Optimal Stopping and Applications

Example 2: American options

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24 March 2009

1 Introduction

1.1 The market model

We consider a financial market consisting of two primary assets, a risk-free bond $B$ and a stock $S$ whose dynamics under the unique risk-neutral measure $\mathbb{P}$ are given by

\begin{align}
\frac{dB(t)}{B(t)} &= rB(t) dt \\
\frac{dS(t)}{S(t)} &= rS(t) dt + \sigma S(t) dW(t) \\
B(0) &= 1 \\
S(0) &= x
\end{align}

(1)

where $r$, $\sigma$ are deterministic constants with $\sigma > 0$ and $W(t)$ is a Brownian motion under $\mathbb{P}$. We refer to $r$ as the interest rate and to $\sigma$ as the volatility of $S$. We denote by $\{\mathcal{F}_t\}_{t \geq 0}$ the natural augmented filtration of $W$. It is easy to verify that (1) under $\mathbb{P}$ has the unique strong solution

\begin{align}
B(t) &= e^{rt} \\
S(t) &= xe^{\sigma W(t) + (r-\sigma^2/2)t}
\end{align}

(2) (3)

It is not difficult to check that $e^{-rt}S(t)$ is a $\{\mathcal{F}_t\}$-martingale under $\mathbb{P}$.

1.2 Pricing formula

**Theorem 1** (Fundamental theorem of asset pricing). Let $T > 0$ and $D$ be a $\mathbb{P}$-integrable and $\mathcal{F}_T$-measurable random variable, which we interpret as the value of some derivative security at time $T$. The arbitrage-free price of $D$ at time $t \in [0;T]$ is given by

$$D(t) = \mathbb{E}[e^{-r(T-t)}D|\mathcal{F}_t]$$

(4)

Moreover $e^{-rt}D(t)$ is a $\{\mathcal{F}_t\}$-martingale under $\mathbb{P}$.

**Proof.** See [8] Chapter 5.
1.3 European and American options

Definition 1. A European [American] call option $C_{Eur}$ [$C_{Am}$] with strike price $K > 0$ and time of maturity $T > 0$ on the underlying asset $S$ is a contract defined as follows

- The holder of the option has, exactly at time $T$ [at any time $t \in [0; T]$], the right but not the obligation to buy one share of the underlying asset $S$ at price $K$ from the underwriter of the option.

Definition 2. A European [American] put option $P_{Eur}$ [$P_{Am}$] with strike price $K > 0$ and time of maturity $T > 0$ on the underlying asset $S$ is a contract defined as follows

- The holder of the option has, exactly at time $T$ [at any time $t \in [0; T]$], the right but not the obligation to sell one share of the underlying asset $S$ at price $K$ to the underwriter of the option.

We fix a strike $K > 0$ and a time of maturity $T > 0$. By theorem 1, the arbitrage-free prices of a European call [put] at time 0 is given by

$$C_{Eur} = E[e^{-rT}(S(T) - K)^+]$$  \hspace{1cm} (5)

$$P_{Eur} = E[e^{-rT}(K - S(T))^+]$$  \hspace{1cm} (6)

which can be expressed in a closed formula, the Black-Scholes formula\(^1\).

Now suppose we are the owner of an American call [put] option. Since we can exercise the option at any time $t \in [0; T]$, we choose an $\{\mathcal{F}_t\}$-stopping time $\tau \in [0, T]$ taking values in $[0, T]$. At time $T$ we own $e^{r(T-\tau)}(K - S(\tau))^2$. Since we may choose any $\{\mathcal{F}_t\}$-stopping time $\tau \in [0, T]$, theorem 1 implies\(^2\) that the arbitrage-free price of an American call [put] option at time 0 is given by

$$C_{Am} = \sup_{\tau \in [0, T]} E[e^{-r\tau}(S(\tau) - K)^+]$$  \hspace{1cm} (7)

$$P_{Am} = \sup_{\tau \in [0, T]} E[e^{-r\tau}(K - S(\tau))^+]$$  \hspace{1cm} (8)

It is obvious that $C_{Am} \geq C_{Eur}$ and $P_{Am} \geq P_{Eur}$, since we can choose the exercise strategy $\tau = T$. The following theorem states in which cases the latter strategy is indeed the best that we can do.

Theorem 2.

1. Suppose $r \geq 0$. Then $C_{Am} = C_{Eur}$.
2. Suppose $r = 0$. Then $P_{Am} = P_{Eur}$.

Proof.

\(^1\)See for instance [1] p 100 et seq.
\(^2\)If we exercise before time $T$, we invest our money for the rest of the time up to $T$ in the risk-less bond
\(^3\)More precisely this follows once we have shown the existence of an optimal stopping time.
1. Fix $\tau \in [0, T]$. Define $g_1 : \mathbb{R}^+ \to \mathbb{R}^+$ by $g_1(x) = (x - e^{-r\tau}K)^+$. Clearly $g_1$ is convex. Jensen’s inequality for conditional expectations, the fact that $e^{-r\tau}S(t)$ is a martingale and the optional sampling theorem yield

$$
C^{Eur} = E[e^{-r\tau}(S(T) - K)^+] = E[g_1(e^{-r\tau}S(T))] \\
= E[E[g_1(e^{-rT}S(T))|\mathcal{F}_\tau]] \geq E[g_1(E[e^{-rT}S(T)|\mathcal{F}_\tau])] \\
= E[g_1(e^{-r\tau}S(\tau))] = E[(e^{-r\tau}S(\tau) - e^{-r\tau}K)^+] \\
\geq E[e^{-r\tau}(S(\tau) - K)^+] 
$$

Since $\tau \in [0, T]$ was arbitrary, we have $C^{Eur} \geq C^{Am}$, which together with $C^{Eur} \leq C^{Am}$ yields the claim.

