# Stochastic analysis of the neutron transport equation

Emma Horton
Inria, Bordeaux

16th September 2021

Based on joint work with Alex Cox (University of Bath), Isaac Gonzalez Garcia (University of Bath), Simon Harris (University of Auckland), Andreas Kyprianou (University of Bath), Denis Villemonais (Université de Lorraine) and Minmin Wang (University of Sussex). Funded by Jacobs.

# Contents

# Contents

# Martingale behaviour

Recall the martingale

$$W_t^1 := \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi, X_t \rangle}{\varphi(r, \upsilon)}.$$

This is a non-negative martingale and thus has an almost sure limit, $W_\infty$.

(H4) There exists $B \subset D$, open and compactly embedded in $D$, such that

$$\inf_{r \in B, \upsilon, \upsilon' \in V} \sigma_{\mathtt{f}}(r, \upsilon) \pi_{\mathtt{f}}(r, \upsilon, \upsilon') > 0.$$

## Theorem (H., Kyprianou, Villemonais)

Suppose that (H1)-(H4) hold. Then

- If $\lambda_* \leq 0$, $W_\infty = 0$, almost surely.
- If $\lambda_* > 0$, then $(W_t^1)_{t \geq 0}$ is $L^2(\mathbb{P})$ convergent.
- Irrespective of the sign of $\lambda_*$, $\{W_\infty = 0\} = \{\zeta < \infty\}$ almost surely, where $\zeta = \inf\{t > 0 : \langle 1, X_t \rangle = 0\}$ is the time of extinction of the NBP.

# Contents

- Let $p_n(r, \upsilon, t_0; R, t_f)$ denote the probability that a neutron with configuration $(r, \upsilon)$ at time $t_0$ will lead to exactly $n$ neutrons in $R \subset D \times V$ at time $t_f$.
- Further let us introduce the probability generating function

$$G(z; r, \upsilon, t) = \sum_{n=0}^{\infty} z^n p_n(r, \upsilon, t; R, t_f).$$

## Pàl-Bell equation

Then $G$ satisfies

$$\begin{aligned} \frac{\partial G(z; r, \upsilon, t)}{\partial t} &= \upsilon \cdot \nabla_r G(z; r, \upsilon, t) - (\sigma_{\mathtt{s}}(r, \upsilon) + \sigma_{\mathtt{f}}(r, \upsilon)) G(z; r, \upsilon, t) \\ &+ \sigma_{\mathtt{s}}(r, \upsilon) \int_V G(z; r, \upsilon', t) \pi_{\mathtt{s}}(r, \upsilon, \upsilon') \mathrm{d}\upsilon' \\ &+ \sigma_{\mathtt{f}}(r, \upsilon) \sum_{\substack{j=1 \\ j \neq 1}}^{M} c_i(r, \upsilon) \left( \int_V G(z; r, \upsilon', t) \mathrm{d}\upsilon' \right)^i \end{aligned} \tag{1}$$

# Non-linear equation

For measurable functions $f : D \times V \to [0,1]$, $(r, \upsilon) \in D \times V$ and $t \geq 0$, define the non-linear semigroup

$$u_t[f](r,\upsilon) = \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\prod_{i=1}^{N_t} f(r_i(t), \upsilon_i(t))\right],$$

and the non-linear branching mechanism

$$\mathcal{G}[f](r,\upsilon) = \sigma_{\mathtt{f}}(r,\upsilon)\mathcal{E}_{(r,\upsilon)}\left[\prod_{i=1}^{N} f(r, \upsilon_i)\right].$$

## Lemma (Harris, H., Kyprianou)

Under (H1) and (H2), $u_t[f](r,\upsilon)$ is the unique solution to

$$\begin{aligned} u_t[g](r,\upsilon) = g(r+\upsilon(t\wedge\kappa_{(r,\upsilon)}),\upsilon) - \int_0^t (\sigma_{\mathtt{s}}(r+\upsilon s,\upsilon) + \sigma_{\mathtt{f}}(r+\upsilon s,\upsilon))u_{t-s}[g](r+\upsilon s,\upsilon)\mathrm{d}s \\ + \int_0^t \sigma_{\mathtt{s}}(r+\upsilon s,\upsilon)\int_V u_{t-s}[g](r+\upsilon s,\upsilon')\pi_{\mathtt{s}}(r+\upsilon s,\upsilon,\upsilon')\mathrm{d}s \\ + \int_0^t \sigma_{\mathtt{f}}(r+\upsilon s,\upsilon)\mathcal{G}[u_{t-s}](r+\upsilon s,\upsilon)\mathrm{d}s. \qquad (2) \end{aligned}$$

