Complete Financial Markets.

$W_t = (W_t^{(1)}, \ldots, W_t^{(D)})$ standard BM on complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $W_0 = 0$ a.s.,

$\mathcal{F}_t$ natural filtration.

<u>Money market</u>: has initial price $S_0(0) = 1$, $S_0(t) e^{rt}$ at time t $(dS_0(t) = S_0 r\,dt)$

<u>N stocks</u> prices per share $S_1(t), \ldots, S_N(t)$ at time t, initial prices $S_1(0), \ldots, S_N(0)$, $S_n(t)$ continuous, strictly positive, satisfies SDE:

$$dS_n(t) = S_n(t)\left[b_n(t)\,dt + \sum_{i=1}^{N} \sigma_{nd}(t)\,dW^{(d)}(t)\right] \quad \forall t \in [0,T]$$

which has solution:

$$S_n(t) = \exp\left\{ \int_0^t \sum_{d=1}^{D} \sigma_{nd}(s)\,dW^{(d)}(s) + \int_0^t b_n(s) - \frac{1}{2}\sum_{d=1}^{D} \sigma_{nd}^2(s)\,ds \right\}$$

<u>Def</u>^{n} (1.3) A financial market $\mathcal{M} = (r, b, \sigma, S(0))$ consists of

(i) a prob. space $(\Omega, \mathcal{F}, \mathbb{P})$
(ii) a D-dim BM W
(iii) a constant $T > 0$, terminal time
(iv) a <u>risk-free</u> interest rate r.
(v) a prog. meas, N-dim mean-rate of return process $b(\cdot)$, $\int_0^T \|b(t)\|\,dt < \infty$ a.s

(vi) a prog. meas., $(N \times D)$-matrix valued <u>volatility process</u> $\sigma(\cdot)$

s.t. $\sum_{n=1}^{N} \sum_{d=1}^{D} \int_0^T \sigma_{nd}^2(t)\, dt < \infty$ a.s.

(vii) a vector of positive, constant initial <u>stock-prices</u> $S(0) = (S_1(0), S_2(0), \dots, S_N(0))$

<u>Def^n</u> (2.1) Portfolio process / gains process.

A portfolio process $(\pi_0(\cdot), \pi_1(\cdot))$ consists of prog. meas. $\mathbb{R}$-valued process $\pi_0(\cdot)$ and $\mathbb{R}^N$-valued p.m. proc. $\pi(\cdot)$

s.t. $\int_0^T |\pi_0(t) + \pi(t) \cdot 1|\, dt < \infty$, $\int_0^T |\pi(t)\,(b(t) - r\,1)|\, dt < \infty$

$$\int_0^T \| \sigma(t)\, \pi(t) \|^2\, dt < \infty.$$

The <u>Gains process</u> $G(\cdot)$ is

$$G(t) := \int_0^t \pi_0(s)\, r\, ds + \int_0^t \pi(s)\, b(s)\, ds + \int_0^t \pi(s)^T \sigma(s)\, dW(s)$$

π is self-financed if $G(t) = \pi_0(t) + \pi(t) \cdot 1$. $t \in [0, T]$.

<u>Doubling Strategies</u> Ex 2.3:

$N = D = 1$, $r = 0$, $b = 0$, $\sigma = 1$. $G(t) = \int_0^T \pi(s)\, dW(s)$.

Consider $I(t) = \int_0^t \sqrt{\frac{1}{T-u}}\, dW(u)$ with

$$\langle I \rangle_t(t) = \int_0^t \frac{1}{T-u}\, du = \log\left(\frac{T}{T-u}\right)$$

Hence $I(t)$ is a martingale. Then $\langle I \rangle^{-1}(s) = T - Te^{-s}$, $s \in (0, \infty)$.

Take the time-changed process $\tilde{I}(s) = I(T - Te^{-s})$ ~~such that~~ which has

$$\langle \tilde{I} \rangle(s) = s \Rightarrow \tilde{I}(s) \text{ is BM}.$$

$$\Rightarrow \limsup_{t \uparrow T} I(t) = \infty = -\liminf_{t \uparrow T} I(t).$$

Set $\tau_\alpha := \inf\{t \in [0, T) : I(t) = \alpha\} \wedge T$

Define $\pi(t) = \sqrt{\frac{1}{T-t}}\, \mathbb{1}\{t \leq \tau_\alpha\}$, $\pi_0(t) = I(t \wedge \tau_\alpha) - \pi(t)$

then $G(t) = \int^{t \wedge \tau_\alpha} \sqrt{\frac{1}{T-t}}\, dW(u) = I(t \wedge \tau_\alpha)$

$$\longrightarrow G(T) = I(T \wedge \tau_\alpha) = \alpha.$$

Then: excess rate of returns, $R(t) = \int_0^t (b(u) - r)\, du + \int_0^t \sigma(u)\, dW(u)$

π is <u>tame</u> if

$$e^{-rt} G(t) = M_0^\pi(t) := \int_0^t e^{-ru} \pi^T(u)\, dR(u)$$

is a.s. bounded below by a const.