2. Fix $\tau \in [0, T]$. Define $g_2 : \mathbb{R}^+ \to \mathbb{R}^+$ by $g_2(x) = (K - x)^+$. Clearly $g_2$ is convex. Jensen’s inequality for conditional expectations, the fact that $S(t)$ is a martingale and the optional sampling theorem yield

$$
C^{Eur} = E[(K - S(T))^+] = E[g_2(S(T))] \\
= E[E[g_2(S(T))|\mathcal{F}_\tau]] \geq E[g_2(E[S(T)|\mathcal{F}_\tau])] \\
= E[g_2(S(\tau))] = E[(K - S(\tau))^+] 
$$

Since $\tau \in [0, T]$ was arbitrary, we have $P^{Eur} \geq P^{Am}$, which together with $P^{Eur} \leq P^{Am}$ yields the claim.

Remark. If $r > 0$ the above argument breaks down for the American put. We will show below that in this case we have $P^{Am} > P^{Eur}$ and we will derive an explicit formula for difference $P^{Am} - P^{Eur}$.

2 Analytical Characterization of the Put Price

2.1 Formal definition of the problem

Let $(\tilde{W}(s))_{s \geq 0}$ be a Brownian motion on some probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathcal{F}}_s, \tilde{\mathbb{P}})$, where $\{\tilde{\mathcal{F}}_s\}_{s \geq 0}$ is the natural augmented filtration of $\tilde{W}$. Let $E := [0, \infty) \times (0, \infty)$ (Perpetual American Put) or $E = [0, T] \times (0, \infty)$; $0 < T < \infty$ (Finite American Put). Set $\Omega = E \times \tilde{\Omega}$, $\mathcal{F} = \mathcal{B}(E) \otimes \tilde{\mathcal{F}}$ and $\mathcal{G} = \{[0, E] \otimes \tilde{\mathcal{F}}\}$. For $s \geq 0$ and $\omega = (t, x, \tilde{\omega}) \in \Omega$ define

$$
W(s)(\omega) = \tilde{W}(s)(\tilde{\omega}) \\
S(s)(\omega) = xe^{rW(s)(\tilde{\omega})+(r-\sigma^2/2)s} \\
X(s)(\omega) = (t + s, S(s)(\omega)) 
$$

where $r, \sigma$ are deterministic constants with $\sigma, r > 0$. Moreover, for $s \geq 0$, let $\mathcal{F}_s = \mathcal{B}(E) \otimes \tilde{\mathcal{F}}_s$ and $\mathcal{G}_s = \{[0, E] \otimes \tilde{\mathcal{F}}_s\}$. Finally define probability measures $\{P_{(t,x)}\}_{(t,x) \in E}$ on $\{\Omega, \mathcal{G}\}$ and $\mathbb{P}$ on $\{\Omega, \mathcal{F}\}$ by $P_{(t,x)} := \delta_{t} \otimes \delta_{x} \otimes \mathbb{P}$ and $\mathbb{P} := \mu \otimes \tilde{\mathbb{P}}$, where $\delta_{t}$ and $\delta_{x}$ denote Dirac measures and $\mu : \{0, E\} \mapsto [0, 1]$ is defined by $\mu([0, \theta]) = 0$; $\mu(E) = 1$. For convenience we set $P_{x} := P_{(0,x)}$. It is not difficult to check that $\{W(s)\}_{s \geq 0}$ is a Brownian motion on $\{\Omega, \mathcal{G}, \mathbb{P}\}$ and $\{S(s)\}_{s \geq 0}$ and $\{X(s)\}_{s \geq 0}$ are strong Markov families on $\{\Omega, \mathcal{F}, \mathcal{F}_s, \{P_{x}\}_{x > 0}\}$ and $\{\Omega, \mathcal{F}, \mathcal{F}_s, \{P_{(t,x)}\}_{(t,x) \in E}\}$. 


Remark. Under $\mathbb{P}_{(t,x)}$ we interpret $S(s)$ as the value of a stock $\tilde{S}$ with volatility $\sigma$ in a financial market with interest rate $r$ at time $t+s$ given that $\tilde{S}(t) = x$.

We fix a strike price $K > 0$. Define the gain function $G : E \mapsto [0, K]$ by $G(t, x) := e^{-rt}(K - x)^+$. For $(t, x) \in E$ define the optimal stopping problem

$$V(t, x) = \sup_{\tau \in [0, T-t]} \mathbb{E}_{(t,x)}[G(X(s))]$$

$$= \sup_{\tau \in [0, T-t]} \mathbb{E}_{(t,x)}[e^{-r(t+\tau)}(K - S(\tau))]]$$

(12)

where $T$ is the upper boundary of the time coordinate of $E$ and $\tau \in [0, T-t]$ is a stopping time\(^4\) taking values in $[0, T-t]$. Since $G$ is bounded, $V(t, x)$ is defined for all $(t, x) \in E$. We call $V$ the value function.

Remark. We interpret $V(t, x)$ as the arbitrage free price of an American put option with strike $K$ and maturity $T$ on $\tilde{S}$ at time $0$ given that $\tilde{S}(t) = x$. Since we have a positive interest rate $r$, we cannot compare prices at different times directly, but need to discount appropriately. The price of an American put option at time $t$ given $\tilde{S}(t) = x$ is given by

$$v(t, x) = e^{rt}V(t, x) = \sup_{\tau \in [0, T-t]} \mathbb{E}_{x}[e^{-r\tau}(K - S(\tau))^+]$$

(13)

We call $v$ the value* function. Similarly we define

$$g(t, x) = e^{rt}G(t, x) = (K - x)^+$$

(14)

which we call the gain* function. Even though $V$ and $G$ are the formal correct objects, which in addition carry the economic interpretation of time value of money, it turns out that $v$ and $g$ are the convenient mathematical objects to work with.

2.2 Elementary properties of the value* function

Lemma 1.

