Optimal consumption & investment with transaction costs:
(Portfolio selection with transaction costs).

- Look at Davis & Norman paper on optimal consumption and investment.

1) No transaction costs

Model: \[ dS_0(t) = (rS_0(t) - c(t)) \, dt \]
\[ dS_1(t) = \sigma S_1(t) \, dt + \sigma S_1(t) \, dZ_f(t) \]

\[ S_0(0) = x, \quad S_1(0) = y. \] drift
volatility

Assumption: Money transferred from bank account to stock and vice-versa is cost-less, and instantaneously.

- Utility function: \[ u(c) = \frac{1}{y} \, c^y, \quad y \in (0, 1). \] (similar results for \( y < 0 \), or \( u(c) = \log(c) \))

- Re-parametrize: \( W(t) = S_0(t) + S_1(t) \), total wealth.

\[ \pi(t) = \frac{S_1(t)}{W(t)} \]

New model: \[ dW(t) = \left[ rW(t) + (\sigma - r)\pi(t) \, W(t) - c(t) \right] dt \]
\[ + \sigma \pi(t) \, W(t) \, dZ_f(t) \]

\[ W(0) = x + y. \]

\[ \Rightarrow \pi \text{ and } c \text{ are control variables.} \]
Policies: a policy is a pair \((c_t, \pi_t)\) of \(\mathcal{F}_t\)-adapted processes s.t.

\[ \begin{align*}
    (i) & \quad c(t, w) \geq 0, \quad \int_0^t c(s, w) \, ds < \infty. \\
    (ii) & \quad |\pi(t, w)| \leq K, \text{ where } K \text{ can depend on the policy.} \\
    (iii) & \quad W(t) \geq 0, \text{ } W \text{ is the unique strong solution to } (*)
\end{align*} \]

\[ \mathcal{U}_t = \{ (\pi, c) \text{ satisfying } (i), (ii), (iii)\}. \]

Control problem:

\[ v(w) = \sup_{(\pi_t, c_t)} \mathbb{E}_{\omega} \left[ \int_0^\infty e^{-\delta t} u(c(t)) \, dt \right] \]

\[ \delta > 0, \text{ discount factor.} \]

Technical condition (Condition A):

\[ \delta > \gamma \left( r + \frac{(\alpha-r)^2}{2\sigma^2(1-\gamma)} \right) \]

Theorem 2.1: Suppose condition A holds. Define

\[ c_t = \frac{1}{1-\gamma} \left[ \delta - \gamma r - \frac{\theta^2}{2(1-\gamma)} \right], \quad \theta = \frac{\alpha-r}{\sigma}. \]

\[ c_t^* = c \omega_t \quad \text{and} \quad \pi_t^* = \frac{\beta}{(1-\gamma)^6}, \]

\[ v(w) = \frac{1}{\gamma} \omega_t^{\gamma-1} \omega_t^{\gamma} \]

Remark: If \( r < \alpha < r + (1 - \gamma) \sigma^2 \Rightarrow \pi^* \in (0, 1) \) "hedging".

If \( \alpha > r + (1 - \gamma) \sigma^2 \Rightarrow \pi^* \in [1, \infty) \) "leverage": borrow to invest in risky asset.

If \( \alpha < r \), \( \pi^* < 0 \) "short selling".

If \( \alpha = r \), \( \pi^* = 0 \) optimally consume initial endowment.

Proof of Theorem 2.1 → See slides!

2) With transaction costs. \( \lambda \in [0, \infty) \), \( \mu \in [0, 1] \)

\[
\begin{align*}
    dS_0(t) &= (rS_0(t) - c(t)) \, dt + \sigma S_0(t) \, dL_t \\
    dS_1(t) &= \alpha S_1(t) \, dt + \sigma S_1(t) \, dB(t) \\
               &+ dL_t - dU_t \quad , \quad S_0(0) = x \\
               &S_1(0) = y .
\end{align*}
\]

\( L_t \) = cumulative purchases of stocks
\( U_t \) = sales
For \( \alpha \), \( \beta \in \mathbb{R}^2 \) define

\[
S_{\alpha, \beta} = \left\{(x, y) \in \mathbb{R}^2 \mid x + (1-\beta)y \geq 0 \text{ and } x + (1+\beta)y \geq 0\right\}.
\]

\[
S = \left\{ (x, y) \in \mathbb{R}^2 \mid x + (1-\beta)y = 0 \right\}.
\]

\[
\tilde{v}_y = (1+\alpha)\tilde{v}_x,
\]

\[
\tilde{v}_x = (1-\beta)\tilde{v}_y.
\]

\[
S_{\tilde{v}_x} = \left\{ (x, y) \in \mathbb{R}^2 \mid x + (1+\alpha)y = 0 \right\}.
\]

\[
\tilde{S}_{\tilde{v}_y} = \left\{ (x, y) \in \mathbb{R}^2 \mid x + (1-\beta)y = 0 \right\}.
\]

- **Policy** \((C_t, L_t, U_t)\) adapted processes s.t. \(Z(t)^{(i)}\) and \(L_t, U_t\) are right-continuous, non-decreasing, \(L_0 = U_0 = 0\).

- **Admissible Policy:** A *stochastic* policy \((c, L, U)\), where

\[
\tau = \inf\{t \geq 0 : (S_t, S_t^{(i)}) \notin S_{\alpha, \beta}\}.
\]

**Fact:** \(U \neq \emptyset\)

**Proof:** *Control Problem:* 

\[
v(x_0, y) = \max_{(c, L, U) \in U} \mathbb{E} \left[ \int_0^\infty e^{-\delta t} u(c(t)) \, dt \right]
\]

**Fact:** \(v\) has the homothetic property:

For \( \rho > 0 \), then 

\[
v(\rho x, \rho y) = \rho^\gamma v(x, y).
\]