<u>Def^n 4.1</u>

A given, tame, self-financed π is an arbitrage opportunity if $G(t) \geq 0$ a.s. and $\mathbb{P}(G(T) > 0) > 0$.

- $\mathcal{M}$ is viable if no arbitrage opportunity exists.

Thm 4.2 (Market price of risk).
$\mathcal{M}$ viable then $\exists$ prog. meas., $\mathbb{R}^D$-valued process $\theta(\cdot)$ called <u>market price of</u> risk s.t. for Leb-a.e. $t \in [0,T]$, the risk premium is related to θ by: $b(t) - \mathbb{1}r = \sigma(t)\,\theta(t)$ a.s.
Conversely, if such a $\theta(\cdot)$ exists, and

$$\int_0^T |\theta(s)|^2 ds < \infty, \quad \mathbb{E}\left[\exp\left\{-\int_0^T \theta(s)\,dW(s) - \frac{1}{2}\int_0^T |\theta(s)|^2 ds\right\}\right] = 1.$$

then $\mathcal{M}$ is viable

<u>Idea</u> - assume π is s.t. $\pi^T(t)\,\sigma(t) = 0$,
$\pi^T(t)\,[b(t) - r\mathbb{1}] \neq 0$

$\longrightarrow$ vector in $\ker(\sigma(t))$ has to be orthogonal to $(b(t) - r\mathbb{1})$.

<u>Then</u> • $R(t) = \int_0^t \sigma(u)\,[\theta(u)\,du + dW(u)]$

<u>Def^n</u> 1.5 : $\mathcal{M}$ is a standard EM if:

(i) viable, (ii) $N \le D$, (iii) θ s.t. $\int_0^T \|\theta(t)\|^2 dt < \infty$

(iv) $Z_0(t) = \exp\left\{-\int_0^t \theta(s)\,dW(s) - \frac{1}{2}\int_0^t \|\theta(s)\|^2 ds\right\}$
$0 \le t \le T$.

Define $W_0(t) = W(t) + \int_0^t \theta(s)\,ds$

is D-dim. B.M. under $\mathbb{P}_0$.

<u>So</u> :
$$R(t) = \int_0^t \theta(u)\, dW_0(u).$$

$$e^{-rt} G(t) = \int_0^t e^{-ru} \pi^T(u)\, dR(u).$$

<u>Thm 5.6</u> : Under $\mathbb{P}_0$ $(e^{-rt} G(t),\ t \in (0,T))$ corresponding to tame, self-financed π, is a local martingale, bounded from below, hence a supermartingale.

In particular, $\mathbb{E}_0[e^{-rT} G(T)] \le 0$.
And $e^{-rt} G(t)$ is a martingale iff mean is 0 $\square$

NB : if $\int_0^t e^{-ru} \pi^T(u)\, dR(u)$ a m'gale, then π is <u>martingale generating</u>.

<u>Completeness</u> :

<u>Def$^{\underline{n}}$ 6.1</u> $\mathcal{M}$ standard FM B, $\mathcal{F}_T$-r.v. s.t. $e^{-rT} B$ a.s. bdd from below and

$$x = \mathbb{E}_0[e^{-rT} B] < \infty \qquad (6.1)$$

i) B financeable if $\exists$ tame, ~~$\exists$~~ x-financed π ~~such that~~ with wealth process $X(t) = x + G(t)$ satisfies $X(T) = B$,

ie.
$$e^{-rT} B = x + \int_0^T e^{-ru} \pi^T(u)\, \sigma(u)\, dW_0(u) \qquad (6.2)$$

(ii) $\mathcal{M}$ is complete if any such D is financeable

Prop 6.2 $\quad \mathcal{M}$ complete $\iff$ B $\mathcal{F}_\phi(T)$-measurable satisfying $\mathbb{E}_0 |Be^{-rT}| < \infty$ and x as above, $\exists$ m'g-gen. x-financed π satisfying (6.2)

Theorem 6.6 : $\mathcal{M}$, a standard FM, is complete iff $N = D$ and $\sigma(t)$ is non-singular Leb-a.e. $t \in [0, T]$.

Sketch Proof:
$M_0(t) = \mathbb{E}_0 [e^{-rT} B | \mathcal{F}_\phi(t)]$ is a martingale, so

$$M_0(t) = \underset{\overset{\parallel}{x}}{M_0(0)} + \int_0^t \phi(s)\, dW_0(s)$$

by Martingale Rep. Thm.

take $\pi(t) = e^{rt} \phi(t) \sigma^{-1}(t)$.

---

§ European Contingent Claims in a complete market.

{ECC "Buyer pays a ~~random~~ (fixed) amount Γ at time 0 and then obtains C at time T".

