Exercise Sheet 2— The Stochastic Integral

Throughout, you should assume that W_t is a real-valued, standard Brownian motion with $W_0 = 0$.

2.1: Show from the definition of the Itô integral that:

$$\int_0^t s\,\mathrm{d}W_s = tW_t - \int_0^t W_s\,\mathrm{d}s.$$

[Hint: for a, b, c, d, show $bd - ac = a(d - c) + d(b - a)$.]

2.2: (a) Let Π^n be a sequence of partitions of $[0, T]$. Show that

$$\sum_i W_{t_i}(W_{t_{i+1}} - W_{t_i})^2 \to \int_0^T W_t\,\mathrm{d}t$$

in $\mathcal{L}^2(\mathbb{P})$ as the mesh of the partition tends to zero.

(b) Show that

$$\sum_i (W_{t_{i+1}} - W_{t_i})^3 \to 0$$

in $\mathcal{L}^2(\mathbb{P})$ as the mesh of the partition tends to zero.

(c) Hence find an expression for $\int_0^t W_s^2\,\mathrm{d}W_s$. [Hint: $(b-a)^3 = b^3 - a^3 - 3a^2(b-a) - 3a(b-a)^2$.

2.3: Prove *i)*–*iv)* of Lemma 3.3 for simple processes. Using Corollary 3.6 show that all these properties, except for path continuity, extend to general $\phi \in \mathcal{V}$.

2.4: A result of Itô gives the following formula for the iterated Itô integral:

$$\int_{u_n=0}^t \int_{u_{n-1}=0}^{u_n} \int_{u_{n-2}=0}^{u_{n-1}} \cdots \int_{u_1=0}^{u_{n-1}} \mathrm{d}W_{u_1}\,\mathrm{d}W_{u_2} \ldots \mathrm{d}W_{u_n} = \frac{t^{n/2}}{n!} h_n\left(\frac{W_t}{\sqrt{t}}\right),$$

where h_n is the Hermite polynomial of degree n, given by

$$h_n(x) = (-1)^n \mathrm{e}^{x^2/2} \frac{d^n}{dx^n}\left[\mathrm{e}^{-x^2/2}\right].$$

Find $h_0, h_1, \ldots, h_3$ and verify that this is true up to $n = 3$.

2.5: Suppose $\varphi, \psi \in \mathcal{V}$. Show that

$$\mathbb{E}\left[\int_0^t \varphi_u\,\mathrm{d}W_u \int_0^t \psi_u\,\mathrm{d}W_u\right] = \mathbb{E}\left[\int_0^t \varphi_u\psi_u\,\mathrm{d}u\right]$$

.

2.6: Directly from the definition of the Stratonovich integral find $\int_0^t W_u \circ \mathrm{d}W_u$ and $\int_0^t W_u^2 \circ \mathrm{d}W_u$.

2.7: Let $\varphi \in \mathcal{V}$ be a function which varies smoothly in t, in the sense that there exists $K, \varepsilon > 0$ such that:

$$\mathbb{E}\left[|\varphi_t - \varphi_s|^2\right] \leq |t-s|^{1+\varepsilon}$$

for $0 \leq s, t \leq T$.

Show that in this case,

$$\int_0^T \varphi_u \, \mathrm{d}W_u = \lim_{\Pi} \sum_i \varphi_{t_i^*}(W_{t_{i+1}} - W_{t_i})$$

where the limit on the right-hand side is in probability, and is taken along partitions, Π with $||\Pi|| \to 0$, and t_i^* is any choice of $t \in [t_i, t_{i+1}]$. [Hint: Show convergence in $\mathcal{L}^1(\mathbb{P})$, which implies convergence in probability.]

Hence deduce that in this case the Itô and Stratonovich integrals coincide.

2.8: **[Hard]:** In this section, we use a result from Martingale Theory to show that the Itô integral has a modification with continuous paths.

(a) A process X_t is called a martingale if (i) $\mathbb{E}[|X_t|] < \infty$ for all $t \geq 0$; (ii) X_t is adapted; (iii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ for all $t \geq s$.
Show that if $\varphi \in \mathcal{S}$, then $\int_0^t \varphi_s \, \mathrm{d}W_s$ is a continuous martingale.

(b) Doob's Martingale Inequality states that if M_t is a martingale with continuous paths, then for all $p \geq 1, T \geq 0$ and $\lambda > 0$

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |M_t| \geq \lambda\right) \leq \frac{1}{\lambda^p}\mathbb{E}\left[|M_T|^p\right].$$

Using this result, show that if $\varphi^n \in \mathcal{S}$ is a sequence of simple integrands approximating φ, and $I_t^n := \int_0^t \varphi_u^n \, \mathrm{d}W_u$, then there exists a subsequence n_k such that

$$\mathbb{P}\left(\sup_{0 \leq t \leq T} |I_t^{n_{k+1}} - I_t^{n_k}| > 2^{-k}\right) < 2^{-k}.$$

(c) By applying the Borel-Cantelli Lemma, show that for $\varphi \in \mathcal{V}$, then $I_t := \int_0^t \varphi_u \, \mathrm{d}W_u$ has a continuous modification.