Optimal consumption & investment with transaction costs:

(Portfolio selection with transaction costs).

- Look at Davis & Norman paper on optimal consumption and investment.

1) No transaction costs

fixed interest rate

consumption rate.

Model: $dS_0(t) = (rS_0(t) - c(t))\,dt$

1-dim BM

$$dS_1(t) = \alpha S_1(t)\,dt + \sigma S_1(t)\,dZ(t)$$

$S_0(0) = x$, $S_1(0) = y$. drift volatility

Assumption: Money transferred from bank account to stock and vice-versa is cost-less, and instantaneously.

- Utility function: $u(c) = \frac{1}{\gamma} c^{\gamma}$, $\gamma \in (0,1)$.

(similar results for $\gamma < 0$, or $u(c) = \log(c)$)

- Re-parametrise: $W(t) = S_0(t) + S_1(t)$, total wealth.

$$\pi(t) = \frac{S_1(t)}{W(t)}$$

New model:

$$(*)\begin{cases} dW(t) = \left[rW(t) + (\alpha - r)\pi(t)W(t) - c(t)\right]dt \\ \qquad\qquad + \sigma\pi(t)W(t)\,dZ(t). \\ W(0) = x+y. \end{cases}$$

$\Rightarrow$ π and c are control variables.

- Policies : a policy is a pair (c_t, π_t) of $\mathcal{F}_t$ - adapted processes s.t

filt^n gen Z

$$\text{(i)} \quad c(t,\omega) \geq 0, \quad \int_0^t c(s,\omega)\,ds < \infty.$$

(ii) $|\pi(t,\omega)| \leq K$, where K can depend on the policy.

(iii) $W(t) \geq 0$, W is the unique strong solution to $(\star)$

$$\mathcal{U} := \{ (\pi, c) \text{ satisfying (i), (ii), (iii).} \}$$

- Control problem:

$$v(\omega) = \sup_{(c,\pi)\in\mathcal{U}} \underbrace{\mathbb{E}_\omega \left[\int_0^\infty e^{-\delta t} u(c(t))\,dt \right]}_{= J_\omega(c,\pi).}$$

$\delta > 0$, discount factor.

- Technical condition (Condition A):

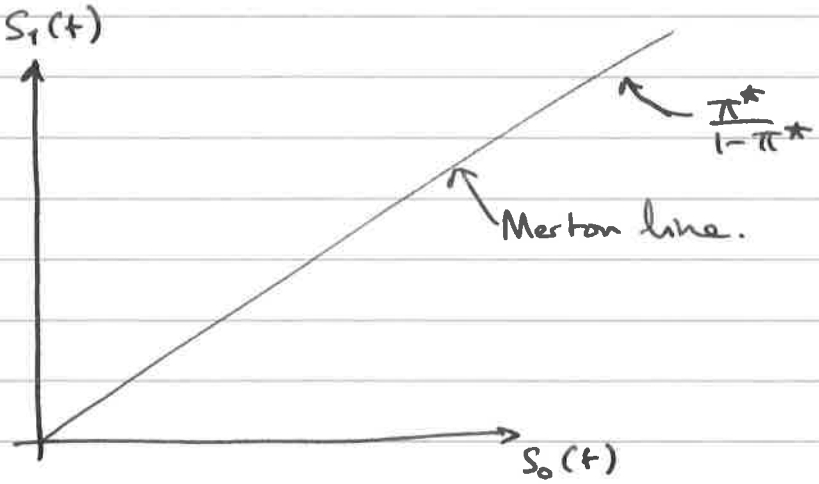
$$\delta > \gamma \left(r + \frac{(\alpha - r)^2}{2\sigma^2(1-\gamma)} \right)$$

Theorem 2.1 : Suppose condition A holds. Define

$$c := \frac{1}{1-\gamma} \left[\delta - \gamma r - \frac{\gamma \beta^2}{2(1-\gamma)} \right], \quad \beta := \frac{\alpha - r}{\sigma}.$$

$$c_t^\star := c\, \omega_t \quad \text{and} \quad \pi_t^\star := \frac{\beta}{(1-\gamma)\sigma}$$

$$v(\omega) = \frac{1}{\gamma} c^{\gamma - 1} \omega^\gamma$$



<u>Remark</u>: If $r < \alpha < r + (1-\gamma)\sigma^2 \Rightarrow \pi^* \in (0,1)$
"hedging".

If $\alpha > r + (1-\gamma)\sigma^2 \Rightarrow \pi^* \in [1,\infty)$
"leverage": borrow to invest in risky asset.