1. If $T = \infty$, the function $t \mapsto v(t, x)$ is constant

2. If $T < \infty$, the function $t \mapsto v(t, x)$ is decreasing with $v(T, x) = (K - x)^+$.

Proof. Let $0 \leq t_1 \leq t_2 \leq T$. Then

$$v(t_1, x) = \sup_{\tau \in [0, T-t_1]} \mathbb{E}_{x}[e^{-r\tau}(K - S(\tau))^+]$$

$$\geq \sup_{\tau \in [0, T-t_2]} \mathbb{E}_{x}[e^{-r\tau}(K - S(\tau))^+]$$

(15)

$$= v(t_2, x)$$

Since $[0, \infty - t_1] = [0, \infty - t_2]$ and clearly $v(T, x) = (K - x)^+$ by definition for $T < \infty$, both assertions follow immediately.

\(^4\) In general we allow $\tau$ to be an $\mathcal{F}_\tau$-stopping time. Note, however, that for fixed $(t, x) \in E$ for each $\mathcal{F}_\tau$-stopping time $\tau_{\mathbb{F}}$ there exists a $\mathcal{F}_{\mathbb{G}}$-stopping time $\tau_{\mathbb{G}}$ with $\tau_{\mathbb{F}} = \tau_{\mathbb{G}} \mathbb{P}_{(t,x)}$-a.s.. If necessary we will work with $\tau_{\mathbb{G}}$ rather than with $\tau_{\mathbb{F}}$, which will always be clear from the context.

\(^5\) $\infty - t := \infty$; moreover we allow $\tau = \infty$ if $T = \infty$

\(^6\) We require $t < \infty$
Lemma 2. The function $x \mapsto v(t, x)$ is convex and continuous

Proof. Fix $t \in [0, T]$\footnote{Again we require $t < \infty$}. For $\tau \in [0, T - t]$, $x > 0$ define

$$u(x, \tau) := e^{-r\tau} (K - xe^{\sigma W(\tau) + (r - \sigma^2/2)\tau})^+$$

It is straightforward to check that $x \mapsto u(x, \tau)$ is convex. By linearity of the integral it follows that $x \mapsto E[u(x, \tau)]$ is convex. Moreover clearly

$$v(x, t) = \sup_{\tau \in [0, T-t]} E[u(x, \tau)]$$

The assertion follows by the well-known facts that the supremum of convex functions is convex again, and that convex functions are continuous.

Lemma 3. The function $(t, x) \mapsto v(t, x)$ is lsc.

Proof. For $\tau \in [0, T]$ and $(t, x) \in E$ define

$$u(t, x, \tau) := e^{-r(\tau \wedge (T-t))} (K - xe^{\sigma W(\tau \wedge (T-t)) + (r - \sigma^2/2)(\tau \wedge (T-t))})^+$$

It is not difficult to check that $(t, x) \mapsto u(t, x, \tau)$ is continuous. By the dominated convergence theorem we get that $(t, x) \mapsto E[u(t, x, \tau)]$ is continuous. Moreover clearly\footnote{Note $\{\tau \wedge (T-t) : \tau \in [0, T]\} = \{\tau : \tau \in [0, T-t]\}$}

$$v(x, t) = \sup_{\tau \in [0, T]} E[u(t, x, \tau)]$$

The assertion follows by the well-known fact that the supremum of lsc functions is lsc again.

2.3 Existence of an optimal stopping time

According to the Markovian approach to optimal stopping problems we define the continuation set

$$C := \{(t, x) \in [0, T) \times (0, \infty) : V(t, x) > G(t, x)\}$$

$$= \{(t, x) \in [0, T) \times (0, \infty) : v(t, x) > g(x)\}$$

and the stopping set

$$D := \{(t, x) \in [0, T] \times (0, \infty) : V(t, x) = G(t, x)\}$$

$$= \{(t, x) \in [0, T] \times (0, \infty) : v(t, x) = g(x)\}$$

Note that $D$ is closed since $v$ is lsc by Lemma 3 and $g$ is continuous. Moreover we define the stopping time$^9$

$$\tau_D := \inf\{s \geq 0 : X_s \in D\}$$

$^7$Again we require $t < \infty$

$^8$Note $\{\tau \wedge (T-t) : \tau \in [0, T]\} = \{\tau : \tau \in [0, T-t]\}$

$^9$This is indeed a stopping time since $D$ is closed and $X$ is continuous.
**Proof.** Let \((t, x) \in [0, T) \times [K, \infty)\) and \(0 < \epsilon < K\). Define the stopping time\(^{10}\)
\[
\tau_\epsilon := \inf\{s \geq 0 : S_s \leq K - \epsilon\} \land (T - t)
\]  
(23)
It is not difficult to show that \(P_{(t, x)}(0 < \tau_\epsilon < T - t) =: \alpha > 0\). Hence we have
\(V(t, x) \geq \alpha e^{-\epsilon T} \epsilon > 0 = G(t, x)\), which establishes the claim.

Now define \(w(t, x) = v(x, t) + x\). Lemma 4 implies
\[
C = \{(t, x) \in [0, T) \times (0, \infty) : w(t, x) > K\}
\]  
(24)
\[
D = \{(t, x) \in [0, T) \times (0, \infty) : w(t, x) = K\} \cup \{T\} \times (0, \infty)
\]  
(25)

**Lemma 5.** The function \(x \mapsto w(t, x)\) is convex and increasing. Moreover
\(\lim_{t \to \infty} w(t, x) = K\).

**Proof.** Convexity follows from convexity of \(x \mapsto v(t, x)\) and \(x \mapsto x\). The obvious inequality \((K - x)^+ + x \leq w(t, x) \leq K + x\), implies \(K \leq w(t, x) \forall x \in (0, \infty)\) as well as \(\lim_{t \to \infty} w(t, x) = K\). These two facts together with convexity of \(x \mapsto w(t, x)\) imply immediately that \(x \mapsto w(t, x)\) is increasing.

