An inverse optimal consumption problem

(Based on joint work w/ D. Hobson & J. Obłój — but see also earlier work of Black, Cox + Leland, He + Huang.)

Consider the classical Black-Scholes model, $S(t)$ an asset price, $\frac{dS(t)}{S(t)} = \sigma(dB(t) + \theta dt) + r\,dt$,

$\sigma, \theta, r > 0$ constant,

θ is market price of risk or Sharpe ratio.

We consider an investor who can consume wealth, and ~~or~~ invest a proportion in the risky asset (the rest is in the bank account). We suppose the investor does this in a manner which optimises utility from consumption over an infinite horizon; that is, she solves:

$$\sup_{c_t, \pi_t : W_t^{c,\pi} \geq 0} \mathbb{E}\left[\int_0^\infty u(t, c_t)\,dt\right],$$

where $W_t^{c,\pi}$ solves:

$$dW_t^{c,\pi} = rW_t^{c,\pi}\,dt - c_t\,dt + \pi_t\,\sigma(dB_t + \theta dt).$$

$$W_0^{c,\pi} = x$$

[Depending on context, we may also write $W_t^{x,c,\pi}$, or W_t]

- However we suppose that we ~~are~~ observe an investor's behaviour and wish to determine ~~whether this is~~ the utility function ~~&~~ and/or whether there exists such a function, or whether it is 'uniquely' determined. [Up to suitable scaling etc. factors]

So. Suppose we are given functions $c(t,w)$, $\pi(t,w)$ which are our consumption and investment functions. What can we say about $u(t,c)$?

— Note that we assume here that our investment + consumption depend on t and current wealth only

To understand what is happening, consider a slightly more general optimal control problem:

Define

$$v(t,w) := \sup_{\pi_s,\, c_s} E\left[\int_t^\infty \big(u(s,c_s) + \Psi(s,W_s) \big)\, ds \,\Big|\, W_t = w \right]$$

Then $$v(t, W_t^{c,\pi}) + \int_0^t \left[u(s,c_s) + \Psi(s, W_s^{c,\pi}) \right] ds$$

should be a supermartingale, ~~and~~ for all c, π, and a martingale for the optimal choice c, π.

Applying Itô, (so assuming v is sufficiently differentiable), we get the drift term:

$$v_t(t,W_t) + v_w(t,W_t)\left(rW_t + \pi_t \sigma\theta - c_t\right) + \tfrac{1}{2}\sigma^2 \pi_t^2 v_{ww}(t,W_t)$$
$$\cancel{v_w t} \quad + u(t,c_t) + \Psi(t,W_t)$$

which should be ≤ 0, and $= 0$ at its maximum.

But we can optimise separately ~~of~~ over ~~π~~ π_t:

$$\sup_{\pi_t} \left[v_w(t,W_t)\, \pi \sigma \theta + \tfrac{1}{2}\sigma^2 \pi^2 v_{ww}(t,W_t) \right]$$

$$= -\frac{1}{2}\theta^2 \frac{v_w^2}{v_{ww}},$$

which occurs at $\pi = -\dfrac{\theta}{\sigma}\dfrac{v_w}{v_{ww}}$.

In particular, since we assume the optimal π is known and a function of t, w, we have:

$$\frac{v_w(t,w)}{v_{ww}(t,w)} = -\frac{\sigma}{\theta}\pi(t,w) \Rightarrow$$

$$v_\omega(t,\omega) = \exp\left\{ A(t) - \int_0^\omega \frac{\theta}{\sigma \pi(t,\xi)} \, d\xi \right\}, \qquad (\dagger)$$

where $A(t)$ is a function to be specified.

In addition, we assume $u(t,c)$ is suitably differentiable, and concave in c (ie. investor is risk-averse).

Then the concave dual, $\tilde{u}(t,\xi) = \sup_x \left(u(t,x) - \xi x \right)$

exists, and we have: $\tilde{u}_\xi(t,\xi) = -x^*$, where x^* is the choice of x which optimises the concave dual.

We therefore have:

$$0 = \tilde{u}(t, v_\omega(t,\omega)) + \frac{1}{2}\pi(t,\omega)\sigma\theta v_\omega(t,\omega) + v_t(t,\omega) + v_\omega(t,\omega) r\omega + \Psi(t,\omega).$$

Differentiating w.r.t. ω:

$$(*) \quad 0 = -v_{\omega\omega}(t,\omega)c(t,\omega) + \frac{1}{2}\pi(t,\omega)\sigma\theta v_{\omega\omega}(t,\omega) + \frac{1}{2}\sigma\theta v_\omega(t,\omega)\pi_\omega(t,\omega) + v_{t\omega}(t,\omega) + v_{\omega\omega}(t,\omega) r\omega + r v_\omega(t,\omega) + \Psi_\omega(t,\omega).$$

