Neutron Transport Equation - discussion

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Monte Carlo Methods for the Neutron Transport Equation
Cox, Harris, Kyprianou & Wang
arXiv:2012.02864

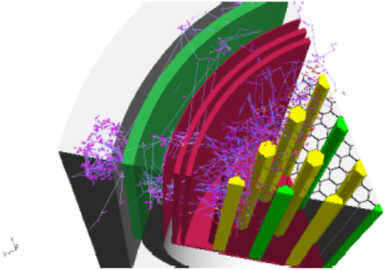

FIGURE 11. *An example of a slice through a mock nuclear reactor design in which some neutron path simulations are depicted. The reactor core displays symmetry.*

Cox et al. 2020 - as per lectures

- **Semigroup** associated with the Neutron Random Walk

$$\phi_t[g](r,v) := \mathrm{E}_{(r,v)}\left[e^{\int_0^t \beta(R_s,\Upsilon_s)\mathrm{d}s} g(R_t,\Upsilon_t)\mathbf{1}(t<\tau_D)\right] \tag{1}$$

- **Principal eigenvalue** $e^{\lambda_\star}$, $\lambda_\star := \lim_{t\to\infty}\frac{1}{t}\log\phi_t[1](r,v)$
- Some **Monte Carlo strategies** to approximate $\phi_t[g](r,v)$ and hence $\lambda_\star$
  1. Approximate the expectation in (1) by iid averaging,
  2. "Importance Sampling": approximate(1) by iid averaging under and suitable change of measure,
  3. Simulate iid replicates of the Neutron Branching Process (NBP) itself,
  4. ...
- **Complexity analysis**: cost/accuracy trade-off as a function t, number of simulation replicates and $\lambda_\star$ itself

Remark 1: Interacting Particle Systems/SMC for approximating $\phi_t[g](r, v)$

$$\phi_t[g](r,v) := \mathsf{E}_{(r,v)}\left[e^{\int_0^t \beta(R_s,\Upsilon_s)\mathrm{d}s} g(R_t,\Upsilon_t)1(t<\tau_D)\right]$$

- **Applicability to NTE, quasi-stationary distributions, Yaglom limits** etc.:
  - Del Moral, P., & Miclo, L. (2003). Particle approximations of Lyapunov exponents connected to Schrödinger operators and Feynman–Kac semigroups. ESAIM: Probability and Statistics, 7, 171-208.
  - Del Moral, P., & Doucet, A. (2004). Particle motions in absorbing medium with hard and soft obstacles. Stochastic Analysis and Applications, 22(5), 1175-1207.
  - Rousset, M. (2006). On the control of an interacting particle estimation of Schrödinger ground states. SIAM journal on mathematical analysis, 38(3), 824-844.

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- **Non-asymptotic and asymptotic error analyses for IPS** relevant to complexity analysis.
- N is number of particles, t is time-horizon
  - **N, t both finite.** Cérou, F., Del Moral, P., & Guyader, A. (2011). A nonasymptotic theorem for unnormalized Feynman-Kac particle models. In Annales de l'IHP Probabilités et statistiques (Vol. 47, No. 3, pp. 629-649).
  - **t finite, and $N \to \infty$.** Too many references to mention - see Pierre's book(s)!
  - **N finite, $t \to \infty$.** Whiteley, N., & Lee, A. (2014). Twisted particle filters. The Annals of Statistics, 42(1), 115-141.
  - **$N \to \infty$ and $t \to \infty$ simultaneously.** Bérard, J., Del Moral, P., & Doucet, A. (2014). A lognormal central limit theorem for particle approximations of normalizing constants. Electronic Journal of Probability, 19, 1-28.