If $\alpha < r$, $\pi^* < 0$ "short selling"

If $\alpha = r$, $\pi^* = 0$ optimally consume initial endowment.

Proof of Theorem 2.1 → See slides!

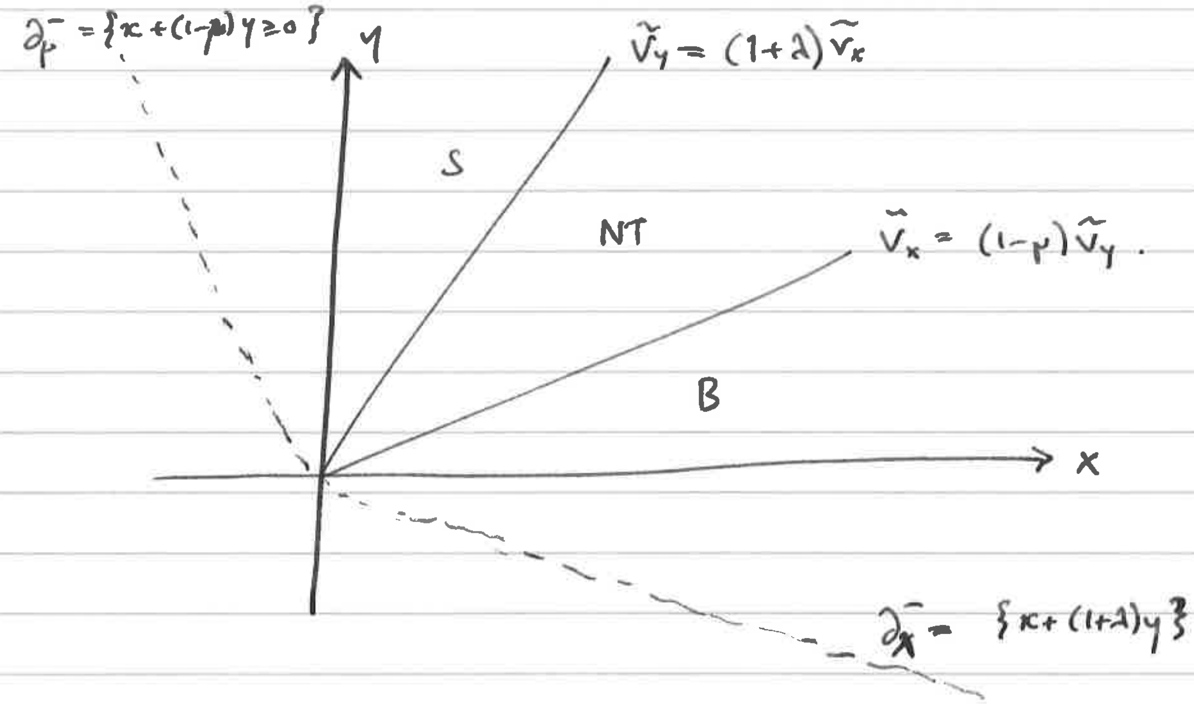
2) With transaction costs. $\lambda \in [0,\infty)$, $\mu \in [0,1]$

$$dS_0(t) = (rS_0(t) - c(t))\,dt - (1-\lambda)\,dL_t + (1-\mu)\,dU_t$$

$$dS_1(t) = \alpha S_1(t)\,dt + \sigma S_1(t)\,dZ(t) + dL_t - dU_t. \quad , \; S_0(0) = x, \; S_1(0) = y.$$

L_t = cumulative purchases of stocks
U_t = ... sales . . . .

$$\mathcal{S}_{\lambda,\mu} = \left\{ (x,y) \in \mathbb{R}^2 \;\middle|\; x + (1-\mu) y \geq 0 \quad \text{and} \quad x + (1+\lambda) y \geq 0 \right\}.$$



- Policy (C_t, L_t, U_t) adapted processes s.t. 2.1(i) and L_t, U_t are right-continuous, non-decreasing, $L_0 = U_0 = 0$.

- Admissible Policy : A ~~triplet~~ policy (c, L, U), ~~where~~ s.t. $\tau = \infty$, where $\tau := \inf\{t > 0 : (S_0(t), S_1(t)) \notin \mathcal{S}_{\lambda,\mu}\}$.

Fact : $\mathcal{U} \neq \emptyset$

~~Prob~~: • Control Problem: $v(x,y) = \max\limits_{(c,L,U) \in \mathcal{U}} \mathbb{E}_{x,y}\left[\int_0^\infty e^{-\delta t} u(c(t))\, dt \right]$

Fact : v has the homothetic property :

For $\rho > 0$, then $v(\rho x, \rho y) = \rho^\gamma v(x,y)$.