Lemma 4 and 5 imply that there exist a function \(b : [0, T) \to [0, K]\) such that
\[
C = \{(t, x) \in [0, T) \times (0, \infty) : x > b(t)\}
\]  
(26)
\[
D = \{(t, x) \in [0, T) \times (0, \infty) : x \leq b(t)\} \cup \{T\} \times (0, \infty)
\]  
(27)

### 2.3.1 Infinite time horizon

For convenience set \(v(x) := v(0, x)\). For \(0 < b < K\) define the stopping time
\(\tau_b = \inf\{s \geq 0 : S_s \leq b\}\) and let
\[
v_b(x) := E_x[e^{-r \tau_b} (K - S(\tau_b))^+]
\]  
(28)
The formula for the Laplace transform for the first passage time of a Brownian motion with drift\(^{11}\) yields after some simple calculations
\[
v_b(x) = \begin{cases}  
K - x & \text{if } 0 < x \leq b \\
(K - b) \left(\frac{x}{b}\right)^{-2r/\sigma^2} & \text{if } x > b
\end{cases}
\]  
(29)
Define \(v^*(x) = \sup_{b \in (0, K)} v_b(x)\). Elementary Calculus yields
\[
v^*(x) = v^*_b(x) = \begin{cases}  
K - x & \text{if } 0 < x \leq b^* \\
(K - b^*) \left(\frac{x}{b^*}\right)^{-2r/\sigma^2} & \text{if } x > b^*
\end{cases}
\]  
(30)
where \(b^* = \frac{2r}{2r + \sigma^2} K\). It is straightforward to check that \(v^* \in C^1((0, \infty))\) and \(v^* \in C^2((0, b) \cup (b, \infty))\) with
\[
v^*_+ (x) = \begin{cases}  
-1 & \text{if } 0 < x \leq b^* \\
\frac{2r}{\sigma^2} v^*(x) & \text{if } x \geq b^*
\end{cases}
\]  
(31)
\[
v^*_{xz} (x) = \begin{cases}  
0 & \text{if } 0 < x < b^* \\
\frac{2r(2r + \sigma^2)}{\sigma^4 x^2} v^*(x) & \text{if } x > b^*
\end{cases}
\]  
(32)
Define \(V^*(t, x) = e^{-rt} v^*(x)\).

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\(^{10}\)This is again a stopping time since \([0, K - \epsilon]\) is closed and \(S\) is continuous.

\(^{11}\)See for instance [8] p 346 et seq (Theorem 8.3.2)
Theorem 3. \( v^*(x) = v(x) \) for \( x \in (0, \infty) \). Moreover \( \tau_{b^*} \) is the optimal stopping time for the Perpetual American Put.

Proof. Since \( V^* \in C^{1,1}(E) \cup C^{1,2}(E\setminus([0, \infty) \times b)) \) and \( \mathbb{P}_x(S(t) = b^*) = 0 \) for all \( x \in (0, \infty) \) and all \( t > 0 \), we can apply a slightly generalized version of Itô’s formula\(^{12}\) to \( V^*(t, S(t)) \) and get

\[
dV^*(t, S(t)) = -rV^*(t, S(t)) \, dt + V^*_x(t, S(t)) \, dS(t) + \frac{1}{2} V^*_xx(t, S(t)) \mathbf{1}_{\{S(t) \neq b\}} \, d(S(t), S(t))
\]

\[
= -e^{-rt}rK \mathbf{1}_{\{S(t) < b^*\}} \, dt + \sigma S(t)V^*_x(t, S(t)) \, dW(t)
\]

(33)

Hence \( V^*(t, S(t)) \) is a \( \{\mathcal{F}_t\}\)-supermartingale\(^{13}\) with \( V^*(t, S(t)) \geq G(t, S(t)) \)^{14}. Let \( \tau \in [0, \infty) \) be a stopping time. Monotonicity of the integral and the optional sampling theorem yield

\[
\mathbb{E}_x[G(\tau, S(\tau))] \leq \mathbb{E}_x[V^*(\tau, S(\tau))] \leq V^*(0, x) = v^*(x)
\]

(34)

Taking the supremum in (34) over \( \tau \in [0, \infty) \) yields \( v^*(x) \geq v(x) \). On the other hand \( v^*(x) \leq v(x) \) by definition. Hence

\[
v^*(x) = v(x) = \mathbb{E}_x[v^*(\tau_b^*, S(\tau_b^*))]
\]

(35)

q.e.d.

2.3.2 Finite time horizon

Since \( V \) is lsc by lemma 3 and \( G \) is continuous, \( \tau_D \) is optimal in (12), since \( \mathbb{P}_t,\pi(\tau_D < \infty) = 1 \) by the main existence theorem of the Markovian approach (Theorem 3.7 of the lecture notes).

2.4 Elementary properties of \( b \) for finite time horizon

Lemma 6. The function \( b \) is increasing with \( b^* \leq b(t) < K \).

Proof. Let \( 0 \leq t_1 < t_2 < T \). By Lemma 1 and the definitions of the functions \( v, g \) and \( b \) we have

\[
g(b(t_1)) = v(t_1, b(t_1)) \geq v(t_2, b(t_1)) \geq g(b(t_1))
\]

(36)

Therefore \( (t_2, b(t_1)) \in D \), which implies \( b(t_2) \geq b(t_1) \). Moreover let \( x \leq b^* \). Then by Theorem 3

\[
v(0, x) \leq \sup_{\tau \in [0, \infty]} \mathbb{E}_x[e^{-r\tau}(K - S(\tau))^+] = K - x = g(x)
\]

(37)

whence \( (0, x) \in D \), which implies \( b(t) \geq b(0) \geq b^* \). Finally, Lemma 4 implies \( b(t) < K \).

\(^{12}\) confer [6] p 74 et seq

\(^{13}\) \( \int_0^t \sigma S(s)V^*_x(s, S(s)) \, dW(s) \) is a proper martingale since \( |V^*_x(s, S(s))| \leq e^{-rs} \leq 1 \).

\(^{14}\) Note that \( v^*(x) \geq (K - x)^+ \) for \( x \in (0, \infty) \).
2.5 Further properties of the value* function

**Lemma 7.** The function \( x \mapsto v(t, x) \) is decreasing and strictly decreasing for \( x \in (0, K] \). Moreover \( \lim_{x \to 0} v(t, x) = K \) and \( \lim_{x \to \infty} v(t, x) = 0 \).

