Complete Financial Markets.

\[ W_t = (W_t^{(1)}, \ldots, W_t^{(N)}) \]  standard BM on complete probability space \((\Omega, \mathcal{F}, P)\) with \(W_0 = 0\) a.s., \(\mathcal{F}_t\) natural filtration.

Money market: \(\) has initial price \(S_0(0) = 1, S_0(t) e^{rt}\) at time \(t\), \(dS_0(t) = S_0(t) \, r \, dt\).

\(N\) stocks prices per share \(S_1(t), \ldots, S_N(t)\) at time \(t\), initial prices \(S_1(0), \ldots, S_N(0), S_n(t)\) continuous, strictly positive, satisfies SDE:

\[ dS_n(t) = S_n(t) \left[ \mu_n(t) \, dt + \sum_{i=1}^{2} \sigma_{n,i}(t) \, dW^{(i)}(t) \right] \quad \forall t \in [0,T] \]

which has solution:

\[
S_n(t) = \exp \left\{ \int_0^t \sum_{i=1}^{D} \sigma_{n,i}(s) \, dW^{(i)}(s) + \int_0^t \mu_n(s) \, ds \right\}
- \frac{1}{2} \sum_{i=1}^{D} \sigma_{n,i}(s)^2 \int_0^t ds \]

Defn (1.3) A financial market \(\mathcal{M} = (\mathcal{S}, \mathcal{B}, \mathcal{S}(0))\) consists of:

(i) a prob. space \((\mathcal{S}, \mathcal{F}, P)\)

(ii) a \(D\)-dim BM \(W\)

(iii) a constant \(T > 0\), terminal time

(iv) a risk-free interest rate \(r\)

(v) a process \(\mathcal{N}\), \(N\)-dim mean-reverting rehnn process \(b(t)\), \(\int_0^t \| b(t) \| \, dt < \infty\)
(vi) a prog. meas., \((N \times D)\)-matrix valued volatility process \(\sigma(.)\)

\[
\operatorname{st} \sum_{n=1}^{N} \sum_{d=1}^{D} \int_0^T \sigma_{nd}^2 (t) \, dt < \infty \quad \text{a.s.}
\]

(vii) a vector of positive, constant initial stock prices

\(S(0) = (S_1(0), S_2(0), \ldots, S_N(0))\)

**Defn (2.1) Portfolio process/gains process.**

A portfolio process \((\Pi_0(\cdot), \Pi(\cdot))\) consists of prog. meas. \(R^N\)-valued process \(\Pi_0(\cdot)\) and \(R^N\)-valued p.m. proc. \(\Pi(.)\) s.t.

\[
\int_0^T |\Pi_0(t) + \Pi(t)| \cdot 1 \, dt < \infty, \quad \int_0^T \| \Pi(t) - (b(t) - r \cdot 1) \| \, dt < \infty
\]

\[
\int_0^T \| \sigma(t) \Pi(t) \| \, dt < \infty.
\]

The **gains process** \(G(.)\) is

\[
G(t) := \int_0^t \Pi_0(\sigma) \, r \, ds + \int_0^t \Pi(s) b(s) \, ds + \int_0^t \Pi(s)^T \sigma(s) \, dW(s)
\]

\(\Pi\) is self-financed if \(G(t) = \Pi_0(t) + \Pi(t) \cdot 1, \quad t \in [0, T]\).

**Doubling Strategies, Ex 2.3:**

\(N = D = 1, \ r = 0, \ b = 0, \ \sigma = 1\). \(G(t) = \int_0^T \Pi(\cdot) \, dW(t)\).

Consider \(I(t) = \int_0^t \sqrt{\frac{1}{T-u}} \, dW(u)\) with
\[ \langle I \rangle_t(t) = \int_0^t \frac{1}{t-u} \, du - \log \left( \frac{t}{t-u} \right) \]

Hence \( I(t) \) is a martingale. Then \( \langle I \rangle_s(s) = T - T e^{-s} \), \( s \in (0, \infty) \).

Take the time-changed process \( \tilde{I}(s) = I(T - T e^{-s}) \)

such that we have
\[ \langle \tilde{I} \rangle(s) = 0 \Rightarrow \tilde{I}(s) \text{ is BM} \]

\[ \Rightarrow \limsup_{t \to T} I(t) = \infty = -\liminf_{t \to T} I(t) \]

Set \( T_\alpha = \inf \{ t \in (0, T) : I(t) = \alpha \} \wedge T \)

Define \( \tau(t) = \frac{1}{t-t} \int_{t}^{t} \mathbb{1} \{ t \leq t \} \, dW(u) = I(t \wedge \tau_\alpha) \)

then \( \tau(t) = \int_{t}^{t} \frac{1}{t-t} \, dW(u) = I(t \wedge \tau_\alpha) \)

\[ \Rightarrow \tau(T) = I(T \wedge \tau_\alpha) = \alpha. \]

Then: excess rate of returns, \( R(t) = \int_0^t (b(u) - r) \, du + \int_0^t \theta(u) \, dB(u) \)

\( \pi \) is tame if
\[ e^{-r \tau} \theta(t) = M_0^\pi(t) := \int_0^t e^{-ru} \pi^\tau(u) \, dB(u) \]

is a.s. bounded below by a constant.