# Non-linear equation

For measurable functions $f : D \times V \to [0,1]$, $(r, \upsilon) \in D \times V$ and $t \geq 0$, define the non-linear semigroup

$$u_t[f](r,\upsilon) = \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\prod_{i=1}^{N_t} f(r_i(t), \upsilon_i(t))\right],$$

and the non-linear branching mechanism

$$\mathcal{G}[f](r,\upsilon) = \sigma_{\mathtt{f}}(r,\upsilon)\mathcal{E}_{(r,\upsilon)}\left[\prod_{i=1}^{N} f(r,\upsilon_i)\right].$$

## Lemma (Harris, H., Kyprianou)

Under (H1) and (H2), $u_t[f](r,\upsilon)$ is the unique solution to

$$\begin{aligned} u_t[g](r,\upsilon) = g(r+\upsilon(t\wedge\kappa_{(r,\upsilon)}),\upsilon) - \int_0^t (\sigma_{\mathtt{s}}(r+\upsilon s,\upsilon) + \sigma_{\mathtt{f}}(r+\upsilon s,\upsilon))u_{t-s}[g](r+\upsilon s,\upsilon)\mathrm{d}s \\ + \int_0^t \sigma_{\mathtt{s}}(r+\upsilon s,\upsilon)\int_V u_{t-s}[g](r+\upsilon s,\upsilon')\pi_{\mathtt{s}}(r+\upsilon s,\upsilon,\upsilon')\mathrm{d}s \\ + \int_0^t \sigma_{\mathtt{f}}(r+\upsilon s,\upsilon)\mathcal{G}[u_{t-s}](r+\upsilon s,\upsilon)\mathrm{d}s. \end{aligned} \tag{2}$$

# Non-linear equation

Now define

$$\mathcal{H}[g](r,\upsilon) := \sigma_{\mathtt{f}}(r,\upsilon)\mathcal{E}_{(r,\upsilon)}\left[1-\prod_{i=1}^{N}\big(1-g(r,\upsilon_i)\big)-\sum_{i=1}^{N}g(r,\upsilon_i)\right].$$

## Lemma (Harris, H., Kyprianou, Wang)

Defining $v_t = 1 - u_t$, we have

$$v_t[g](r,\upsilon) = \psi_t[1-g](r,\upsilon) + \int_0^t \psi_s\Big[\mathcal{H}[v_{t-s}[g]]\Big](r,\upsilon)\mathrm{d}s, \qquad (3)$$

for any measurable $g : D \times V \to [0,1]$ and $(r,\upsilon) \in D \times V$.

# Contents

# One more assumption

(H5) Define

$$\mathbb{W}[g](r,\upsilon) = \mathcal{E}_{(r,\upsilon)}\Big[\sum_{\substack{i,j=1\\ i\neq j}}^{N} g(r,\upsilon_i)g(r,\upsilon_j)\Big]$$

There exists a constant $C > 0$ such that for all $g \in L^+_\infty(D \times V)$,

$$\langle \tilde{\varphi}, \sigma_{\mathtt{f}} \mathbb{W}[g]\rangle \geq C\langle \tilde{\varphi}, \bar{g}^2\rangle,$$

where $\bar{g} : D \to [0,\infty) : r \mapsto \displaystyle\int_V g(r,\upsilon)\mathrm{d}\upsilon.$

# Yaglom theorem

**Theorem (Harris, H., Kyprianou, Wang)**

Suppose $(H1) - (H5)$ hold and $\lambda_* = 0$. For $f \in L^+_\infty(D \times V)$ and for all $r \in D$ and $\upsilon \in V$,

$$\lim_{t\to\infty} \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\exp\left(-\theta\frac{\langle f, X_t\rangle}{t}\right)\middle| N_t > 0\right] = \frac{1}{1 + \langle\tilde{\varphi}, f\rangle\Sigma\theta/2},$$

where $\Sigma = \langle\tilde{\varphi}, \sigma_{\mathtt{f}}\mathbb{W}[\varphi]\rangle$. In other words, the law of the process conditioned on survival is asymptotically equivalent to an exponential distribution with parameter $(\Sigma/2)\langle\tilde{\varphi}, f\rangle$.

Plan: show that the moments of the conditioned distribution converge to the moments of an exponential distribution.