**Proof.** The claim is trivial for \( t = T \), so assume \( t < T \). Lemma 5 implies \( \lim_{x \to 0} v(t, x) = \lim_{x \to \infty} w(t, x) = K \). Moreover by (32) for \( x \geq K \)

\[
v(x, t) = \sup_{\tau \in [0, T - t]} E_x [\text{e}^{-rt} (K - S(\tau))^+] \leq \sup_{\tau \in [0, \infty]} E_x [\text{e}^{-rt} (K - S(\tau))^+] = (K - b^*) \left( \frac{x}{b^*} \right)^{-2r/\sigma^2}
\]

which implies \( \lim_{x \to \infty} v(t, x) = 0 \). Since \( 0 \leq v(t, x) \leq K \) by definition and \( x \mapsto v(t, x) \) is convex by Lemma 2, \( x \mapsto v(t, x) \) is decreasing. Moreover clearly \( v(t, x) > 0 \) for \( x < K \). Again convexity of \( x \mapsto v(t, x) \) implies that \( x \mapsto v(t, x) \) is strictly decreasing for \( x \in (0, K] \).

**Lemma 8.** The function \( v \) is continuous in \( E \).

**Proof.** For \( t \geq 0 \) define \( M(t) := \sup_{0 \leq s \leq t} |W(s)| \). Fix \( x \in (0, \infty) \) and let \( 0 \leq t_1 < t_2 \leq T \). Denote by \( \tau_1 \) the \( \{G_s\}\)-optimal stopping time for \( v(t_1, x) \) and define \( \tau_2 := \tau_1 \wedge (T - t_2) \). Clearly \( \tau_1 \geq \tau_2 \) with \( \tau_1 - \tau_2 \leq t_2 - t_1 \). By stationarity and independent increments \( \{W(\tau_2 + t) - W(\tau_2)\}_{t \geq 0} \) is independent of \( G_{\tau_2} \) and equal in law to \( \{W(t)\}_{t \geq 0} \). Recalling that \( \text{e}^{-rt} S(t) \) is a martingale and \( v(t_1, x) \geq v(t_2, x) \) we get

\[
0 \leq v(t_1, x) - v(t_2, x) \\
\leq E_x[\text{e}^{-\tau_1}(K - S(\tau_1))^+] - E_x[\text{e}^{-\tau_2}(K - S(\tau_2))^+] \\
\leq E_x[\text{e}^{-\tau_2}(K - S(\tau_1))^+ - (K - S(\tau_2))^+] \\
\leq E_x[\text{e}^{-\tau_2}(S(\tau_2) - S(\tau_1))^+] \\
\leq E_x[\text{e}^{-\tau_2} S(\tau_2) (1 - \text{e}^{\sigma(W(\tau_1) - W(\tau_2)) + (r - \sigma^2/2)(\tau_1 - \tau_2)} )^+] \\
\leq E_x[\text{e}^{-\tau_2} S(\tau_2) E[(1 - \text{e}^{\sigma(W(\tau_1) - W(\tau_2)) + (r - \sigma^2/2)(\tau_1 - \tau_2)} )^+ | G_{\tau_2} ]] \\
\leq E[\text{e}^{-\tau_2} S(\tau_2) E[(1 - \text{e}^{\sigma(W(\tau_1 - \tau_2) + (r - \sigma^2/2)(\tau_1 - \tau_2)} )^+]] \\
\leq \text{e}^{-\sigma M(t_2 - t_1) - (r - \sigma^2/2)||t_2 - t_1||^+] \tag{39}
\]

Define the function \( h : \mathbb{R} \to \mathbb{R} \) by \( h(t) := E[(1 - \text{e}^{-\sigma M(|t|) - (r - \sigma^2/2)||t||^+}] \). By dominated convergence \( h \) is continuous at 0 with \( h(0) = 0 \). Now fix \( (t_0, x_0) \in E \) and let \( \{t_n, x_n\}_{n \geq 1} \) be a sequence in \( E \) with \( \lim_{n \to \infty} (t_n, x_n) = (t_0, x_0) \). Then by Lemma 2 and (39) we have

\[
\limsup_{n \to \infty} |v(t_n, x_n) - v(t_0, x_0)| \leq \limsup_{n \to \infty} |v(t_n, x_n) - v(t_0, x_n)| + \limsup_{n \to \infty} |v(t_0, x_n) - v(t_0, x_0)| \\
\leq \limsup_{n \to \infty} x_n h(t_n - t_0) + 0 = 0 \tag{40}
\]

Hence \( v \) is continuous in \( E \).
Lemma 9. The function $v$ is $C^{1,2}$ in $C$ and satisfies there $v_x \leq 0$ and $v_t \leq 0$ as well as $v_{xx}(t,x) \geq \frac{2r}{\sigma^2} v(x,t)$.

Proof. Denote by $L_X$ the infinitesimal generator of $X$. It is not difficult to establish that

$$L_X = \frac{\partial}{\partial t} + r_x \frac{\partial}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2}{\partial x^2}$$ (41)

Now fix $(t_0,x_0) \in C$ and let $r_0 > 0$ such that $B := \overline{B}_{r_0}(t_0,x_0) \subset C$. Now consider the following PDE

$$L_X U = 0 \quad \text{in } B$$
$$U = V \quad \text{on } \partial B$$ (42)

Since $V$ is continuous by Lemma 8, standard PDE results\textsuperscript{15} state, that there exist a unique solution $U$ of (42) in $C^{1,2}(B) \cap C^0(B)$. Let $(t,x) \in B$ and $\epsilon > 0$ be arbitrary. Since $B$ is compact and $U, V \in C^0(B)$ with $U = V$ on $\partial B$ there exists $0 < r_1 < r_0$ such that $(t,x) \in B_\epsilon := B_{r_1}(t_0,x_0)$ and $|U - V| \leq \epsilon$ on $\partial B_\epsilon$. Let $U^* : E \mapsto \mathbb{R}$ be a $C^{1,2}$-extension of $U|_{B_\epsilon}$\textsuperscript{16}. Now applying Itô's formula to $U^*(X_s)$ yields

$$dU^*(X_s) = L_X U^*(X_s) \, ds + \sigma S(s) U^*(X_s) \, dW(s)$$ (43)