**Def. 4.1**

A given, tame, self-financed \( \pi \) is an arbitrage opportunity if \( \theta(t) > 0 \) a.s. and \( \mathbb{P}(\theta(t) > 0) > 0. \)
\begin{center}

\textbf{Theorem 4.2 (Market price of risk)}

\(M\) viable \iff \(\exists\) program \(\theta(t)\) \((R^d\)-valued process\) s.t. called \textit{market price of risk}\ s.t. for \(2\)-ae. \(t \in [0, T]\), the risk premium is related to \(\theta\) by:

\[
\theta(t) - \frac{1}{2} \sigma(t)^T \sigma(t) = \sigma(t) \theta(t), \quad \text{a.s.}
\]

Conversely, if such a \(\theta(t)\) exists, and

\[
\int_0^T |\sigma(s)|^2 \, ds < \infty, \quad \mathbb{E} \left[ \exp \left\{ -\int_0^T \sigma(s) \, dW(s) - \frac{1}{2} \int_0^T (\sigma(s))^2 \, ds \right\} \right] = 1.
\]

then \(M\) is viable.

\textbf{Idea - assume} \(\pi\) is s.t.

\[
\pi^T(t) \sigma(t) = 0,
\]

\[
\pi^T(t) \left[ b(t) - r \, 1 \right] \neq 0
\]

\(\rightarrow\) vector in \(\ker(\sigma(t))\) has to be orthogonal to \((b(t) - r \, 1)\).

\[
\text{Then} \quad \mathcal{R}(t) = \int_0^t \sigma(u) \left[ \sigma(u) \, du + dW(u) \right]
\]

\textbf{Definition 1.5 :} \(M\) is a standard EM if:

(i) viable, (ii) \(\sigma \in \mathcal{D}\), (iii) \(\theta\) s.t.

\[
\int_0^T \|\theta(t)\|^2 \, dt < \infty
\]

(iv)

\[
Z_0(t) = \exp \left\{ -\int_0^t \sigma(s) \, dW(s) - \frac{1}{2} \int_0^t \|\sigma(s)\|^2 \, ds \right\}
\]

\(0 \leq t \leq T.

Define \(W_0(t) = W(t) + \int_0^t \sigma(s) \, ds\)

is \(\mathcal{D}\)-dim. B.M. under \(\mathbb{P}_0\).
So: \[ R(t) = \int_0^t \theta(u) \, dW_0(u). \]

\[ e^{-rt} G(t) = \int_0^t e^{-ru} \pi^*(u) \, dR(u). \]

**Thm 5.6:** Under \( P_0 \), \( (e^{-rt} G(t), \, t \in (0, T)) \) corresponding to some self-financed \( \Pi \), is a local martingale, bounded from below, hence a supermartingale.

In particular, \( E_0 [e^{-rT} G(T)] \leq 0 \).
And \( e^{-rt} G(t) \) is a martingale iff mean is 0.

**Note:** If \( \int_0^t e^{-ru} \pi^*(u) \, dR(u) \) is a martingale, then \( \Pi \) is a martingale generating

**Completeness:**

**Defn 6.1:** \( M \) standard FM \( B, \, \mathcal{F}_T \)-r.v. s.t. \( e^{-rt} B \) a.s. limit from below and
\[ r = E_0 [e^{-rT} B] < \infty \] (6.1)

i) \( B \) financeable if \( \exists \) tame, \( \theta \)-financed \( \Pi \) such that \( B \) wealth process \( X(t) = \Pi + G(t) \) satisfies \( X(T) = B \), i.e.
\[ e^{-rT} B = \theta + \int_0^T e^{-ru} \pi^*(u) \sigma(u) \, dW_0(u) \] (6.2)
Prop 6.2 \( M \) complete \( \iff \) \( B \) \( F^+_\tau \)-measurable
\( \text{satisfying } E_0 \left| B e^{-r\tau} \right| < \infty \) \\
\( \text{and } x \text{ as above, } \exists \text{ m.g.} \) \\
\( x \)-financed \( \Pi \) satisfying (6.2)

Theorem 6.6: \( M \), a standard FM, is complete \( \iff \) \( N=0 \)
\( \sigma(t) \) is non-singular \( \text{Leb-a.e.} \), \( t \in [0, T] \).

Sketch Proof:
\[
M_0(t) = E_0 \left[ e^{-r\tau} B \mid F^+_\tau(t) \right]
\]

\( M_0(t) = M_0(0) + \int_0^t \phi(s) \, dW_0(s) \)
\[
\frac{\Pi}{x} \quad \text{by Martingale rep. Thm.}
\]
take \( \tau(t) = e^{r\tau} \phi(t) \sigma^{-1}(t) \).