Note that

$$\lim_{t\to\infty}\frac{1}{t^k}\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k | N_t > 0] = \lim_{t\to\infty}\frac{1}{t^k}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k]}{\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)}.$$

# Yaglom theorem

## Theorem (Harris, H., Kyprianou, Wang)

Suppose $(H1) - (H5)$ hold and $\lambda_* = 0$. For $f \in L^+_\infty(D \times V)$ and for all $r \in D$ and $\upsilon \in V$,

$$\lim_{t\to\infty} \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\exp\left(-\theta\frac{\langle f, X_t\rangle}{t}\right)\Bigg| N_t > 0\right] = \frac{1}{1 + \langle\tilde{\varphi}, f\rangle\Sigma\theta/2},$$

where $\Sigma = \langle\tilde{\varphi}, \sigma_{\mathtt{f}}\mathbb{V}[\varphi]\rangle$. In other words, the law of the process conditioned on survival is asymptotically equivalent to an exponential distribution with parameter $(\Sigma/2)\langle\tilde{\varphi}, f\rangle$.

Plan: show that the moments of the conditioned distribution converge to the moments of an exponential distribution.

Note that

$$\lim_{t\to\infty}\frac{1}{t^k}\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k | N_t > 0] = \lim_{t\to\infty}\frac{1}{t^k}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k]}{\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)}.$$

# Yaglom theorem

## Theorem (Harris, H., Kyprianou, Wang)

Suppose $(H1) - (H5)$ hold and $\lambda_* = 0$. For $f \in L^+_\infty(D \times V)$ and for all $r \in D$ and $\upsilon \in V$,

$$\lim_{t\to\infty} \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\exp\left(-\theta\frac{\langle f, X_t\rangle}{t}\right)\middle| N_t > 0\right] = \frac{1}{1 + \langle\tilde{\varphi}, f\rangle\Sigma\theta/2},$$

where $\Sigma = \langle\tilde{\varphi}, \sigma_{\mathtt{f}}\mathbb{W}[\varphi]\rangle$. In other words, the law of the process conditioned on survival is asymptotically equivalent to an exponential distribution with parameter $(\Sigma/2)\langle\tilde{\varphi}, f\rangle$.

Plan: show that the moments of the conditioned distribution converge to the moments of an exponential distribution.

Note that

$$\lim_{t\to\infty} \frac{1}{t^k}\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k | N_t > 0] = \lim_{t\to\infty} \frac{1}{t^k}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k]}{\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)}.$$

# Yaglom theorem

## Theorem (Harris, H., Kyprianou, Wang)

Suppose $(H1) - (H5)$ hold and $\lambda_* = 0$. For $f \in L^+_\infty(D \times V)$ and for all $r \in D$ and $\upsilon \in V$,

$$\lim_{t\to\infty} \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\exp\left(-\theta\frac{\langle f, X_t\rangle}{t}\right)\Bigg| N_t > 0\right] = \frac{1}{1 + \langle\tilde\varphi, f\rangle\Sigma\theta/2},$$

where $\Sigma = \langle\tilde\varphi, \sigma_{\mathtt{f}}\mathbb{V}[\varphi]\rangle$. In other words, the law of the process conditioned on survival is asymptotically equivalent to an exponential distribution with parameter $(\Sigma/2)\langle\tilde\varphi, f\rangle$.

Plan: show that the moments of the conditioned distribution converge to the moments of an exponential distribution.

Note that

$$\lim_{t\to\infty}\frac{1}{t^k}\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k | N_t > 0] = \lim_{t\to\infty}\frac{1}{t^k}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f, X_t\rangle^k]}{\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)}.$$

# Step 1: Study the moments of the NBP

- Recall that $v_t[f](r, \upsilon) = 1 - \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\prod_{i=1}^{N_t} f(r_i(t), \upsilon_i(t))\right]$.

- Suppose we take $f = \mathrm{e}^{-\theta g}$. Then

$$v_t[f](r, \upsilon) = 1 - \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\mathrm{e}^{-\theta \sum_{i=1}^{N_t} g(r_i(t), \upsilon_i(t))}\right] = 1 - \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\mathrm{e}^{-\theta \langle g, X_t \rangle}\right].$$

- Differentiating $k \geq 1$ times and setting $\theta = 0$ yields

$$\frac{\partial^k}{\partial \theta^k} v_t[\mathrm{e}^{-\theta g}](r, \upsilon)\Big|_{\theta=0} = (-1)^{k+1} \mathbb{E}_{\delta_{(r,\upsilon_j)}}\left[\langle g, X_t \rangle^k\right]$$

- What happens if we differentiate the evolution equation satisfied by $v_t$?

