Incomplete Markets: Hedging under constraints.

<u>Money market</u>: $dS_0(t) = r S_0(t)\, dt \Rightarrow S_0(t) = e^{rt}$

N stocks: $dS_n(t) = b_n S_n(t)\, dt + \sum_{d=1}^{D} \sigma_{nd}\, S_n(t)\, dW^d(t)$,

for $t \in [0,T]$, $n = 1, \dots, N$.

$(\mathcal{F}_t) = (\mathcal{F}_t^N) \cup \{N_T\}$

A portfolio process $(\pi_0(\cdot), \pi(\cdot))$, $(\mathcal{F}_t)$-prog. meas.

For a self-financed port. proc. the gains process $G(t)$ is given by $dG(t) = \frac{G(t)}{S_0(t)} dS_0(t) + \pi'(t)\, dR_t$,

the excess yield process $R_t = (b - r \cdot 1)t + \sigma W(t)$.

$e^{-rt} G(t)$ is tame, i.e. a.s bdd below, by a constant.

$X(t) = x + G(t)$

A self-financed portfolio process $\pi(\cdot)$ is an arbitrage opportunity if $G(T) \geq 0$ a.s., $\mathbb{P}(G(T) > 0) > 0$.

$\mathcal{M}$ is a financial market: $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$

<u>Theorem 4.2</u>: $\mathcal{M}$ is viable iff market price of risk exists: $\theta = (b - r1)\sigma^{-1}$.

$$\mathbb{P}_0(A) := \mathbb{E}[Z_0(T)\mathbb{1}_A] \qquad \forall A \in \mathcal{F}_T$$

$$Z_0(t) = \exp\{-\theta' W(t) - \tfrac{1}{2}\|\theta\|^2 t\}.$$

$$W_0(t) = W(t) + \theta t \qquad \forall t \in [0,T].$$

$e^{-rt}G(t)$, $e^{-rt}X(t)$ are M-gales under P_0.

$B \in \mathcal{F}_T$ s.t. $e^{-rT}B$ a.s. bdd from below,
$x = \mathbb{E}_0[e^{-rT}B] < \infty$.

B is financeable if there is a tame, x-financed portfolio process $(\pi_0(\cdot), \pi(\cdot))$ where the associated wealth process X satisfies $X(t) = B$ a.s. and

$$e^{-rT}B = x + \int_0^T e^{-ru}\pi'(u)\,\sigma\, dW_0(u) \quad \text{a.s.}$$

A financial market is complete if every $\mathcal{F}_T$-meas. contingent claim B is financeable.

<u>Theorem</u> 1·6·6 : A standard (= viable) financial market is complete iff $N=D$ and σ is non-singular.

<u>Incomplete markets</u>:

Not possible to replicate/hedge every contingent claim perfectly due to portfolio constraints, regardless of the initial cash available.

Therefore we want to find a superreplicating portfolio process and an initial wealth x, ie. $X^{x,\pi}(T) \geq B$ a.s., $X^{x,\pi}(t) \geq 0$ $\forall t \in [0,T]$ a.s.

$$h_{up}(K) = \text{upper hedging price}$$
$$= \inf\{x \geq 0 : \exists\ \pi \in \mathcal{A}(x;K) \text{ with } X^{x,\pi}(T) \geq B \text{ a.s.}\}$$

A portfolio process is admissible for the initial wealth $x \geq 0$ and constraint set K, is denoted by $\pi \in \mathcal{A}(x;K)$, if
- K is a non-empty, convex, closed subset of $\mathbb{R}^N$

A contingent claim B is ~~incomplete~~ K-attainable if $h_{up}(K) < \infty$ and $\pi \in \mathcal{A}(h_{up}(K), K)$ with $X^{h_{up}(K),\pi} = B$ a.s.

$$p(t) = \begin{cases} \dfrac{\pi(t)}{X^{x,\pi}(t)} & \text{for } X^{x,\pi}(t) \geq 0 \text{ a.s.} \\ p_* \in K & \text{for } X^{x,\pi}(t) = 0 \text{ a.s.} \end{cases}$$

1-dim case: price a call option with strike $q \geq 0$, but $K = [\alpha, \beta]$, $-\infty \leq \alpha \leq 0 \leq \beta \leq \infty$.