Define the stopping time

$$\tau_{B_\epsilon} := \inf\{s \geq 0 : X_s \in B_\epsilon^c\}$$ (44)

Note that $L_X = 0$ in $B_\epsilon$ by (42). Hence (43) and the Optional Sampling theorem yield\textsuperscript{17}

$$U(t,x) = U^*(t,x) = E_{(t,x)}[U^*(X_{\tau_{B_\epsilon}^x})] = E_{(t,x)}[U(X_{\tau_{B_\epsilon}^x})]$$ (45)

On the other hand, since $\tau_{B_\epsilon} \leq \tau_D$ we get by the Strong Markov property

$$E_{(t,x)}[V(X_{\tau_{B_\epsilon}^x})] = E_{(t,x)}[E_{X_{\tau_{B_\epsilon}^x}}[G(X_{\tau_D})]]$$
$$= E_{(t,x)}[E_{(t,x)}[G(X_{\tau_{B_\epsilon}^x} + \tau_D) | \mathcal{F}_{\tau_{B_\epsilon}^x}]]$$
$$= E_{(t,x)}[G(X_{\tau_{B_\epsilon}^x} + \tau_D)] = E_{(t,x)}[G(X_{\tau_D})] = V(t,x)$$ (46)

Putting (45) and (46) together yields

$$|V(t,x) - U(t,x)| = |E_{(t,x)}[V(X_{\tau_{B_\epsilon}^x})] - U(X_{\tau_{B_\epsilon}^x})|$$
$$\leq E_{(t,x)}[|V(X_{\tau_{B_\epsilon}^x}) - U(X_{\tau_{B_\epsilon}^x})|]$$
$$\leq E_{(t,x)}[\epsilon] = \epsilon$$ (47)

Since $\epsilon > 0$ was arbitrary, we get $V(t,x) = U(t,x)$. Thus $V = U$ in $\overline{B}$, which in particular implies that $V$ is $C^{1,2}$ in $(t_0,x_0)$. Since $(t_0,x_0) \in C$ was chosen

\textsuperscript{15}See for instance [3] for a proof

\textsuperscript{16}More precisely $U^* \in C^{1,2}(E)$ and $U^* = U$ on $B_\epsilon$.

\textsuperscript{17}Note that a priori we only know that $\int_0^t \sigma S(s) U^*_x(X_s) \, dW(s)$ is local martingale. Therefore we use a localizing sequence \{${\tau_n}$\}\textsubscript{n$\geq$0} of stopping times and observe that $U_x$ is bounded on $B_\epsilon$. The dominated convergence theorem then yields the desired result.
Lemma 10. The function \( x \mapsto v(t, x) \) is differentiable at \( b(t) \) with \( v_x = g_x \).

Proof. Fix \( t^* \in [0, T] \). Since \( x \mapsto v(t^*, x) \) is convex by Lemma 2, the right-hand derivative \( \frac{\partial^+ v}{\partial x}(t^*, x^*) \) exists for all \( x \in (0, \infty) \). Denote \( x^* = b(t^*) < K \). Then

\[
\frac{\partial^+ v}{\partial x}(t^*, x^*) = \lim_{\epsilon \downarrow 0} \frac{v(t^*, x^* + \epsilon) - v(t^*, x^*)}{\epsilon} \\
\geq \lim_{\epsilon \downarrow 0} \frac{g(x^* + \epsilon) - g(x^*)}{\epsilon} = -1 \tag{49}
\]

Define \( \tau_{x^*} := \inf\{ s \geq 0 : S(s) \leq x^* \} \) and denote by \( \tau^\xi \) the \( \{\mathcal{G}_s\} \)-optimal stopping time for \( v(t^*, x^* + \xi) \) for \( \xi \geq 0 \). Since \( b \) is increasing by Lemma 6, clearly \( \tau^\xi \leq \tau_{x^*} \) under \( P(t^*, x^* + \xi) \) for all \( \xi \geq 0 \). Moreover by (29) we have

\[
\liminf_{\xi \downarrow 0} \mathbb{E}[e^{-r\tau^\xi}] \geq \liminf_{\xi \downarrow 0} \mathbb{E}[v(t^*, x^* + \xi)]e^{-r\tau^\xi} = \lim_{\xi \downarrow 0} \left( \frac{x^* + \xi}{x^*} \right)^{-2r/\sigma^2} = 1 \tag{50}
\]

This implies that for any sequence \( \{\xi_n\}_{n \geq 1} \) in \( \mathbb{R}^+ \) with \( \lim_{n \to \infty} \xi_n = 0 \) we have \( \lim_{n \to \infty} \tau_{x^*} = 0 \) in probability. For \( t \geq 0 \) define \( M(t) := \sup_{0 \leq s \leq t} |W(s)| \) and for convenience set \( \Sigma(t) := e^{\sigma W(t) + (r - \sigma^2/2)t} \) and \( \Theta(t) := e^{\sigma M(t) + [(r - \sigma^2/2)t]}. \)