## Proposition (Gonzalez, H., Kyprianou)

Fix $k \geq 2$. Then

$$\mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^k\right] = \psi_t[f^k](x) + \int_0^t \psi_s\left[\beta\eta_{t-s}^{(k-1)}[f]\right](r,\upsilon)\,\mathrm{d}s, \qquad t \geq 0, \tag{4}$$

where

$$\eta_{t-s}^{(k-1)}[f](r,\upsilon) = \mathcal{E}_{(r,\upsilon)}\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^{k_j}\right]\right],$$

and $[k_1,\dots,k_N]_k^2$ is the set of all non-negative $N$-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the $k_i$ are strictly positive.

# Step 1: Study the moments of the NBP

## Proposition (Gonzalez, H., Kyprianou)

Fix $k \geq 2$. Then

$$\mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^k\right] = \psi_t[f^k](x) + \int_0^t \psi_s\left[\beta\eta_{t-s}^{(k-1)}[f]\right](r,\upsilon)\,\mathrm{d}s, \qquad t \geq 0, \tag{4}$$

where

$$\eta_{t-s}^{(k-1)}[f](r,\upsilon) = \mathcal{E}_{(r,\upsilon)}\left[\sum_{[k_1,\dots,k_N]_k^2}\binom{k}{k_1,\dots,k_N}\prod_{j=1}^N \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^{k_j}\right]\right],$$

and $[k_1,\dots,k_N]_k^2$ is the set of all non-negative $N$-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the $k_i$ are strictly positive.

## Theorem (Gonzalez, H., Kyprianou)

Uniformly, for $(r,\upsilon) \in D \times V$ and $f \in L_+^\infty$, we have

$$\lim_{t\to\infty}\left|\varphi(r,\upsilon)^{-1}t^{-(\ell-1)}\mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^\ell\right] - \ell!\,\langle f,\tilde{\varphi}\rangle^\ell(\Sigma/2)^{\ell-1}\right| = 0.$$

# Step 2: Survival probability

**Theorem (Harris, H., Kyprianou, Wang)**

For all $r \in D$ and $\upsilon \in V$

$$\lim_{t\to\infty} t\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0) = \frac{2\varphi(r,\upsilon)}{\Sigma}.$$

- Note that by setting $f = \mathbf{0}$, we have

$$v_t[\mathbf{0}](r,\upsilon) = 1 - \mathbb{E}_{\delta_{(r,\upsilon)}}\left[\prod_{i=1}^{N_t} \mathbf{0}\right] = \mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)$$

  and so $\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t > 0)$ is a solution to (3).

- Study the non-linear equation to get coarse upper and lower bounds of the form $C/t$.
- We them bootstrap these bounds to obtain a precise limit.

**Step 3**: The previous two steps imply that

$$\begin{aligned}\lim_{t\to\infty}\frac{1}{t^k}\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f,X_t\rangle^k|N_t>0] &= \lim_{t\to\infty}\frac{1}{t^k}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f,X_t\rangle^k]}{\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t>0)}\\ &= \lim_{t\to\infty}\frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle f,X_t\rangle^k]/t^{k-1}}{t\mathbb{P}_{\delta_{(r,\upsilon)}}(N_t>0)}\\ &= \frac{\varphi(r,\upsilon)k!(\Sigma/2)^{k-1}\langle f,\tilde{\varphi}\rangle^k}{2\varphi(r,\upsilon)/\Sigma}\\ &= k!\left(\frac{\langle f,\tilde{\varphi}\rangle\Sigma}{2}\right)^k.\end{aligned}$$

# Contents

# So far...

- From the Perron-Frobenius result, we have

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle]}{\varphi(r,\upsilon)} = \langle \tilde{\varphi}, g\rangle.$$

- Can we obtain a stochastic analogue of this result:

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t\rangle}{\varphi(r,\upsilon)} = ??$$

- It turns out that studying the martingale

$$W_t := \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi, X_t\rangle}{\varphi(r,\upsilon)}, \quad t \geq 0$$

and its limit, $W_\infty$, provides the answer.