(Lagrangian approach!)

$$V^* = \sup_{x \in X} f(x) \quad \text{subject to } g(x) \geq 0.$$

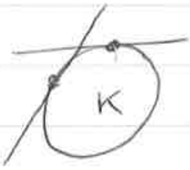
$$\sup_{\substack{x \in X \\ g(x) \geq 0}} f(x) = \sup_{\substack{x \in X \\ \cancel{(g(x) \geq 0)}}} \inf_{\lambda \geq 0} \{f(x) + \lambda g(x)\}$$

$$\leq \inf_{\lambda \geq 0} \sup_{x \in X} \{f(x) + \lambda g(x)\} = V_*.$$

Unconstrained problem.

$$h_{up}(K) = \inf\{x \geq 0 : \exists\, \pi \in \mathcal{A}(x;K),\ X^{x,\pi}(T) \geq B \text{ a.s.}\}$$

Supporting hyperplane theorem:



Support function of set K: $\zeta(\nu) = \sup_{p \in K} (-p'\nu)$.

Effective domain: $\hat{K} := \{\nu \in \mathbb{R}^N \mid \zeta(\nu) < \infty\}$

$$p \in K \iff \zeta(\nu) + p'\nu \geq 0 \quad \forall \nu \in \hat{K}.$$

$$p \notin K \iff \zeta(\nu) + p'\nu < 0, \text{ some } \nu \in \tilde{K}.$$

$\mathcal{H}$ is Hilbert space of $(\mathcal{F}_t)$-p.m. $\nu : [0,T] \times \Omega \to \mathbb{R}^N$,

$$\langle \nu_1, \nu_2 \rangle := \mathbb{E}\left[\int_0^T \nu_1(t)\, \nu_2(t)\, dt\right] < \infty.$$

$D \subset \mathcal{H}$, where $\nu : [0,T] \times \Omega \longrightarrow \hat{K}$, $\mathbb{E}\left[\int_0^T \zeta(\nu(t))\,dt\right] < \infty$.

$$\inf_{p \in K}\{x(p)\} = \inf_{p \in K} \sup_{\nu \in D} \left\{x(p) \cdot \exp\left\{-\left(\zeta(\nu(t)) + p(t)'\nu(t)\right)\right\}\right\}.$$

$$\geq \sup_{\nu \in D} \inf_{p} \left\{x(p) \cdot \exp\left\{-\left(\zeta(\nu(t)) + p(t)'\nu(t)\right)\right\}\right\}$$

$$= \sup_{\nu \in D} u_\nu$$

Auxillary Market : $\mathcal{M}_\nu$, $r_\nu(t) = r + \zeta(\nu(t))$
$b_\nu(t) = b(t) + \zeta(\nu(t))\mathbf{1}$.
for $t \in [0,T]$.

$$S_0^{(\nu)}(t) = S_0(t) \exp\left\{ \int_0^t \zeta(\nu(s))\,ds \right\}$$

$$S_n^{(\nu)}(t) = S_n(t) \exp\left\{ \int_0^t (\zeta(\nu(s)) + \nu_n(s))\,ds \right\}$$

$$\Theta_\nu(t) = \Theta(t) + \sigma^{-1}(t)\,\nu(t)$$

$$W_\nu(t) = W(t) + \int_0^t \Theta_\nu(s)\,ds = W_0(t) + \int_0^t \sigma^{-1}(s)\,\nu(s)\,ds$$

is a B.M. under P^ν, $P_\nu(A) = E[Z_\nu(T); A]$, $A \in \mathcal{F}_T$.

In $\mathcal{M}_\nu$, new disc. wealth process $e^{-rt - \int_0^t \zeta(\nu_s)\,ds} X_\nu^{x,\pi}(t)$

$$= x + \int_0^t e^{-rs - \int_0^s \zeta(\nu(u))\,du} \pi'(s)\,\sigma(s)\,dW_\nu(s)$$

Main hedging result : (Thm 5.6.2)

Unconstrained hedging price of B in $\mathcal{M}_\nu$ is $u_\nu = E^\nu\left[\frac{B}{S_0^{(\nu)}(T)}\right]$

$u_\nu < \infty$, the unconstrained hedging portfolio, π_ν, is any portfolio process that satisfies $X_\nu^{u_\nu,\pi_\nu}(t) = E^\nu\left[\frac{S_0^{(\nu)}(t)}{S_0^{(\nu)}(T)} B \,\middle|\, \mathcal{F}_t\right]$.