Differentiating $(\dagger)$ w.r.t. t, we see that:

$$v_{t\omega}(t,\omega) = \left(A'(t) + \int_0^\omega \frac{\theta}{\sigma\pi(t,\xi)^2} \pi_t(t,\xi)\, d\xi \right) v_\omega(t,\omega)$$

So $(*)$ becomes:

$$0 = \left(\frac{1}{2}\pi(t,\omega)\sigma\theta + r\omega - c(t,\omega) \right) v_{\omega\omega}(t,\omega) + \Psi_\omega(t,\omega)$$
$$+ \left(\frac{1}{2}\sigma\theta\pi_\omega(t,\omega) + r + A'(t) + \int_0^\omega \frac{\theta}{\sigma\pi(t,\xi)^2} \pi_t(t,\xi)\, d\xi \right) v_\omega(t,\omega)$$

Dividing through by V_ω,

$$-\frac{\Psi_\omega(t,\omega)}{V_\omega(t,\omega)} = \left[\frac{1}{2}\pi(t,\omega)\sigma\theta + r\omega - c(t,\omega)\right]\frac{V_{\omega\omega}(t,\omega)}{V_\omega(t,\omega)}$$
$$+\frac{1}{2}\sigma\theta\pi_\omega(t,\omega) + r + A'(t) + \int_0^\omega \frac{\theta}{\sigma\pi(t,\xi)^2}\pi_t(t,\xi)d\xi$$

$$\Rightarrow \Psi_\omega(t,\omega)\exp\left\{-A(t) + \int_1^\omega \frac{\theta}{\sigma\pi(t,\xi)}d\xi\right\}$$
$$= \frac{1}{2}\theta^2 + \frac{r\omega\theta}{\sigma\pi(t,\omega)} - \frac{\theta c(t,\omega)}{\pi(t,\omega)\sigma} - \frac{1}{2}\sigma\theta\pi_\omega(t,\omega) - r - A'(t)$$
$$-\int^\omega \frac{\theta}{\sigma\pi(t,\xi)^2}\pi_t(t,\xi)d\xi$$

But we're generally interested in $\Psi(t,\omega)=0$, so (fixing A) this happens if

$$\frac{\theta^2}{2} + \frac{r\theta\omega}{\sigma\pi(t,\omega)} - \frac{\theta c(t,\omega)}{\sigma\pi(t,\omega)} - \frac{\sigma\theta}{2}\pi_\omega(t,\omega) - r - \int_1^\omega \frac{\theta}{\sigma\pi(t,\xi)^2}\pi_t(t,\xi)d\xi$$

is independent of ω. If we differentiate again, this is equivalent to:

$$\pi_t(t,\omega) = -\frac{\sigma^2}{2}\pi(t,\omega)^2\pi_{\omega\omega}(t,\omega) + (c(t,\omega)-r\omega)\pi_\omega(t,\omega)$$
$$-\pi(t,\omega)c_\omega(t,\omega) + r\pi(t,\omega).$$

This is Black's Equation (essentially due to Black, '68).

OR, in integrated form:

$$\beta(t) = \int_1^\omega \frac{\pi_t(t,\xi)}{(\pi(t,\xi))^2}d\xi + \frac{\sigma^2}{2}\pi_\omega(t,\omega) + \frac{c(t,\omega)}{\pi(t,\omega)} - \frac{\sigma^2}{2}\pi_{\omega\omega}(t,\omega)$$

This gives us a consistency condition that π, c must satisfy in order for them to correspond to an optimal consumption problem. Is this the only condition?

The answer will essentially be yes, but we need to be more careful about the budget constraints.
Clearly natural to assume that $\pi(t, 0) = c(t, 0) = 0 \quad \forall t \geq 0$, but this does not mean that we will consume "all" our wealth. (and we need to be more careful what this means on an infinite horizon).

Recall instead:

A given consumption strategy C_t can be financed with initial wealth x iff

$$\mathbb{E}\left[\int_0^\infty C_t Z_t \, dt\right] \leq x$$

$$\text{where} \quad Z_t = \exp\left\{-rt - \theta B_t - \frac{\theta^2 t}{2}\right\},$$

and clearly if we expect a strategy to be optimal, we must have equality [Otherwise increase consumption over initial period $[0, \varepsilon]$ say].