# So far...

- From the Perron-Frobenius result, we have

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle]}{\varphi(r,\upsilon)} = \langle \tilde{\varphi}, g\rangle.$$

- Can we obtain a stochastic analogue of this result:

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t\rangle}{\varphi(r,\upsilon)} = ??$$

- It turns out that studying the martingale

$$W_t := \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi, X_t\rangle}{\varphi(r,\upsilon)}, \quad t \geq 0$$

and its limit, $W_\infty$, provides the answer.

# So far...

- From the Perron-Frobenius result, we have

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\mathbb{E}_{\delta_{(r,\upsilon)}}[\langle g, X_t\rangle]}{\varphi(r,\upsilon)} = \langle\tilde{\varphi}, g\rangle.$$

- Can we obtain a stochastic analogue of this result:

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t\rangle}{\varphi(r,\upsilon)} = ??$$

- It turns out that studying the martingale

$$W_t := \mathrm{e}^{-\lambda_* t} \frac{\langle \varphi, X_t\rangle}{\varphi(r,\upsilon)}, \quad t \geq 0$$

and its limit, $W_\infty$, provides the answer.

# Strong law of large numbers

## Theorem (Harris, H., Kyprianou)

Under the assumptions (H1)-(H3), for all measurable, non-negative and directionally continuous[a] functions $g$ on $D \times V$ such that, up to a multiplicative constant, $g \leq \varphi$, then for any $(r, \upsilon) \in D \times V$, we have,

$$\mathrm{e}^{-\lambda_* t} \frac{\langle g, X_t \rangle}{\varphi(r, \upsilon)} \to \langle g, \tilde{\varphi} \rangle W_\infty$$

$\mathbb{P}_{\delta_{(r,\upsilon)}}$-almost surely and in $L^2(\mathbb{P})$, as $t \to \infty$.

[a] By this, we mean functions $g$ such that $\lim_{s \to 0} g(r + \upsilon s, \upsilon) = g(r, \upsilon)$.

# Moments

**Theorem (Gonzalez, H., Kyprianou)**

Suppose $\lambda > 0$. Uniformly for $(r, \upsilon) \in D \times V$ and $f \in L^\infty_+(D \times V)$, we have

$$\lim_{t\to\infty} \left|\varphi(r,\upsilon)^{-1}\mathrm{e}^{-\ell\lambda t}\mathbb{E}_{\delta_{(r,\upsilon)}}\left[\langle f, X_t\rangle^\ell\right] - \ell!\langle f, \tilde{\varphi}\rangle^\ell L_\ell\right| = 0,$$

where $L_1 = 1$ and we define iteratively for $k \geq 2$

$$L_k = \frac{1}{\lambda(k-1)}\left\langle \tilde{\varphi}, \beta\mathcal{E}.\left[\sum_{[k_1,\dots,k_N]_k^2} \prod_{\substack{j=1\\ j:k_j>0}}^{N} \varphi(r,\upsilon_j)L_{k_j}(r,\upsilon_j)\right]\right\rangle,$$

where $[k_1,\dots,k_N]_k^2$ is the set of all non-negative $N$-tuples $(k_1,\dots,k_N)$ such that $\sum_{i=1}^N k_i = k$ and at least two of the $k_i$ are strictly positive

# Contents

## Theorem (Gonzalez, H., Kyprianou)

Suppose $\lambda < 0$. Then uniformly for $(r, \upsilon) \times D \times V$ and $f \in L_+^\infty(D \times V)$, we have

$$\lim_{t \to 0} \left| \varphi(r, \upsilon)^{-1} \mathrm{e}^{-\lambda t} \mathbb{E}_{\delta_{(r,\upsilon)}} \left[ \langle f, X_t \rangle^\ell \right] - \ell! \langle f, \tilde{\varphi} \rangle^\ell L_\ell \right| = 0,$$

where we define iteratively $L_1 = \langle f, \tilde{\varphi} \rangle$ and for $k \geq 2$,

$$L_k = \frac{\langle f^k, \tilde{\varphi} \rangle}{\langle f, \tilde{\varphi} \rangle^k k!} - \left\langle \beta \mathcal{E}. \left[ \sum_{n=2}^{k} \frac{1}{\lambda(n-1)} \sum_{[k_1,\ldots,k_N]_k^n} \prod_{\substack{j=1 \\ j: k_j > 0}}^{N} \varphi(r, \upsilon_j) L_{k_j} \right], \tilde{\varphi} \right\rangle,$$

where $[k_1, \ldots, k_N]_k^n$ is the set of all non-negative $N$-tuples $(k_1, \ldots, k_N)$ such that $\sum_{i=1}^{N} k_i = k$ and exactly $2 \leq n \leq k$ of the $k_i$ are strictly positive.