Let \( \epsilon > 0 \) be arbitrary and consider \( \{\xi_n\}_{n \geq 1} \) a sequence in \( \mathbb{R}^+ \) with \( \lim_{n \to \infty} \xi_n = 0 \). After possibly discarding a subsequence we may assume that \( \lim_{n \to \infty} \tau_{x^*} = 0 \) a.s. Then it holds

\[
\frac{\partial^+ v}{\partial x}(t^*, x^*) = \limsup_{n \to \infty} \frac{v(t^*, x^* + \xi_n) - v(t^*, x^*)}{\xi_n} \\
\leq \limsup_{n \to \infty} \mathbb{E}[e^{-r\tau_{x^*}} ((K - (x^* + \xi_n)\Sigma(\tau_{x^*}))^+ - (K - x^*\Sigma(\tau_{x^*}))^+)/\xi_n] \\
\leq \limsup_{n \to \infty} \mathbb{E}[e^{-r\tau_{x^*}} (-\Sigma(\tau_{x^*}))\mathbb{I}_{(x^* + \xi_n, \Sigma(\tau_{x^*}) < K)} \mathbb{I}_{(\tau_{x^*} < \epsilon)}] \\
= e^{-r\tau_{x^*}} \limsup_{n \to \infty} \mathbb{E}[\Theta^+(\epsilon)]\mathbb{I}_{(x^* + \xi_n, \Sigma(\tau_{x^*}) < K)} \mathbb{I}_{(\tau_{x^*} < \epsilon)}] \\
= -e^{-r\tau_{x^*}} \mathbb{E}[\Theta^+(\epsilon)]\mathbb{I}_{(x^* + \xi_n, \Sigma(\tau_{x^*}) < K)} \mathbb{I}_{(\tau_{x^*} < \epsilon)}] \tag{51}
\]

Letting \( \epsilon \downarrow 0 \) in (51) we get by dominated convergence\(^{18}\)

\[
\frac{\partial^+ v}{\partial x}(t^*, x^*) \leq -1 \tag{52}
\]

\(^{18}\)Clearly \( \lim_{t \to 0} \Theta^+(\epsilon) = 1 \); recall moreover that \( x^* < K \).
which together with (49) implies \( \frac{\partial^2 v}{\partial x^2}(t^*, x^*) = -1 \). Finally we have

\[
\frac{\partial^2 v}{\partial x^2}(t^*, x^*) = \lim_{\epsilon \to 0} \frac{v(t^*, x^* - \epsilon) - v(t^*, x^*)}{-\epsilon}
= \lim_{\epsilon \to 0} \frac{g(x^* - \epsilon) - g(x^*)}{-\epsilon} = -1
\]  

(53)

Hence \( x \mapsto v(t, x) \) is differentiable at \( b(t) \) with \( v_x = g_x \) (smooth fit).

**Lemma 11.** The function \( x \mapsto v(t, x) \) is \( C^1 \) with \(-1 \leq v_x(t, x) \leq 0 \).

**Proof.** Fix \( t^* \in [0, T) \). For \( x > b(t^*) \) the assertion follows by lemma 7; for \( x < b(t^*) \) this follows by the fact that \( v(t, x) = g(x) \) in \((0, b(t^*)) \) and clearly \( g(x) \in C^1((0, b(t^*)) \). Now let \( x = b(t^*) \). Since \( x \mapsto v(t^*, x) \) is differentiable at \( b(t^*) \) by lemma 10 with \( v_x(t^*, b(t^*)) = -1 = \lim_{x \uparrow b(t^*)} v_x(t^*, x) \), it remains to show that

\[
\lim_{x \downarrow b(t^*)} v_x(t^*, x) = -1
\]  

(54)

Since \( x \mapsto v(t^*, x) \) is differentiable we clearly have

\[
v_x(t^*, x) = \frac{\partial^2 v}{\partial x^2}(t^*, x)
\]  

(55)

Since \( x \mapsto v(t^*, x) \) is convex, the function \( x \mapsto \frac{\partial^2 v}{\partial x^2}(t^*, x) \) is right-continuous\(^{19}\). This fact together with (55) immediately establishes (54). Hence \( x \mapsto v(t^*, x) \) is \( C^1 \). Finally, clearly \( v_x(t^*, x) = -1 \) for \( x \in (0, b(t^*)) \) and hence by continuity of \( x \mapsto v_x(t^*, x) \), convexity of \( x \mapsto v(t^*, x) \) and lemma 9 we get \(-1 \leq v_x(t^*, x) \leq 0 \) for \( x \in (0, b(t^*)) \).

### 2.6 Further properties of \( b \) for finite time horizon

**Lemma 12.** The function \( b \) is continuous with \( \lim_{t \uparrow T} b(t) = K \).

**Proof.**

- **Right-continuity:** Let \( t \in [0, T) \). Since \( b \) is increasing by Lemma 6, the right-hand limit \( b(t+) \) exists with \( b(t) \leq b(t+) < K \). Moreover by definition \((t, b(t)) \in D \) for \( t \in [0, T) \). Since \( D \) is closed, it follows that \((t, b(t)) \in D \). This together with Lemma 7 implies

\[
0 \leq v(t, b(t+)) - v(t, b(t)) = (K - b(t+)) - (K - b(t)) = b(t) - b(t+) \leq 0
\]  

(56)

Hence \( b(t) = b(t+) \) and \( t \mapsto b(t) \) is right-continuous.

- **Left-continuity:** Let \( t \in (0, T] \). For convenience set \( b(T) := K \). Since \( b \) is increasing, the left-hand limit \( b(t-) \) exists with \( b(t-) \leq b(t) \leq K \).\(^{20}\) Moreover \((t, b(t)) \in D \) for \( t \in [0, T) \). Since \( D \) is closed, it follows that

\(^{19}\) For a proof see [5] p 142 et seq [Satz 7.7 iv].
\(^{20}\) Recall that \( b(t) < K \) for \( t \in [0, T) \).
Lemma 13. The function \( (t, b(-)) \in D \). Seeking a contradiction, suppose that \( b(t-) < b(t) \). Set \( x^* := (b(t-) + b(t))/2 \) and let \( t' < t \). Then

\[
b(t') \leq b(t-) < x^* < b(t) \leq K
\]

which implies in particular that \( (b(t'), x^*) \subset C \) and \( (t, x^*) \in D \). By definition of \( C \) and Lemma 10 we have

\[
v(t', b(t')) - g(b(t')) = 0 \quad \text{and} \quad v_x(t', b(t')) - g_x(b(t')) = 0
\]