Then the Lagrangian formulation is:

$$\sup_{C_t} \mathbb{E}\left[\int_0^\infty u(t, C_t)\, dt - \lambda\left(\int_0^\infty C_t Z_t \, dt - x\right)\right]$$

for some $\lambda = \lambda(x)$.

$$\text{Optimality} \Rightarrow \quad u_c(t, c(t, W_t)) = \lambda Z_t.$$

If we write $I(t, \cdot)$ for inverse in space of $u_c(t, \cdot)$, then

$$c(t, W_t) = I(t, \lambda Z_t)$$

But consumption should be increasing in wealth, so:

$$W_t = f(t, \lambda Z_t), \quad \text{some fn } f.$$

Then Itô $\Rightarrow$

$$dW_t = \lambda f_z(t, \lambda Z_t)\, dZ_t + f_t(t, \lambda Z_t)\, dt + \tfrac{1}{2} f_{zz}(t, \lambda Z_t) \lambda^2 \, d\langle Z \rangle_t$$

$$\Rightarrow \quad dW_t = -\theta\lambda Z_t f_z \, dB_t + \{ f_t + \tfrac{1}{2}\theta^2\lambda^2 Z_t^2 f_{zz} - r\lambda Z_t f_z \} dt$$

$$\Rightarrow \quad \sigma\pi(t, f(t,z)) = -\theta z f_z(t,z) \qquad (z = \lambda Z_t)$$

$$\& \quad r f(t,z) - c(t, f(t,z)) + \theta\sigma\pi(t, f(t,z))$$

$$= f_t(t,z) + \tfrac{1}{2}\theta^2 z^2 f_{zz}(t,z) - r z f_z(t,z).$$

We can similarly manipulate these two equations to ~~get~~ eliminate f and get Black's equation again.

Now suppose $F(t,w)$ is the (spatial) inverse to $f(t,z)$, so:

$$F(t, f(t,z)) = z, \quad \Rightarrow \quad F_w(t,w) = \frac{1}{f_z(t, F(t,w))}$$

$$\Rightarrow \quad F_w(t,w) = -\frac{\theta F(t,w)}{\sigma\pi(t,w)}$$

$$\Rightarrow \quad F(t,w) = e^{A(t)} \exp\left\{ -\frac{\theta}{\sigma}\int_1^w \frac{d\xi}{\pi(t,\xi)} \right\}.$$

So if $Y(t, c(t,w)) = w$, then

$$u_c(t, c(t,W_t)) = \lambda Z_t = F(t, W_t)$$

$$\Rightarrow \quad u_c(t,b) = F(t, Y(t,b)).$$

and we can recover u from π, c (up to various constants).
$\left[H(t,c) := \int_1^c F(t, Y(t,b))\, db \right]$

Def$^{\underline{n}}$
- $u : [0,\infty)^2 \to [-\infty, \infty)$ is a regular utility fn if ~~$u(t,\cdot)$ is~~ C^2, strictly concave, increasing, and satisfies Inada condition $u_c(t,0) = \infty$, $u_c(t,\infty) = 0$, + diff'bility cond.s
  Then $\mathcal{A}^u(x)$ is set of admissible (c,π): either $\int u(t,c_t)_+ dt < \infty$ or $\int u(t,c_t)_- dt < \infty$
- π, c a regular consumption/investment pair if
  - $\forall t \geq 0$, $c(t,0) = 0$, $c(t,\cdot)$ strictly increasing, unbdd, ~~[illegible]~~ + diff'bility cond.

– for each $t \geq 0$, $\pi(t, 0) = 0$, $\pi(t,\cdot)$ strictly positive,

$$\int_0^1 \frac{d\xi}{\pi(t,\xi)} = \infty = \int_1^\infty \frac{d\xi}{\pi(t,\xi)} \quad + \text{ diff'bility conds.}$$

<u>Theorem:</u> $\forall\, x > 0$ TFAE:

(i) $c(t, W_t^x)$, $\pi(t, W_t^x)$ achieve a finite max in

$$\max_{c,\pi \in \mathcal{A}^u(x)} \mathbb{E}\left[\int_0^\infty u(t, c_t)\, dt\right]$$

for a regular utility fn u, for which

$$\exists\, \lambda > 0 \text{ s.t. } x = \mathbb{E}\left[\int_0^\infty Z_t\, I(t, \lambda Z_t)\, dt\right]$$

(ii) c, π regular consumption/investment pair s.t. Black's equation holds &

$$\mathbb{E}\left[\int_0^\infty Z_t\, c(t, W_t^x)\, dt\right] = x,$$

and for some $0 < x_0 \leq x$ $\mathbb{E}\left[\left|H(t, c(t, W_t^{x_0}))\right|\right] < \infty$, a.e. $t \geq 0$ & $\int_0^\infty \mathbb{E}\left[H(t, c(t, W_t^{x_0})) - h(t)\right]^+ dt < \infty$, where $h(t) = \mathbb{E}\left[H(t, c(t, W_t^{x_0}))\right]$.

Moreover, we then have $u_c(t,c) = H_c(t,c)$, $\mathcal{A}^u(x) = \underbrace{\mathcal{A}(x)}_{\text{admissible strat.}}$, and in (i), we can take $u(t,c) = H(t,c) - h(t)$.