# Some open questions/future research directions

- Genealogical stucture
- Clustering
- Central limit theorems
- Law of large numbers in terms of initial population size
- Precise rates of convergence for the moments $\rightsquigarrow$ Monte Carlo error bounds

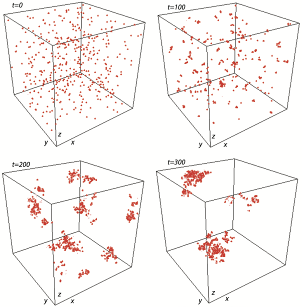


Figure: Image provided by Eric Dumonteil

# Contents

# $k_{\text{eff}}$ eigenvalue problem

Find a triple $(k_{\text{eff}}, \phi, \tilde{\phi})$ such that

$$(\overleftarrow{T} + \overleftarrow{S})\phi(r, \upsilon) = -\frac{1}{k_{\text{eff}}}\overleftarrow{F}\phi(r, \upsilon) \qquad \langle \tilde{\phi}, (\overleftarrow{T} + \overleftarrow{S})f\rangle = -\frac{1}{k_{\text{eff}}}\langle \tilde{\phi}, \overleftarrow{F} f\rangle,$$

where we have the following regimes

$$k_{\text{eff}} \begin{cases} < 1, & \text{system is subcritical} \\ = 1, & \text{system is critical} \\ > 1, & \text{system is supercritical.} \end{cases}$$

$\rightsquigarrow$ This involves studying the growth of the process at branching times.

# $k_{\text{eff}}$ eigenvalue problem

Find a triple $(k_{\text{eff}}, \phi, \tilde{\phi})$ such that

$$(\overleftarrow{T} + \overleftarrow{S})\phi(r, \upsilon) = -\frac{1}{k_{\text{eff}}}\overleftarrow{F}\phi(r, \upsilon) \qquad \langle\tilde{\phi}, (\overleftarrow{T} + \overleftarrow{S})f\rangle = -\frac{1}{k_{\text{eff}}}\langle\tilde{\phi}, \overleftarrow{F}f\rangle,$$

where we have the following regimes

$$k_{\text{eff}} \begin{cases} < 1, & \text{system is subcritical} \\ = 1, & \text{system is critical} \\ > 1, & \text{system is supercritical.} \end{cases}$$

$\rightsquigarrow$ This involves studying the growth of the process at branching times.

# Neutron generational process

- In place of $(X_t, t \geq 0)$, we consider the process $(\mathcal{X}_n, n \geq 0)$, where, for $n \geq 1$, $\mathcal{X}_n$ given by

$$\mathcal{X}_n = \sum_{i=1}^{\mathcal{N}_n} \delta_{(r_i^{(n)}, \upsilon_i^{(n)})},$$

  where $\{(r_i^{(n)}, \upsilon_i^{(n)}), i = 1, \cdots \mathcal{N}_n\}$ are the position-velocity configurations of the $\mathcal{N}_n$ particles that are $n$-th in their genealogies to be the result of a fission event.

- In place of the semigroup $\psi_t$, we consider

$$\Psi_n[g](r, \upsilon) := \mathbb{E}_{\delta_{(r,\upsilon)}} [\langle g, \mathcal{X}_n \rangle] .$$

- The associated evolution equation is given by

$$\Psi_n[g](r, \upsilon) = \int_0^\infty \mathbb{E}_{(r,\upsilon)} \left[ \mathrm{e}^{-\int_0^s \sigma_{\mathrm{f}}(R_u, \Upsilon_u)\mathrm{d}u} \overleftarrow{F} \Psi_{n-1}[g](R_s, \Upsilon_s) \right] \mathrm{d}s,$$

  where $(R_s, \Upsilon_s)$ is the $\sigma_{\mathrm{s}}\pi_{\mathrm{s}}$-NRW.

# Perron Frobenius behaviour

## Theorem (Cox, H., Kyprianou, Villemonais)

Suppose

- the cross-sections are uniformly bounded from above and
- $\inf_{r\in D, \upsilon, \upsilon' \in V} \sigma_{\mathtt{f}}(r,\upsilon)\pi_{\mathtt{f}}(r,\upsilon,\upsilon') > 0$.

