A duality formula + Particle Gibbs for Feynman-Kac many body measures

M. Arnaudon, P. Del Moral

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Neutron Transport Days, Sept. 16th 2021

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- `this talk` $\rightsquigarrow$ `Arxiv (18)` & EJP (20), see also for discrete time + Kohn-Patras `Arxiv (14), CRAS (15), IHP (16)`
- FK+Moran particle: +Miclo SPA (00), Schröd. op. ESAIM (03), Rousset SIAM (06), . . . , stability in Ferré-Rousset-Stoltz SPDE (21)

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Integration (Lebesgue or Riemann)

$$\eta(f) = \int \eta(dx) f(x)$$

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In this talk: Measure estimates w.r.t. total variation distance

$$\mu^\epsilon = \mu \pm \epsilon \quad \Longleftrightarrow \quad \|\mu^\epsilon - \mu\|_{tv} \leq c\ \epsilon \quad \text{for some constant } c \perp \epsilon$$

X_t Markov $\in$ metric space S and potential $V(X_t) \geq 0$

Historical processes

$$\widehat{X}_t := (X_s)_{s \leq t} \in \widehat{S} := \cup_{t \geq 0} S_t \quad \text{and} \quad \widehat{V}_t(\widehat{X}_t) = V_t(X_t)$$

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  $\rightsquigarrow$ Obvious stochastic calculus (Genealogies [AAP-01], FK-04)

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  $\rightsquigarrow$ Discrete time + density argument on cylindrical functions

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  $\rightsquigarrow$ Infinitesimal generator processes/FK on measurable spaces [+ Miclo SPA-00, Séminaire Probab.-00, SAA-07]

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  ⇝ Ito-functional calculus ∈ [Dupire-09,Cont-Fournié-13,Jazaerli-Saporito-17, Saporito-14]

FK-measures on path space $:= FK(\widehat{X}, \widehat{V})$

$$d\mathbb{Q}_t = \frac{1}{Z_t} \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right] d\mathbb{P}_t$$

with the normalizing constant

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$$\Updownarrow$$

Weak formulation on test functions F on S_t:

$$\mathbb{Q}_t(F) = \int \ F(x) \ \mathbb{Q}_t(dx) \propto \mathbb{E}\left(F(\widehat{X}_t) \ \exp\left[-\int_0^t \widehat{V}_s(\widehat{X}_s)ds\right]\right)$$

Time marginal FK-measures $= FK(X, V)$

Path-space:

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Same $(\mathbb{Q}_t, \eta_t) \rightarrow$ different forms (Choice of X and V not unique)

$FK(\widehat{X}, \widehat{V})$ - particle sampler $=$ N-path particles

$$\xi_t := (\xi_t^i)_{1 \leq i \leq N} \in S_t^N \quad \text{with ancestral lines} \quad \xi_t^i = (\xi_{s,t}^i)_{1 \leq s \leq t}$$

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Ancestral line & Occupation measure of the ancestral tree

$$\mathbb{X}_t \sim m(\xi_t) := \frac{1}{N} \sum_{1 \leq i \leq N} \delta_{\xi_t^i}$$

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- (killing + Jump) $\simeq$ N-Selection at each time unit t_n:

$$\underbrace{e^{-\widehat{V}(\xi^i_{t_n})\Delta}\ \delta_{\xi^i_{t_n}}}_{\text{no jump}} + \underbrace{\left(1 - e^{-\widehat{V}(\xi^i_{t_n})\Delta}\right)}_{\simeq_{\Delta\simeq 0}\text{jump rate }\ \widehat{V}(\xi^i_t)dt}\ \underbrace{\sum_{1\leq j\leq N} \frac{e^{-\widehat{V}(\xi^j_{t_n})\Delta t}}{\sum_{1\leq k\leq N} e^{-\widehat{V}(\xi^k_{t_n})\Delta t}}\ \delta_{\xi^j_{t_n}}}_{\simeq_{\Delta\simeq 0}\text{uniformly on the path-pool}=m(\xi_t)}$$

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Genetic type particle sampler

(on geo-clocks +Jacob+Lee+Murray+Peters SAA (13))

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(+ many other variants when $V = V_+ - V_-$)

Genetic algo mimics the branching process
with FK-intensity (i.e. first moment unbiased)

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Many other lively buzzwords (how to make new with old :-)):
Reconfiguration, resampling, bootstrapping, spawning, cloning, pruning, replenish, splitting, enrichment, recycling, go with the winner, look-ahead, weighted dynamics, quantum teleportation, Fleming-Viot, quasi-stationary Method/Monte Carlo, ...

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(please suggest new lively ideas for collector updates)

Some key results (when $FK(X, V)$ stable)

Fluctuation/clt + expo. concentration + ldp :

$$m(\xi_t) = \mathbb{Q}_t + \frac{1}{\sqrt{N}} \mathbb{V}_t \qquad \text{with} \quad \mathbb{V}_t \simeq \text{Gaussian field}$$

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Bias/propagation of chaos:

$$\text{Law}(\mathbb{X}_t) = \mathbb{Q}_t \pm \frac{t}{N} \quad \text{and} \quad \text{Law}(\mathbb{X}_{s,t}) = \mathbb{Q}_t \pm \frac{1 + (t-s)}{N}$$

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Unbiasedness property :

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↳ *not so new:* ⇝ [discrete time MPRF-96, continuous time SPA-00]

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- *Independent Metropolis-Hastings - obvious*

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$$\Downarrow$$

Gibbs sampler on $(\mathbb{S}_1 \times \mathbb{S}_2)$ with target Π_t

$$\underbrace{((1) \mid (2)) = \left(\mathbb{X}_t \mid \widehat{\xi}_t\right) = m(\xi_t)}_{\textit{easy to sample !}}$$

