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When $\pi$ and $c$ are constants, then the generator of $w_t$ acts on $\tilde{v} \in C^2$ by

$$
(A^{c,\pi} \tilde{v})(w) = ((r + (\alpha - r)\pi)w - c)\tilde{v}'(w) + \frac{1}{2}w^2\pi^2\sigma^2\tilde{v}''(w).
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The HJB-equation reads

$$\max_{c,\pi}\{ (A^c,\pi \tilde{v})(w) + \frac{c^\gamma}{\gamma} - \delta \tilde{v}(w) \} = 0 \quad \text{for all } w > 0.$$
The maxima are achieved at

\[ c = \tilde{v}'(w) \frac{-1}{1-\gamma} \quad \text{and} \quad \pi = \frac{-\beta \tilde{v}'(w)}{w \sigma \tilde{v}''(w)} \]
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and hence the HJB-equation is equivalent to

\[ rw\tilde{v}' - \frac{\beta^2}{2} \frac{(\tilde{v}')^2}{\tilde{v}''} + \frac{1-\gamma}{\gamma} (\tilde{v}')^{-\gamma/(1-\gamma)} - \delta \tilde{v} = 0. \]
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It is easy to see that \( v(w) = \gamma^{-1} C^{\gamma-1} w^\gamma \) is solution of this differential equation.
Step ??:
Let \((c_t, \pi_t) \in \mathcal{U}\) be an arbitrary policy and define the process

\[
x_t := \int_0^t \sigma \pi_u \, dz_u.
\]

Then \(w_t\) is given explicitly (proof: Itô’s formula) by

\[
w_t = \left( w - \int_0^t c_s f_s \, ds \right) \mathcal{E}(x_t) \exp \left( rt + \int_0^t (\alpha - r) \pi_u \, du \right)
\]

where \(\mathcal{E}\) is the stochastic exponential of \(x_t\) and

\[
f_s := \exp \left( - rs - \int_0^s \left( (\alpha - r) \pi_u - \frac{1}{2} \sigma^2 \pi_u^2 \right) \, du - \int_0^s \sigma \pi_u \, dz_u \right).
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\(\Rightarrow\) \(w_t\) has moments of all orders by Holder’s inequality and since \(\pi_t\) is bounded.
Step ??:
Define for any policy \((c_t, \pi_t)\) the process

\[
M_t := \int_0^t e^{-\delta s} u(c_s) \, ds + e^{-\delta t} v(w_t),
\]

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\[
M_t = M_0 + \int_0^t e^{-\delta s} \left( (A^c, \pi^c)(w_s) + \frac{c_s^\gamma}{\gamma} - \delta v(w_s) \right) \, ds
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\[ + \sigma C^{\gamma-1} \int_0^t e^{-\delta s} \pi_s w_s^\gamma \, dz_s. \]

\( \Rightarrow \) \( M_t \) is a supermartingale and if \((c_t, \pi_t) = (c^*_t, \pi^*_t)\) it is a martingale. Thus,

\[ v(w) = M_0 \geq \mathbb{E}_w[M_t] = \mathbb{E}_w[\int_0^t e^{-\delta s} u(c_s) \, ds] + \mathbb{E}_w[e^{-\delta t} v(w_t)]. \]
The proof is complete if we can show that

$$\lim_{t \to \infty} \mathbb{E}_w[e^{-\delta t}v(w_t)] = 0$$

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for any \((c, \pi) \in \mathcal{U}\). To this end, observe that by Itô’s formula we may write

$$e^{-\delta t} w^\gamma_t = w^\gamma_0 \mathcal{E}(\gamma x_t) \exp \left( \int_0^t a_s \, ds \right),$$

where

$$a_s = \gamma \left( r + (\alpha - r) \pi_s - \frac{c_s}{w_s} - \frac{1}{2} (1 - \gamma) \pi_s^2 \sigma^2 \right) - \delta.$$
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for any \((c, \pi) \in \mathcal{U}\). To this end, observe that by Itô’s formula we may write

$$e^{-\delta t} w_t^\gamma = w_0^\gamma \mathcal{E}(\gamma x_t) \exp \left( \int_0^t a_s \, ds \right),$$

where

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Since \(a_s \leq -(1 - \gamma) C\) the claim follows. This completes the proof.
Guessing solution for problem with transaction costs.