§ European Contingent Claims in a complete market.

(\text{ECC}) "Buyer pays a (fixed) amount \( \Pi \) at time 0 and then obtains \( C \) at time \( T \)."

Seller's gain process
\[
\Gamma'(t) = \begin{cases} 
\Gamma(0) & t < T \\
\Pi-C & t = T 
\end{cases}
\]

Find \( \Gamma'(t) \)-financed \( \Pi \) s.t. \( X(T) = \Pi-C + G(t) \) a.s.
Sellers wealth process
\[ X(t) = \Gamma(0) - C \mathbb{1}_{\{t=T\}} + G(t) \]
and
\[ e^{-rt} X(t) = \Gamma(0) - e^{-rT} C \mathbb{1}_{\{T=t\}} + \int_0^t e^{-ru} \pi^T(u) \sigma(u) \, dw(u) \]

- Suppose \( \pi \) is no-gain, \( X(T) = 0 \) a.s.

\[ \rightarrow x = \mathbb{E}_0 [C e^{-rT}] \leq \Gamma(0) \]

\[ \rightarrow \text{lower bound for } \Gamma(0). \]

- Suppose seller charged \( x \), by prop. 6.2, \( \exists \) no-gain generating \( \pi \) s.t.

\[ e^{-rT} C = x + \int_0^T e^{-ru} \pi^T(u) \sigma(u) \, dw(u) \]
Define
\[ e^{-rt} X(t) = x + e^{-rT} C \mathbb{1}_{\{t=T\}} + \int_0^t e^{-ru} \pi^T(u) \sigma(u) \, dw(u) \]

By (2.6), \( \text{RHS} = 0 \), thus \( X(t) = 0 \) a.s.

\[ e^{-rT} X(T) = e^{-rT} X(t) - e^{-rT} C \mathbb{1}_{\{t<T\}} \]
\[ + \int_T^T e^{-ru} \pi^T(u) \sigma(u) \, dw(u) \]

Def. (2.2): The value of \( C \) at time \( t \), \( V^{\text{EEC}}(t) \)
is the smallest \( \mathcal{F}_t \) measurable RV \( \tilde{C} \) s.t.
if \( X(t) = \tilde{C} \) above, then for some \( \tilde{C} \)

\( \text{no-gain. } \tilde{C} \), \( X(T) \geq 0 \) a.s.
Prop 2.3 \[ \text{VECC}(t) = e^{-r(t-t')} E_0 \left[ C \mathbb{1}_{\{ t' < T \}} | F_t \right]. \]

In particular, \[ \text{VECC}(0) = e^{-rT} E_0[C]. \]

If \( X(t) = 0 \), by (2.8),
\[ e^{-rT} X(t) = E_0 \left[ e^{-rT} C \mathbb{1}_{\{ t' < T \}} | F_t \right]. \]

\[ \Rightarrow \text{lower bound for VECC}(t) \]

\[ \Rightarrow \text{get equality above if we choose } X \]

\[ \Rightarrow \hat{X} \text{ is called hedging portfolio.} \]

European option in a constant coefficient market.
\[ h_n(t,x,y) = p_n \exp \left\{ (r - \frac{1}{2} \sigma^2 \alpha n) t + \alpha n \right\} \]

\[ S_n(u) = h_n(u - t, S(t), \sigma W(u - t) - W(t)) \quad 0 \leq u \leq T. \]

ECC: \[ C = \phi(S(T)). \text{ By prop 2.3} \]
\[ \text{VECC}(t) = e^{-r(t-t')} E_0 \left[ \phi(S(T)) | F_t \right] \]
\[ = e^{-r(t-t')} E_0 \left[ \phi(h(T-t, S(t), \sigma W(t-t'))) \right] \]
\[ = e^{-r(t-t')} \int_{\mathbb{R}^n} \phi(h(T-t, S(t), \sigma z)) \frac{1}{(2\pi (T-t))^{n/2}} e^{-\frac{1}{2} z^2} dz \]
\[ u(s, x) = \int_{\mathbb{R}^N} e^{-rs} \phi(h(s, x, \sigma z)) (2\pi \sigma)^{-\frac{N}{2}} e^{-\frac{1}{2\sigma^2} h^2} \, dz \quad s \geq 0, \]

\[ \phi(x) \quad s = 0. \]

\[ V_{\text{elec}}(t) = u(T-t, S(t)) \]

\[ u(t, x) = E_x \left[ e^{-\eta t} \phi(x_t) \right] \quad \text{solved} \quad u_t = Au - qu, \]

\[ u(0, x) = \phi(x), \]

\[ A = \frac{1}{2} \sum_{n=1}^{N} \sum_{k=1}^{N} \sigma_{nk} x_n x_k \frac{\partial^2}{\partial x_n \partial x_k} + \sum_{i=1}^{N} \frac{r}{x_i} \frac{\partial}{\partial x_i} \]

\[ \frac{\partial}{\partial t} = S_n(t) \frac{\partial}{\partial x_n} (T-t, S(t)). \]