) \in D_A \quad \forall t$,

$$\left.\begin{aligned} \frac{\partial u}{\partial t} &= Au & t>0,\ x \in \mathbb{R}^n \\ u(0,x) &= f(x) & x \in \mathbb{R}^n. \end{aligned}\right\} \textcircled{A}$$

Moreover, if $w \in C^{1,2}(\mathbb{R} \times \mathbb{R}^n)$ a bounded fn satisfying $\textcircled{A}$, then $w = u$.

Sketch proof: Fix $t$, let $g(x) = u(t,x)$. Then:

$$\frac{\mathbb{E}^x[g(X_r)] - g(x)}{r} = \frac{1}{r}\,\mathbb{E}^x\left[\mathbb{E}^{X_r}[f(X_t)] - \mathbb{E}^x f(X_t)\right]$$

$$= \frac{1}{r}\mathbb{E}^x\left[\mathbb{E}^x[f(X_{t+r}) \mid \mathcal{F}_r] - \mathbb{E}^x[f(X_t) \mid \mathcal{F}_r]\right]$$

$$= \frac{1}{r}\mathbb{E}^x[f(X_{t+r}) - f(X_t)]$$

$$= \frac{u(t+r,x) - u(t,x)}{r} \rightarrow \frac{\partial u}{\partial r} \text{ as } r \downarrow 0$$

But $Au = \lim_{r\downarrow 0} \dfrac{\mathbb{E}^x[g(X_r)] - g(x)}{r}$, so $Au = \dfrac{\partial u}{\partial r}$.

Uniqueness: Apply Dynkin's formula to $Y_t$ in $\mathbb{R}^{n+1}$,

$$Y_t = (s-t,\ X_t^{0,x}),$$

which has generator $\tilde{A}w = -\frac{\partial w}{\partial t} + Aw$,

so $$w(s,x) = \mathbb{E}^{s,x}[w(Y_t)]$$

$$t=s \Rightarrow w(s,x) = \mathbb{E}^{s,x}[f(X_s)].$$

□

Note that the trick of moving from time-homogeneous diffusions to space-time diffusions similar to above is quite useful!

Similar results---

Feynman-Kac:

$f \in C_0^2(\mathbb{R}^n)$, $q \in C(\mathbb{R}^n)$, $q$ bounded below.

Then $$v(t,x) := \mathbb{E}^x\left[\exp\left\{-\int_0^t q(X_s)ds\right\} f(X_t)\right]$$

solves $$\frac{\partial v}{\partial t} = Av - qv \qquad t>0,\ x\in\mathbb{R}^n$$

and $$v(0,x) = f(x).$$

Poisson-Dirichlet:

Suppose $D$ a domain in $\mathbb{R}^n$, $\phi \in C(\partial D)$ bounded,

$g \in C(D)$, $\mathbb{E}^x \int_0^{\tau_D} |g(X_t)|dt$, $\forall x$,

$$\tau_D = \inf\{t \geq 0 : X_t \notin D\},$$

and $w \in C^2(D)$ bounded, solving

(i) $\quad Lw = -g \quad$ in $D$

(ii) $\quad \lim_{t \uparrow \tau_D} w(X_t) = \phi(X_{\tau_D}) \mathbb{1}_{\{\tau_D < \infty\}}$

$\mathbb{P}^x$ a.s. $\forall x$,

Then $$w(x) = \mathbb{E}^x\left[\phi(X_{\tau_D}) \mathbb{1}_{\{\tau_D < \infty\}}\right] + \mathbb{E}^x\left[\int_0^{\tau_D} g(X_t)\,dt\right]$$

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## §5. Girsanov's Theorem:

Let $Y_t \in \mathbb{R}^n$ be an Itô process of form:

$$dY_t = \beta_t\,dt + \theta_t\,dB_t, \qquad t \leq T$$

$B_t \in \mathbb{R}^m$, $\beta_t \in \mathbb{R}^n$, $\theta_t \in \mathbb{R}^{n \times m}$, such that there exist progressively meas. $u_t, \alpha_t$ where

$$\theta_t u_t = \beta_t - \alpha_t .$$

Set $$M_t = \exp\left\{-\int_0^t u_s\,dB_s - \frac{1}{2}\int_0^t u_s^2\,ds\right\}$$

If $M_t$ a martingale

$\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$ on $\mathcal{F}_T$, then $\mathbb{Q}$ a probability measure, and under $\mathbb{Q}$,

$$\hat{B}_t := \int_0^t u_s\,ds + B_t$$

is a Brownian motion, and

$$dY_t = \alpha_t\,dt + \theta_t\,d\hat{B}_t$$

Note: $\frac{d\mathbb{Q}}{d\mathbb{P}} = M_T$ on $\mathcal{F}_T$ means, for

$$A \in \mathcal{F}_T, \quad \mathbb{Q}(A) = \mathbb{E}^{\mathbb{P}}[\mathbb{1}_A \cdot M_T].$$

Sketch Proof: First note that

$$dM_t = -u_t M_t \, dB_t + \tfrac{1}{2} u_t^2 M_t \, dt - \tfrac{1}{2} u_t^2 M_t \, dt$$

$$= -u_t M_t \, dB_t$$

So $M_t$ a $\mathbb{P}$-martingale, modulo (assumed) integrability conditions, and then $M_0 = 1 \Rightarrow$ $\mathbb{Q}(\Omega) = \mathbb{E}^{\mathbb{P}}[1 \cdot M_T] = 1$, other properties of p.m. also follow.

Now, consider 1-D, and

$$d(M_t \hat{B}_t) = M_t \, d\hat{B}_t + \hat{B}_t \, dM_t + dM_t \, d\hat{B}_t$$

$$= M_t (dB_t + u_t \, dt) - \hat{B}_t u_t M_t \, dB_t - u_t M_t \, dt$$

$$= (M_t - \hat{B}_t u_t M_t) \, dB_t.$$

So, under $\mathbb{Q}$, $\hat{B}_t$ is a continuous (local-) martingale. Moreover: $d(\hat{B}_t^2) = 2\hat{B}_t \, d\hat{B}_t + dt$

$$d(M_t \hat{B}_t^2) = 2M_t \hat{B}_t \, d\hat{B}_t - M_t \, dt - u_t M_t \hat{B}_t^2 \, dB_t - 2M_t u_t \hat{B}_t \, dt$$

$$= 2M_t \hat{B}_t \, dB_t + 2M_t \hat{B}_t u_t \, dt + M_t \, dt$$

$$- u_t M_t \hat{B}_t^2 \, dB_t - 2M_t \hat{B}_t u_t \, dt$$

$$\Rightarrow \quad \mathbb{E}^{\mathbb{Q}}\left[\hat{B}_t^2\right] = \mathbb{E}^{P}\left[M_t B_t^2\right]$$

$$= \mathbb{E}\left[\int_0^t M_t \, dt\right]$$

$$= \int_0^t \mathbb{E}[M_t] \, dt = \int_0^t 1 \, dt = t.$$

Hence, under $\mathbb{Q}$, $\hat{B}_t$, $\hat{B}_t^2 - t$ are martingales

$$\Rightarrow \quad \hat{B}_t \text{ a } \mathbb{Q}-\text{BM}$$

$$dY_t = \beta_t \, dt + \theta_t \, dB_t$$

$$= \beta_t \, dt + \theta_t \left( d\hat{B}_t - u_t \, dt \right)$$

$$= \left( \beta_t - \theta_t u_t \right) dt + \theta_t \, d\hat{B}_t$$

$$= \alpha_t \, dt + \theta_t \, d\hat{B}_t.$$

$\square$