Ansatz: try $L$ and $U$ absolutely continuous with bounded derivatives, that is,

$$L_t = \int_0^t l_s \, ds, \quad U_t = \int_0^t u_s \, ds, \quad 0 \leq l_s, u_s \leq \kappa$$
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The HJB-equation reads

$$\max_{c,l,u} \left\{ \frac{1}{2} \sigma^2 y^2 \ddot{v}_{yy} + rx \ddot{v}_x + \alpha y \ddot{v}_y + \frac{1}{\gamma} c^\gamma - c \ddot{v}_x \right.\left. \right.$$\left.\right.$$

$$\left.\left.\left. - (1 + \lambda) \ddot{v}_x + \ddot{v}_y \right) l + ((1 - \mu) \ddot{v}_x - \ddot{v}_y) u - \delta \ddot{v} \right\} = 0.$$
Since $\tilde{v}_x$ and $\tilde{v}_y$ are positive (extra wealth gives increased utility), we see that the maxima are attained as follows:

\[
\begin{align*}
c & = (\tilde{v}_x)^{1/(\gamma-1)}, \\
l & = \begin{cases} 
\kappa, & \text{if } \tilde{v}_y \geq (1 + \lambda)\tilde{v}_x, \\
0, & \text{if } \tilde{v}_y < (1 + \lambda)\tilde{v}_x,
\end{cases} \\
u & = \begin{cases} 
0, & \text{if } \tilde{v}_y > (1 - \mu)\tilde{v}_x, \\
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\end{cases}
\]

This indicates that the optimal transaction policies are “bang-bang”: buying and selling either take place at maximum rate or not at all, and the solvency region splits into three regions

- $B$, the region in which stocks are bought,
- $S$, the region in which stocks are sold,
- $NT$ the region where no transactions take place.
Let us analyse the boundary

\[ \tilde{v}_y = (1 + \lambda)\tilde{v}_x \]

between $S$ and $NT$ (a similar argument applies for the boundary between $NT$ and $B$). To this end assume that $\tilde{v} \in C^1$ and that it is homothetic which implies that

\[ \tilde{v}_x(\rho x, \rho y) = \rho^{\gamma-1}\tilde{v}_x(x, y). \]
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It follows that if \( \tilde{v}_y(x, y) = (1 + \lambda)\tilde{v}_x(x, y) \) for some point \((x, y)\), then the same is true for all points along the ray through \((x, y)\).
Heuristic argument suggests so far:

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Heuristic argument suggests so far:

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- boundaries between transaction and no-transaction regions are straight lines through the origin,
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- the finite transaction in $S$ or $B$ moves the portfolio down or up a line of slope $-1/(1 - \mu)$ or $-1/(1 + \lambda)$.
- after the initial transaction, all further transactions must take place at the boundaries, and this suggests a “local time” type of transaction policy,
- meanwhile, consumption takes place at rate $(v_x)^{1/(\gamma - 1)}$. 
In $NT$ the value function $v(x, y)$ satisfies the HJB-equation with $l = u = 0$:

$$\max_c \left\{ \frac{1}{2} \sigma^2 y^2 v_{yy} + (rx - c)v_x + \alpha y v_y + \frac{1}{\gamma} c^\gamma - \delta v \right\} = 0,$$

i.e.,

$$\frac{1}{2} \sigma^2 y^2 v_{yy} + (rx - c)v_x + \alpha y v_y + \frac{1 - \gamma}{\gamma} v_x^{-\gamma/(1-\gamma)} - \delta v = 0.$$
In NT the value function \( v(x, y) \) satisfies the HJB-equation with \( l = u = 0 \):

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\]

The final step now consists of reducing this equation to an equation in one variable. In order to do so, define

\[
\psi(x) := v(x, 1).
\]

By the homothetic property it follows that \( v(x, y) = y^\gamma \psi(x/y) \).
If our conjectured optimal policy is correct then $\nu$ is constant along lines of slope $(1 - \mu)^{-1}$ in $S$ and along lines of slope $(1 + \lambda)^{-1}$ in $B$, and this implies by homothetic property that

$$
\psi(x) = \frac{1}{\gamma}(x + 1 - \mu)^\gamma, \quad x \leq x_0,
$$

$$
\psi(x) = \frac{1}{\gamma}(x + 1 + \lambda)^\gamma, \quad x \geq x_T,
$$

for some constants $A, B$ and $x_0$ and $x_T$ as in the picture.
If our conjectured optimal policy is correct then \( v \) is constant along lines of slope \((1 - \mu)^{-1}\) in \( S \) and along lines of slope \((1 + \lambda)^{-1}\) in \( B \), and this implies by homothetic property that