Theorem 6.2: For any contingent claim B, we have the representation

$$h_{up}(K) = \sup_{v \in D} u_v \,. \quad ⊛$$

Further, if $\hat{u} := \sup_{v \in D} u_v < \infty$, there exists a portf. proc. $\hat{\pi} \in \mathcal{A}(\hat{u}; K)$ with wealth process
$$X^{\hat{u},\hat{\pi}}(t) = \operatorname{esssup}_{v \in D} E^v\left[\frac{S_0^{(0)}(t)}{S_0^{(v)}(T)} B \,\Big|\, \mathcal{F}_t\right]$$

for $t \in [0,T]$.

In particular, $X^{\hat{u},\hat{\pi}}(T) = B$ a.s.

Sketch proof

1.) $h_{up}(K) \geq \hat{u}$.

Suppose there exists $\pi_t \in \mathcal{A}$ s.t. $p(t) \in K$, a.e. $t \in [0,T]$, and $X^{x,\pi}(T) \geq B$, a.s. $(\pi_t \in \mathcal{A}(x; K))$.

$$e^{-rt} X_v^{x,\pi}(t) = x + \int_0^t e^{-rs}\Big[\big(X_v^{x,\pi}(s) \delta(v(s)) + \pi'(s)' v(s) \big) ds + \pi'(s)\, \sigma(s)\, dW_0(s) \Big]$$

$$= x + \int_0^t e^{-rs} X_v^{x,\pi}(s)\Big[\underbrace{\big(\delta(v(s)) + p'(s)' v(s)\big)}_{\geq 0 \text{ if } p(t) \in K} ds + p(s)'\, \sigma(s)\, dW_0(s) \Big]$$

$$e^{-rt} X^{x,\pi}(t) = x + \int_0^t e^{-ru}\, \pi'(u)\, \sigma(u)\, dW_0(u) \qquad \left(p(t) = \frac{\pi(t)}{X^{x,\pi}(t)} \right)$$

Therefore $X_v^{x,\pi}(T) \geq X^{x,\pi}(T) \geq B$ a.s.; and

$$E^v\left[S_0^{(v)}(T)^{-1} B \right] \leq x. \Rightarrow x \geq \sup_{v \in D} u_v =: \hat{u}.$$

2) $h_{up}(K) \le \hat{u}$.

Suppose $\hat{u} = u_{\hat{\nu}}$, some $\hat{\nu} \in \mathcal{D}$.
(ie. we assume that $\exists$ optimal dual process!)

Define $\hat{X}(t) := \mathbb{E}^{\hat{\nu}}\left[\frac{B}{S_0^{(\hat{\nu})}(T)} S_0^{(\hat{\nu})}(t) \Big| \mathcal{F}_t \right]$

$= X^{\hat{\pi}, \hat{u}}(t)$, some $\hat{\pi}$, NTS $\hat{\pi} \in K$.

Then $\frac{\hat{X}(\cdot)}{S_0^{(\mu)}(\cdot)}$ is a $\mathbb{P}_\mu$ - supermartingale (by a DPP-type argument)

$$\frac{d\left(S^{(\mu)}(t)^{-1} \hat{X}(t)\right)}{S^{(\mu)}(t)^{-1} \hat{X}(t)} = \left(\zeta(\hat{\nu}(t)) - \zeta(\mu(t))\right) dt + p(t)'\sigma(t)\, dW_{\hat{\nu}}(t)$$

$$= \left(\zeta(\hat{\nu}(t)) - \zeta(\mu(t)) + p(t)'(\nu(t) - \mu(t))\right) dt$$
$$+ p(t)'\sigma(t)\, dW_\mu(t)$$

As .. $\mathbb{P}_\mu$ - supermartingale, follows that $\forall \mu \in \mathcal{D}$, $\nu = \hat{\nu}$

$$\left(\zeta(\nu(t)) - \zeta(\mu(t)) + p(t)'(\nu(t) - \mu(t))\right) \le 0,$$

$\mu = 0 \Rightarrow \quad \zeta(\nu(t)) + p(t)'\nu(t) \le 0$

$\mu = N \cdot \nu \in \mathcal{D} \quad -(N-1)\left(\zeta(\nu(t)) + p(t)'\nu(t)\right) \le 0$