Seller's gain process
$$\Gamma(t) = \begin{cases} \Gamma(0) & t < T \\ \Gamma(0) - C & t = T. \end{cases}$$

Find $\Gamma(t)$-financed π s.t. $X(T) = \Gamma(0) - C + G(t) \geq 0$ a.s.

Sellers wealth process

$$X(t) = \Gamma(0) - C\,\mathbb{1}\{t=T\} + G(t)$$

and $$e^{-rt}X(t) = \Gamma(0) - e^{-rT}C\,\mathbb{1}\{T=t\} + \int_0^t e^{-ru}\pi^T(u)\sigma(u)\,dW(u)$$

- Suppose π is m'g-gen, $X(T) \geq 0$ a.s.

$\longrightarrow$ $x = \mathbb{E}_0[Ce^{-rT}] \leq \Gamma(0)$.

$\longrightarrow$ lower-bounds for $\Gamma(0)$.

- Suppose seller charges x, by prop. 6.2, $\exists$ m'gale generating $\hat{\pi}$ s.t.

(2.6). $$e^{-rT}C = x + \int_0^T e^{-ru}\pi^T(u)\sigma(u)\,dW_0(u)$$

Define $$e^{-rt}\hat{X}(t) = x + e^{-rT}C\,\mathbb{1}\{t=T\} + \int_0^t e^{-ru}\hat{\pi}^T\sigma(u)\,dW_0(u)$$

By (2.6), RHS $=0$, thus $\hat{X}(T) = 0$ a.s.

$$e^{-rT}X(T) = e^{-rt}X(t) - e^{-rT}C\,\mathbb{1}\{t<T\} + \int_t^T e^{-ru}\pi^T(u)\sigma(u)\,dW_0(u)$$

Def^n (2.2): The value of C at time t, $V^{ECC}(t)$ is the smallest $\mathcal{F}_t$-measurable RV ξ s.t. if $X(t) = \xi$ above, then for some ~~prog~~ mart.gen. π $X(T) \geq 0$ a.s.

Prop 2.3 $\quad V^{ECC}(t) = e^{-r(T-t)} \mathbb{E}_0 [C \mathbb{1}_{\{t<T\}} \mid \mathcal{F}_t].$

In particular, $V^{ECC}(0) = e^{-rT} \mathbb{E}_0 [C].$

If $X(T) \geq 0$, by (2.8),

$$e^{-rt} X(t) \geq \mathbb{E}_0 [e^{-rT} C \mathbb{1}\{t<T\} \mid \mathcal{F}_t]$$

$\Rightarrow$ lower bound for $V^{ECC}(t)$

$\rightarrow$ get equality above if we choose $\hat{X}$

$\rightarrow$ $\hat{\pi}$ is called hedging portfolio.

European options in a constant coefficient market.

$$h_n(t,p,y) = p_n \exp\{(r - \tfrac{1}{2} \underbrace{a_{nn}}_{a = \sigma\sigma^T})t + y_n\}$$

$$S_n(u) = h_n(u-t, S(t), \sigma(W_0(u) - W_0(t)) \qquad 0 \leq t \leq u \leq T.$$

ECC : $\quad C = \phi(S(T))$. By prop 2.3

$$\cancel{V^{ECC}(t)} \quad V^{ECC}(t) = e^{-r(T-t)} \mathbb{E}_0 [\phi(S(T)) \mid \mathcal{F}(t)]$$

$$= e^{-r(T-t)} \mathbb{E}_0 [\phi(h(T-t, S(t), \sigma(W_0(T) - W_0(t))) \mid \mathcal{F}_t]$$

$$= e^{-r(T-t)} \int_{\mathbb{R}^N} \frac{\phi(h(T-t, S(t), \sigma z))\, e^{-\|z\|^2 / 2(T-t)}}{(2\pi(T-t))^{N/2}} \, dz$$

$$u(s,x) = \begin{cases} e^{-rs} \int_{\mathbb{R}^N} \phi(h(s,x,\sigma z)) \, (2\pi s)^{-N/2} e^{-\|z\|^2/2} \, ds & s>0 \\ \phi(x) & s=0. \end{cases}$$

$$\Rightarrow \quad V^{ELC}(t) = u(T-t, S(t))$$

$$u(t,x) = \mathbb{E}_x\left[e^{-qt} \phi(X_t) \right] \text{ solves } u_t = Au - qu$$

$$u(0,x) = \phi(x),$$

$$A = \frac{1}{2} \sum_{n=1}^{N} \sum_{\ell=1}^{N} a_{n\ell} \, x_n x_\ell \frac{\partial^2}{\partial x_n \partial x_\ell} + \sum_{i=1}^{N} r \, x_n \frac{\partial}{\partial x_n}$$

$$\hat{\pi}_n(t) = S_n(t) \frac{\partial u}{\partial x_n}(T-t, S(t)).$$