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The Π_t-historical/ancestral process $\widehat{\xi}_t$ given $\mathbb{X}_t$ $\neg$

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2 \leq i \leq N}$ and say $\xi_s^1 = \mathbb{X}_s$

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Nb.: without the blue

the above process coincides with the FK-particle sampler

Interacting jump process: $\xi_s^- := (\xi_s^i)_{2 \leq i \leq N}$ and say $\xi_s^1 = \mathbb{X}_s$

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Given $\mathbb{X}_t = x = (x_u)_{0 \leq u \leq t} \sim \mathbb{Q}_t$

Coincides with the particle sampler $\zeta_s^- := (\zeta_s^i)_{2 \leq i \leq N}$ **of FK measures:**

$$\begin{array}{lll} \widehat{V}_s & \rightsquigarrow & (1 - 1/N)\ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \oplus \text{extra jump rate } (2/N)\ \widehat{V}_s \text{ onto } (x_u)_{0 \leq u \leq s} \end{array}$$

Equivalent formulation - Duality formula

Theo.: [+Arnaudon ⇝ Arxiv (18)]

$$\mathbb{E}\left(F(\widehat{\xi}_t, \mathbb{X}_t)\ e^{-\int_0^t m(\xi_s)(\widehat{V}_s)ds}\right) = \mathbb{E}\left(F(\widehat{\zeta}_t, \widehat{X}_t)\ e^{-\int_0^t \widehat{V}_s(\widehat{X}_s)ds}\right)$$

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- = many-body/twisted distribution on $\widehat{\xi}_t$ and

$$(\widehat{\xi}_t \mid \mathbb{X}_t = x) = (\widehat{\zeta}_t \mid \widehat{X}_t = x) \qquad \mathbb{Q}_t - a.e.\ x$$

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- *discrete time version*

$\subset$ +Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)

$\Longrightarrow$ Gibbs-sampler with target $\mathbb{Q}_t$

$$\mathbb{X}_t = x \longrightarrow \left[\ \widehat{\zeta}_t^x \ \sim \ \left(\widehat{\zeta}_t \mid \widehat{X}_t = x\right)\ \right] \longrightarrow \overline{\mathbb{X}}_t = \overline{x} \ \sim \ m(\zeta_t^x)$$

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$$\Updownarrow$$

$\mathbb{Q}_t$-reversible Markov transition:

$$\mathbb{K}_t(f)(x) = \mathbb{E}(f(\overline{\mathbb{X}}_t) \mid \mathbb{X}_t = x) = \mathbb{E}\left[m(\zeta_t^x)(f)\right]$$

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Key observation for the convergence analysis:

$$m(\zeta_t^x) = \frac{1}{N} \ \delta_x + \left(1 - \frac{1}{N}\right) \ m(\zeta_t^{x,-})$$

with $\widehat{\zeta}_t^{x,-} = (\zeta_s^{x,-})_{0 \leq s \leq t}$ = particle sampler of an FK measure:

$$\begin{array}{lll} \widehat{V}_s & \rightsquigarrow & (1 - 1/N) \ \widehat{V}_s \\ \widehat{X}_s & \rightsquigarrow & \widehat{X}_s \oplus \text{extra jump rate } (2/N) \ \widehat{V}_s \text{ onto } x_s \end{array}$$

The FK-bias/propagation of chaos

$$\mathbb{K}_t(f)(x) = \mathbb{E}\left[m(\zeta_t^x)(f)\right] = \mathbb{Q}_t^x(f) \pm \frac{t}{N}$$

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"A little" perturbation analysis

$$\mathbb{Q}_t^x = \mathbb{Q}_t \pm \frac{t}{N} \Longrightarrow \delta_x \mathbb{K}_t = \mathbb{Q}_t \pm \frac{t}{N} \Longrightarrow \delta_x \mathbb{K}_t - \delta_y \mathbb{K}_t = \frac{t}{N}$$

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For any starting path $x \in S_t$ after n iterations we have

$$\delta_x\mathbb{K}_t^n = \mathbb{Q}_t \pm \left(c\ \frac{t}{N}\right)^n$$

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More refined (discrete time)

Theo. [+Kohn+Patras, Arxiv (14), CRAS (15)+IHP (16)]

$$\mathbb{K}_t = \mathbb{Q}_t + \sum_{1 \leq k < l} \left(\frac{t}{N}\right)^k \partial^{(k)}\mathbb{K}_t + \left(\frac{t}{N}\right)^l \overline{\partial}^l \mathbb{K}_t$$

$(\forall l \geq 1) \oplus$ Differential operators $\sim$ coalescent/colored trees and s.t.

$$\left\|\partial^{(k)}\mathbb{K}_t\right\| \vee \left\|\overline{\partial}^{(k)}\mathbb{K}_t\right\| \leq k^{2k} \quad \text{as soon as } l^2 t/N < 1.$$

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$\Downarrow$ [*direct*]

$$\left| \| \mathbb{K}_t^{\mathbf{n}}(f) - \mathbb{Q}_t(f) \|_{\mathbb{L}_p(\mathbb{Q}_t)} - \left(\frac{t}{N}\right)^{\mathbf{n}} \| [\partial \mathbb{K}_t]^{\mathbf{n}}(f) \|_{\mathbb{L}_p(\mathbb{Q}_t)} \right| \leq \left(c \, \frac{t}{N}\right)^{\mathbf{n}+1}$$

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Continuous time ? $\rightsquigarrow$ *Open problem/question*

"Hint/solution" $\rightsquigarrow$ Taylor/Bias [+ Rubenthaler-Patras] JTP (09)

$\oplus$ backward analysis using prop 2.1 in $\rightsquigarrow$ Arxiv (18)