Then there exist $k_* \in \mathbb{R}$, a positive right eigenfunction $\varphi \in L^+_\infty(D\times V)$ and a left eigenmeasure, $\eta$, on $D\times V$, both having associated eigenvalue $k_*^n$. Moreover, $k_*$ is the leading eigenvalue in the sense that, for all $g \in L^+_\infty(D\times V)$,

$$\langle \eta, \Psi_n[g]\rangle = k_*^n \langle \eta, g\rangle \quad (\text{resp. } \Psi_n[\varphi] = k_*^n\varphi) \quad n \geq 0, \tag{5}$$

and there exists $\gamma > 1$ such that, for all $g \in L^+_\infty(D\times V)$,

$$\sup_{g\in L^+_\infty(D\times V):\|g\|_\infty\leq 1} \left\| k_*^{-n}\varphi^{-1}\Psi_n[g] - \langle \eta, g\rangle \right\|_\infty = O(\gamma^{-n}),\ n\geq 0. \tag{6}$$

# Many-to-one

- Recall that $m(r, \upsilon) := \int_V \pi_{\mathtt{f}}(r, \upsilon, \upsilon')\mathrm{d}\upsilon'$. Further let $\sigma = \sigma_{\mathtt{s}} + \sigma_{\mathtt{f}}$.
- Consider a $\sigma\varpi$-NRW, where

$$\varpi(r, \upsilon, \upsilon') = \frac{\sigma_{\mathtt{s}}(r, \upsilon)}{\sigma(r, \upsilon)}\pi_{\mathtt{s}}(r, \upsilon, \upsilon') + \frac{\sigma_{\mathtt{f}}(r, \upsilon)}{\sigma(r, \upsilon)}\frac{\pi_{\mathtt{f}}(r, \upsilon, \upsilon')}{m(r, \upsilon)}, \qquad r \in D, \upsilon, \upsilon' \in V.$$

- We can think of the $\sigma\varpi$-NRW as equal in law to the following process.
  - the NRW $(R, \Upsilon)$ scatters for the $k$-th time at $(r, \upsilon)$ with rate $\sigma(r, \upsilon)$;
  - a coin is tossed and the random variable $\mathrm{I}_k(r, \upsilon)$ takes the value 1 with probability $\sigma_{\mathtt{f}}(r, \upsilon)/\sigma(r, \upsilon)$ and its new velocity, is selected according to an independent copy of the random variable $\Theta_k^{\mathtt{f}}(r, \upsilon)$, whose distribution has probability density $\pi_{\mathtt{f}}(r, \upsilon, \upsilon')/m(r, \upsilon)$;
  - On the other hand, with probability $\sigma_{\mathtt{s}}(r, \upsilon)/\sigma(r, \upsilon)$; the random variable $\mathrm{I}_k(r, \upsilon)$ takes the value 0 and its new velocity, is selected according to an independent copy of the random variable $\Theta_k^{\mathtt{s}}(r, \upsilon)$, whose distribution has probability density $\pi_{\mathtt{s}}(r, \upsilon, \upsilon')$.
- As such, the velocity immediately after the $k$-th scatter of the NRW, given that the position-velocity configuration immediately before is $(r, \upsilon)$, is coded by the random variable

$$\mathrm{I}_k(r, \upsilon)\Theta_k^{\mathtt{f}}(r, \upsilon) + (1 - \mathrm{I}_k(r, \upsilon))\Theta_k^{\mathtt{s}}(r, \upsilon).$$

# Many-to-one

- Recall that $m(r,\upsilon) := \int_V \pi_{\mathtt{f}}(r,\upsilon,\upsilon')\mathrm{d}\upsilon'$. Further let $\sigma = \sigma_{\mathtt{s}} + \sigma_{\mathtt{f}}$.
- Consider a $\sigma\varpi$-NRW, where

$$\varpi(r,\upsilon,\upsilon') = \frac{\sigma_{\mathtt{s}}(r,\upsilon)}{\sigma(r,\upsilon)}\pi_{\mathtt{s}}(r,\upsilon,\upsilon') + \frac{\sigma_{\mathtt{f}}(r,\upsilon)}{\sigma(r,\upsilon)}\frac{\pi_{\mathtt{f}}(r,\upsilon,\upsilon')}{m(r,\upsilon)}, \qquad r \in D, \upsilon,\upsilon' \in V.$$