$\Rightarrow \left(\zeta(\nu(t)) + p(t)'\nu(t)\right) = 0$

$\Rightarrow -\zeta(\mu(t)) - p(t)'\mu(t) \le 0 \quad \forall \mu \in \mathcal{D} \Rightarrow p \in K.$

Now consider a Black-Scholes market:

$$S(t) = S(0) \exp\{\mu t + \sigma W_t - \tfrac{1}{2}\sigma^2 t\}$$
$$S_0(t) = \exp\{rt\}$$

Suppose $K = [\alpha, \beta]$, $\alpha < 0 < \beta$, so $\zeta(v) = -\alpha v_+ + \beta v_-$.

Suppose we want to find the upper hedging price of $B = \phi(S(T)) \geq 0$ i.e. want

$$\mathbb{E}^{\mathbb{Q}^v}\left[B / S_0^v(T) \right], \quad S_0^v(t) = \exp\left\{ \int_0^t (r + \zeta(s))\, ds \right\}$$

and $$S^v(t) = S(0) \exp\left\{ \mu t + \int_0^t (\zeta(v_s) + v_s)\, ds + \sigma W_t - \tfrac{1}{2}\sigma^2 t \right\}$$

$$\Rightarrow \mathbb{E}^{\mathbb{Q}^v}\left[B / S_0(T) \right] = \mathbb{E}^{\mathbb{Q}^v}\left[e^{-rT - \int_0^T \zeta(v_s)\, ds} \phi\left(S_0 e^{\mu t + \sigma W_T - \frac{1}{2}\sigma^2 T} \right) \right]$$
$$= \mathbb{E}^{\mathbb{Q}^v}\left[e^{-rT - \int_0^T \zeta(v_s)\, ds} \phi\left(S_0\, e^{rt + \sigma W^v(T) - \frac{1}{2}\sigma^2 T - \int_0^T v_s\, ds} \right) \right]$$

But if we write $\tilde{v}_T = \int_0^T v_s\, ds$, since ζ is convex:

$$\int_0^T \zeta(v_s)\, ds \geq \zeta\left(\int_0^T v_s\, ds \right) = \zeta(\tilde{v}_T)$$

$$\Rightarrow \sup_v \mathbb{E}^{\mathbb{Q}^v}\left[B / S_0(T) \right] \leq \sup_{\tilde{v}_T} \left\{ \mathbb{E}^{\mathbb{Q}^v}\left[e^{-rT} e^{-\zeta(\tilde{v}_T)} \phi\left(S_0 e^{rT + \sigma W^v(T) - \frac{1}{2}\sigma^2 T} \times e^{-\tilde{v}_T} \right) \right] \right.$$
$$\leq \mathbb{E}^{\mathbb{Q}^v}\left[\sup_{\tilde{v}_T} \left[e^{-rT} e^{-\zeta(\tilde{v}_T)} \phi(- -) \right] \right]$$

If we consider specifically $\phi(x) = (x-k)_+$, and suppose $\beta > 1$ (other cases easier):

We get:

$$\hat{\phi}(x) = \sup_{v} \left[e^{-\mathfrak{Z}(v)} (x e^{-v} - k)_+ \right] = \begin{cases} (x-k)_+ & x \geq \frac{k\beta}{\beta-1} \\ \left(\frac{x}{\beta}\right)\left(\frac{\beta-1}{k}\right)^{\beta-1} & x \leq \frac{k\beta}{\beta-1} \end{cases}$$

$$\| $$

$$\sup_{v \leq 0} \left[e^{\beta v} (x e^{-v} - k)_+ \right]$$

i.e. Upper-hedging price $= E^{\mathbb{Q}}\left[e^{-rT} \hat{\phi}\left(S_0 e^{rT + \sigma W^{\mathbb{Q}}(T) - \frac{1}{2}\sigma^2 T} \right) \right]$

(W_0 a $\mathbb{Q}$-BM)

The price of a modified payoff under the original model!