Remark 2: Dealing with hard killing

- In Neutron Branching Process a particle is **killed if it leaves the domain** D.
- This corresponds to the indicator function in

$$\phi_t[g](r,v) := \mathrm{E}_{(r,v)}\left[e^{\int_0^t \beta(R_s,\Upsilon_s)\mathrm{d}s} g(R_t,\Upsilon_t)1(t<\tau_D)\right]$$

- **Variance reduction**. Theoretically well-grounded IPS/SMC algorithms designed to deal with killing:
  - Le Gland, F. & Oudjane, N. (2006). A sequential particle algorithm that keeps the particle system alive. In Stochastic Hybrid Systems : Theory and Safety Critical Applications, (H. Blom & J. Lygeros, Eds), Lecture Notes in Control and Information Sciences 337, 351–389, Springer: Berlin
  - Moral, P. D., Jasra, A., Lee, A., Yau, C., & Zhang, X. (2015). The alive particle filter and its use in particle Markov chain Monte Carlo. Stochastic Analysis and Applications, 33(6), 943-974.
  - Persing, A., & Jasra, A. (2016). Twisting the alive particle filter. Methodology and Computing in Applied Probability, 18(2), 335-358.

Algorithm 1: Alive Particle Filter

1. At time 1. For $j = 1, 2, \ldots$ until $j =: T_1$ is reached such that $G_1(x_1^j) = 1$ and $\sum_{i=1}^{j} G_1(x_1^i) = N$:
   - Sample x_1^j from $M_1(x_0, \cdot)$.
2. At time $1 < p \leq n$. For $j = 1, 2, \ldots$ until $j =: T_p$ is reached such that $G_p(x_p^j) = 1$ and $\sum_{i=1}^{j} G_p(x_p^i) = N$:
   a. Sample a_{p-1}^j uniformly from $\{k \in \{1, \ldots, T_{p-1} - 1\} : G_{p-1}(x_{p-1}^k) = 1\}$.
   b. Sample x_p^j from $M_p(x_{p-1}^{a_{p-1}^j}, \cdot)$.

Remark 3: Towards parallel/distributed computing

- Interaction in IPS/resampling in SMC $\rightsquigarrow$ **time-uniform control on errors associated with empirical measure**
- ...but makes algorithm **difficult to parallelize**
  - Bolic, M., Djuric, P. M., & Hong, S. (2005). Resampling algorithms and architectures for distributed particle filters. IEEE Transactions on Signal Processing, 53(7), 2442-2450.
  - Vergé, C., Dubarry, C., Del Moral, P., & Moulines, E. (2015). On parallel implementation of sequential Monte Carlo methods: the island particle model. Statistics and Computing, 25(2), 243-260.
  - Whiteley, N., Lee, A., & Heine, K. (2016). On the role of interaction in sequential Monte Carlo algorithms. Bernoulli, 22(1), 494-529.
  - Lee, A., & Whiteley, N. (2016). Forest resampling for distributed sequential Monte Carlo. Statistical Analysis and Data Mining: The ASA Data Science Journal, 9(4), 230-248.
  - Heine, K., & Whiteley, N. (2017). Fluctuations, stability and instability of a distributed particle filter with local exchange. Stochastic Processes and their Applications, 127(8), 2508-2541.
  - Sen, D., & Thiery, A. H. (2019). Particle filter efficiency under limited communication. arXiv preprint arXiv:1904.09623.

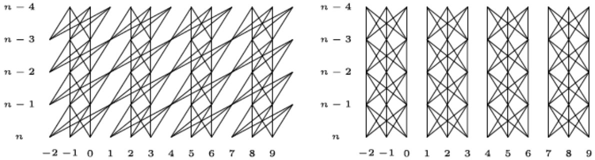
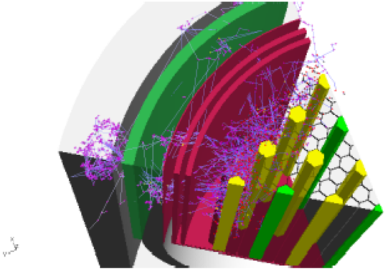


Fig. 2. Some of the paths assigned positive probability by α_∞ for the (a) LEPF and (b) IBPF. In both cases $M = 3$ and in (a) $\theta = 1$.

Remark 4: beyond computing the eigen-triple - what is of interest?

- Alex highlighted the two applications areas:
  - **Criticality**
  - **Shielding Problem**
- Beyond just calculating eigen-quantities
  - optimization?
  - sensitivity of e.g. $\lambda_\star$ to physical parameters? $\nabla_\theta \lambda_\star(\theta)$ of interest?
  - sensitivity of rare event probabilities to physical parameters?
- Beyond simulation
  - inference?
  - incorporation of measured data?



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