- We can think of the $\sigma\varpi$-NRW as equal in law to the following process.
  - the NRW $(R,\Upsilon)$ scatters for the $k$-th time at $(r,\upsilon)$ with rate $\sigma(r,\upsilon)$;
  - a coin is tossed and the random variable $\mathtt{I}_k(r,\upsilon)$ takes the value 1 with probability $\sigma_{\mathtt{f}}(r,\upsilon)/\sigma(r,\upsilon)$ and its new velocity, is selected according to an independent copy of the random variable $\Theta^{\mathtt{f}}_k(r,\upsilon)$, whose distribution has probability density $\pi_{\mathtt{f}}(r,\upsilon,\upsilon')/m(r,\upsilon)$;
  - On the other hand, with probability $\sigma_{\mathtt{s}}(r,\upsilon)/\sigma(r,\upsilon)$; the random variable $\mathtt{I}_k(r,\upsilon)$ takes the value 0 and its new velocity, is selected according to an independent copy of the random variable $\Theta^{\mathtt{s}}_k(r,\upsilon)$, whose distribution has probability density $\pi_{\mathtt{s}}(r,\upsilon,\upsilon')$.
- As such, the velocity immediately after the $k$-th scatter of the NRW, given that the position-velocity configuration immediately before is $(r,\upsilon)$, is coded by the random variable

$$\mathtt{I}_k(r,\upsilon)\Theta^{\mathtt{f}}_k(r,\upsilon) + (1-\mathtt{I}_k(r,\upsilon))\Theta^{\mathtt{s}}_k(r,\upsilon).$$

# Generational many-to-one

Define sequentially, $T_0 = 0$ and, for $n \geq 1$,

$$T_n = \inf\{t > T_{n-1} : \Upsilon_t \neq \Upsilon_{t-} \text{ and } \mathtt{I}_{k_t}(R_t, \Upsilon_{t-}) = 1\},$$

where $(k_t, t \geq 0)$ is the process counting the number of scattering events of the NRW up to time $t$.

## Many-to-one

Suppose $(H1)$, $(H2)$ and $(H4)$ hold. The solution to (2) among the class of expectation semigroups is unique for $g \in L^\infty_+(D \times V)$ and the semigroup $(\Psi_n, n \geq 0)$ may alternatively be represented as

$$\Psi_n[g](r, \upsilon) = \mathbf{E}_{(r,\upsilon)}\left[\prod_{i=1}^{n} m(R_{T_i}, \Upsilon_{T_i-}) g(R_{T_n}, \Upsilon_{T_n}) \mathbf{1}_{(T_n < \kappa^D)}\right], \qquad r \in D, \upsilon \in V, n \geq 1, \quad (7)$$

(with $\Psi_0[g] = g$), where $(R_t, \Upsilon_t)_{t \geq 0}$ is the $\sigma\varpi$-NRW marked at times $(T_i, i \geq 1)$, and

$$\kappa^D := \inf\{t > 0 : R_t \notin D\}.$$

# Generational many-to-one

Define sequentially, $T_0 = 0$ and, for $n \geq 1$,

$$T_n = \inf\{t > T_{n-1} : \Upsilon_t \neq \Upsilon_{t-} \text{ and } \mathtt{I}_{k_t}(R_t, \Upsilon_{t-}) = 1\},$$

where $(k_t, t \geq 0)$ is the process counting the number of scattering events of the NRW up to time $t$.

## Many-to-one

Suppose $(H1)$, $(H2)$ and $(H4)$ hold. The solution to (2) among the class of expectation semigroups is unique for $g \in L^\infty_+(D \times V)$ and the semigroup $(\Psi_n, n \geq 0)$ may alternatively be represented as

$$\Psi_n[g](r, \upsilon) = \mathbf{E}_{(r,\upsilon)}\left[\prod_{i=1}^{n} m(R_{T_i}, \Upsilon_{T_i-}) g(R_{T_n}, \Upsilon_{T_n}) \mathbf{1}_{(T_n < \kappa^D)}\right], \qquad r \in D, \upsilon \in V, n \geq 1, \quad (7)$$

(with $\Psi_0[g] = g$), where $(R_t, \Upsilon_t)_{t \geq 0}$ is the $\sigma\varpi$-NRW marked at times $(T_i, i \geq 1)$, and

$$\kappa^D := \inf\{t > 0 : R_t \notin D\}.$$

# Open questions/future research directions

- Do it all again for "discrete time".

# Open questions/future research directions

- Do it all again for "discrete time".
- Interactions with the environment
- Effects of temperature, pressure, ...
- Not just fissile systems: shielding problems, rare event simulation, ...
- Not just nuclear reactors: health care, space, ...

Thank you!