Finally, by Lemma 9 we have for \( x \in (b(t'), x^*) \)

\[
v_{xx}(t', x) \geq \frac{2r}{\sigma^2 x^2} v(t', x) \geq \frac{2r}{\sigma^2 x^2} (K - x) \\
\geq \frac{2r}{\sigma^2 b(t')^2} (K - x^*) =: \gamma > 0
\]

A double application of the Fundamental Theorem of Calculus together with (58) yields

\[
v(t', x^*) - g(x^*) = \int_{b(t')}^{x^*} (v_x(t', y) - g_x(y)) \, dy \\
= \int_{b(t')}^{x^*} \int_{b(t')}^{y} (v_{xx}(t', z) - g_{xx}(z)) \, dz \, dy \\
\geq \int_{b(t')}^{x^*} \int_{b(t')}^{y} \gamma \, dz \, dy = \gamma \frac{(x^* - b(t'))^2}{2}
\]

Taking the limit \( t' \uparrow t \) and using that \( v \) is continuous yields

\[
v(t, x^*) - g(x^*) \geq \gamma \frac{(x^* - b(t-))^2}{2} > 0
\]

Hence \( (t, x^*) \notin D \) in contradiction to \( (t, x^*) \in D \). Thus \( b(t) = b(t-) \) and \( t \mapsto b(t) \) is left-continuous with \( \lim_{t \uparrow T} b(t) = K \).

**Lemma 13.** The function \( t \mapsto b(t) \) is convex and satisfies

\[
\lim_{t \uparrow T} \frac{\log(b(t)/K)}{\sigma \sqrt{(T-t)}(-\log(8\pi \tau^2(T-t)/\sigma^2))} = 1
\]

**Proof.** See [2].

**Remark.** This result will not be used in the following.

### 2.7 Early exercise premium representation

The above lemmata imply\(^{22}\) that \( V \in C^{0,1}(E \setminus \{K\}) \cap C^{1,2}(E \setminus \Gamma(b(t))) \).\(^{23}\) Moreover, since \( P_{(t,x)}(X(s) = b(t+s)) = 0 \) for all \( (t, x) \in (0, T) \times (0, \infty) \) and

\(^{21}\) Note that \( t' < T \).

\(^{22}\) Note that clearly \( V \in C^{1,2}(\text{Int}(D)) \).

\(^{23}\) \( \Gamma(b(t)) := \{(t, x) \in [0, T] \times (0, \infty) : x = b(t)\} \)
all $0 < s < T - t$, we can apply a slightly generalized version of Itô’s formula to $V(X_s)$ and get
\[
    dV(X_s) = LV(X_s) \mathbb{1}_{X(s) \neq b(t+s)} \, ds + V_x(X(s)) \, dS(t)
    = -\sigma r K \mathbb{1}_{(s) < b(t+s)} \, dt + \sigma S(t) V_x(X_s) \, dW(t)
\]
(63)
From (63) get immediately for $(t, x) \in E$
\[
    E_{(t,x)}[V(X(T-t))] = V(t,x) - rK \int_0^{T-t} e^{-r(t+s)} \mathbb{P}_{(t,x)}(S(s) < b(t+s)) \, ds
\]
(64)
By the Markov Property using $(T-t) + \tau_D = T - t$ under $\mathbb{P}_{(t,x)}$ we have
\[
    E_{(t,x)}[V(X(T-t))] = E_{(t,x)}[E_{X(T-t)}[G(X(T))]]
    = E_{(t,x)}[E_{(t,x)}[G(X((T-t) + \tau_D))] \mathcal{F}_{T-t}]}
    = E_{(t,x)}[G(X((T-t) + \tau_D)])
    = E_{(t,x)}[G(X((T-t)))]
\]
(65)
multiplying (64) with $e^{-rt}$ yields using (65)
\[
    v(t,x) = e^{-r(T-t)} E_{(t,x)}[g(S(T-t))]
    + rK \int_0^{T-t} e^{-rs} \mathbb{P}_{(t,x)}(S(s) < b(t+s)) \, ds
\]
(66)
Plugging in $t = 0$ in (66) yields after some algebra
\[
    V(0, x) = E_x[e^{-rT}(K - S(T))^+]
    + rK \int_0^T e^{-rs} \Phi \left( \frac{1}{\sigma \sqrt{s}} \left( \log \left( \frac{b(s)}{b(0)} \right) - \left( r - \frac{\sigma^2}{2} \right) s \right) \right) \, ds
\]
(67)
where $\Phi$ denotes the cdf of a standard normal.

**Remark.** Formula (67) is called the *early exercise premium representation* of the value function. It shows that the value of an American put option with strike price $K$ and maturity $T$ is the sum of the value of an European put option with the same strike and maturity and the so-called *early exercise premium*.

### 2.8 Free boundary equation for $b(t)$

**Theorem 4.** The function $t \mapsto b(t)$ is the unique solution in the class of continuous increasing functions $c : [0, T] \to \mathbb{R}$ satisfying $0 < c(t) < K$ for all $0 < t < \infty$ of the following free-boundary integral equation
\[
    K - b(t) = e^{-r(T-t)} \int_0^K \Phi \left( \frac{1}{\sigma \sqrt{T-t}} \left( \log \left( \frac{K-x}{b(t)} \right) - \left( r - \frac{\sigma^2}{2} \right) (T-t) \right) \right) \, dx
    + rK \int_0^{T-t} e^{-rs} \Phi \left( \frac{1}{\sigma \sqrt{s}} \left( \log \left( \frac{b(t+s)}{b(t)} \right) - \left( r - \frac{\sigma^2}{2} \right) s \right) \right) \, ds
\]
(68)

25 Note that $\int_0^t \sigma S(s) V_x(s, S(s)) \, dW(s)$ is a proper zero-mean martingale since $|V_x(X_s)| \leq e^{-rt} < 1$. 

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Proof.

- $t \mapsto b(t)$ is a solution of (68): Plugging $(t, b(t))$ in (66) and noting that $v(t, b(t)) = K - b(t)$ yields after some lengthy calculation (68).

- $t \mapsto b(t)$ is the unique solution of (68): See [6] p 386 - 392.

References