\[
\psi(x) = \frac{1}{\gamma} (x + 1 - \mu)\gamma, \quad x \leq x_0, \\
\psi(x) = \frac{1}{\gamma} (x + 1 + \lambda)\gamma, \quad x \geq x_T,
\]

for some constants \( A, B \) and \( x_0 \) and \( x_T \) as in the picture. Using the homothetic property again, one can show that \( \psi \) satisfies for \( x \in [x_0, x_T] \),

\[
\beta_3 \psi''(x) + \beta_2 x \psi'(x) + \beta_1 \psi(x) + \frac{1 - \gamma}{\gamma} (\psi'(x))^{-\gamma/(1-\gamma)} = 0,
\]

where \( \beta_1 = -\frac{1}{2} \sigma^2 \gamma (1 - \gamma + \alpha \gamma) - \delta, \beta_2 = \sigma^2 (1 - \gamma) + r - \alpha, \beta_3 = \frac{1}{2} \sigma^2 \).
Theorem (4.1, follows from [?])

Take $0 < x_0 < x_T$ and let $NT$ be the closed wedge shown in the picture, with upper and lower boundaries $\partial S, \partial B$ respectively. Let $c : NT \to [0, \infty)$ be any Lipschitz continuous function and let $(x, y) \in NT$. Then there exists a unique process $s_0, s_1$ and continuous increasing processes $L, U$ such that for $t < \tau = \inf\{t \geq 0 : (s_0(t), s_1(t)) = 0\}$

$$ds_0(t) = (rs_0(t) - c(s_0(t), s_1(t)))dt - (1 + \lambda)dL_t + (1 - \mu)dU_t, \quad s_0(0) = x,$$

$$ds_1(t) = \alpha s_1(t)dt + \sigma s_1(t)dz_t - dU_t, \quad s_1(0) = y,$$

$$L_t = \int_0^t 1\{(s_0(\xi), s_1(\xi)) \in \partial B\}dL_\xi,$$

$$U_t = \int_0^t 1\{(s_0(\xi), s_1(\xi)) \in \partial S\}dU_\xi.$$

The process $\tilde{c}_t := c(s_0(t), s_1(t))$ satisfies condition (2.1)(i).
Define the set of policies that do not involve short selling:

\[ \mathcal{U}' = \{(c, L, U) \in \mathcal{U} : (s_0(t), s_1(t)) \in \mathcal{S}'_\mu \text{ for all } t \geq 0\}, \]

where \( \mathcal{S}'_\mu = \{(x, y) \in \mathbb{R}^2 : y \geq 0 \text{ and } x + (1 - \mu)y \geq 0\}. \)
Theorem (4.2, proof in [?])

Let $0 < \gamma < 1$ and assume Condition A holds. Suppose there are constants $A, B, x_0, x_T$ and a function $\psi : [-1(1-\mu), \infty) \rightarrow \mathbb{R}$ such that

$$0 < x_0 < x_T < \infty,$$

$\psi$ is $C^2$ and $\psi'(x) > 0$ for all $x$,

$$\psi(x) = \frac{1}{\gamma} A(x + 1 - \mu)^\gamma \text{ for } x \leq x_0,$$

$$\beta_3 \psi''(x) + \beta_2 x \psi'(x) + \beta_1 \psi(x) + \frac{1 - \gamma}{\gamma} (\psi'(x))^{-\gamma/(1-\gamma)} = 0 \text{ for } x \in [x_0, x_T],$$

$$\psi(x) = \frac{1}{\gamma} B(x + 1 + \lambda)^\gamma \text{ for } x \geq x_T.$$
Theorem

Let $N_T$ denote the closed wedge

$\{(x, y) \in \mathbb{R}_+^2 : x_T^{-1} \leq yx^{-1} \leq x_0^{-1}\}$

and let $B$ and $S$ denote the regions below and above $N_T$ as in the picture. For $(x, y) \in N_T \setminus \{(0, 0)\}$ define

$c^*(x, y) = y\psi'(x/y)^{-1/(1-\gamma)}$.

Let $\tilde{c}_t^* = c^*(s_0(t), s_1(t))$ where $(s_0, s_1, L^*, U^*)$ is the unique solution of (4.1) with $c := c^*$. Then the policy $(\tilde{c}_t^*(t), L^*(t), U^*(t))$ is optimal in the class $\mathcal{U}'$ for any initial endowment $(x, y) \in N_T$. If $(x, s) \notin N_T$ then an immediate transaction to the closest point in $N_T$ followed by application of this policy is optimal in $\mathcal{U}'$. The maximal expected utility is

$v(x, s) = y^{\gamma}\psi(x/y)$.