Long term behaviour: From particle dynamics through kinetic equations to fractional diffusion equations

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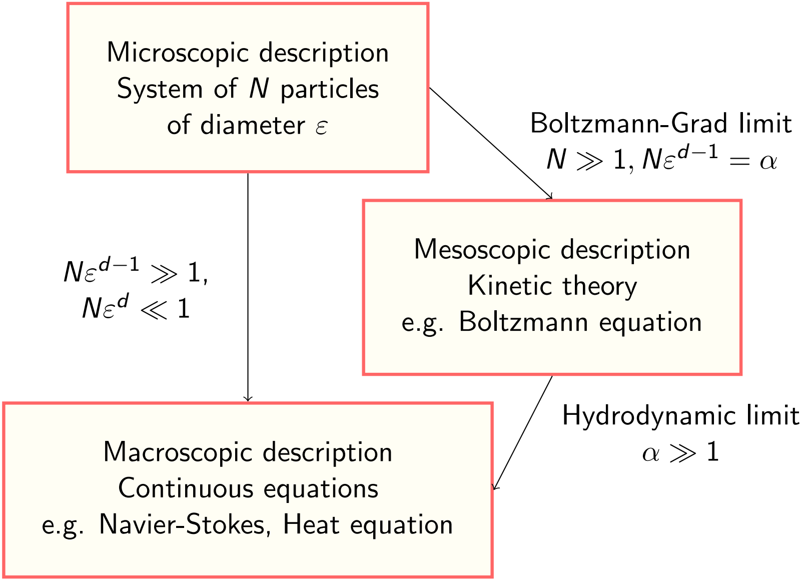
Microscopic to Macroscopic description



Microscopic to Mesoscopic

N particle system

Boltzmann-Grad limit
$N \gg 1, N\varepsilon^{d-1} = \alpha$

Boltzmann equation

- N hard spheres particle system obey the Newton laws

$$\frac{dx_i(t)}{dt} = v_i, \quad \frac{dv_i(t)}{dt} = 0, \text{ unless collision}$$

- The Boltzmann equation

$$\partial_t f(t,x,v) + v \cdot \nabla_x f(t,x,v) = Q(f,f)(t,x,v)$$

- This convergence is known for small time interval, of the order of $\frac{1}{\alpha}$. *O. E. Lanford '74*

Mesoscopic to Macroscopic

Kinetic theory

Hydrodynamic limit

$\alpha \gg 1$

Continuous equations

It is known in some situations, i.e.

- The renormalised solutions f_α of a scaled Boltzmann equation

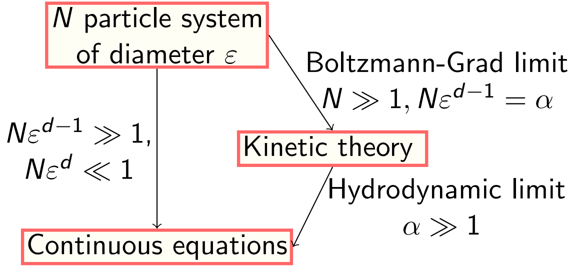
$$\frac{1}{\alpha}\partial_t f + v \cdot \nabla_x f = \alpha Q(f, f)$$

  converges as α goes to infinity to $g(v, u)$, where u satisfies the incompressible Navier-Stokes equations

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu \Delta u = -\nabla p, \\ \operatorname{div} u = 0 \ . \end{cases}$$

F. Golse and L. Saint-Raymond '04

The full derivation



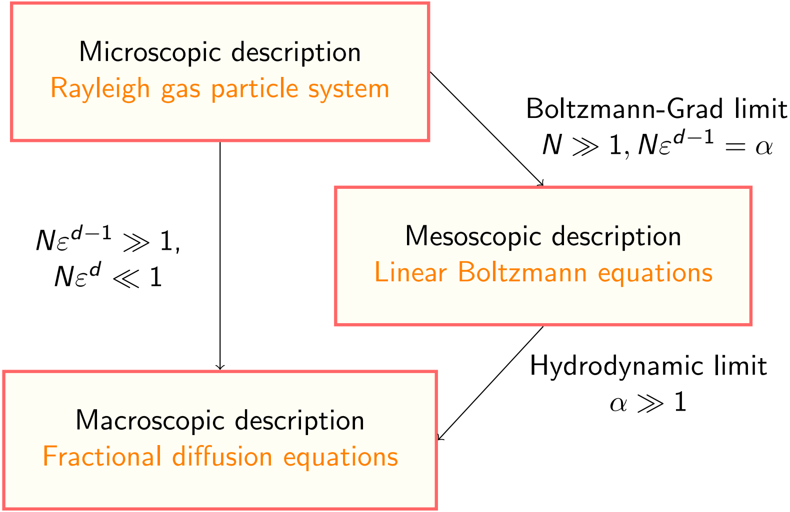
- The distribution of a tagged particle in a background at equilibrium satisfies a heat equation

$$\partial_t u - \kappa_\beta \Delta_x u = 0$$

  at the limit $N \to \infty$, using the linear Boltzmann equation as an intermediate step with $\alpha = N\varepsilon^{d-1}$ going slowly to infinity with N.

T. Bodineau, I. Gallagher and L. Saint-Raymond '16

Description of the problem



Particle model (Microscopic description)

- Short range potential: The particles carry a force that affects only nearby particles.
  The Hamiltonian equations of motions are given by

$$\frac{dx_i(t)}{dt} = v_i, \quad \frac{dv_i(t)}{dt} = -\sum_{j \neq i} \nabla\Phi(x_i - x_j),$$

  where $\Phi : \mathbb{R}^3 \to \mathbb{R}$ is the interaction potential, radial, supported in the ball of $\mathbb{R}^3$ of radius R, class C^2 in $\{x \in \mathbb{R}^3, 0 < |x| < R\}$ and goes to zero at $|x| = R$.
- Conservation of momentum and kinetic energy:

$$v_i + v_j = v_i' + v_j',$$
$$|v_i|^2 + |v_j|^2 = |v_i'|^2 + |v_j'|^2.$$

- Rayleigh gas: The background particles are of equal mass with the tagged particle and the background particles interact only with the tagged particle and not with each other.

Linear Boltzmann equation (Mesoscopic description)

- The linear Boltzmann equation

$$\partial_t f + v \cdot \nabla_x f = Q(f) \ , \quad x \in \mathbb{T}^3, \ v \in \mathbb{R}^3, \ t \geq 0$$

- $f = f(t,x,v)$: probability density in $\mathbb{T}^3 \times \mathbb{R}^3$
- $Q(f)$: linear Boltzmann collision operator, local in (t,x), describes the interactions of the particles with the surrounding medium

$$Q(f)(v) = \int_{\mathbb{R}^3} \int_{\mathbb{S}^2} (f(v')g(v_1') - f(v)g(v_1)) \ B(v - v_1, \omega) \ d\omega \ dv_1$$

- g: distribution of the background particles
- B is the collision kernel
- $v' = v + \omega \cdot (v - v_1)\omega, \quad v_1' = v_1 - \omega \cdot (v - v_1)\omega$

Fractional diffusion limit (Macroscopic description)

The fractional diffusion equation of order $\gamma < 2$:

$$\partial_t \rho + \kappa(-\Delta_x)^{\frac{\gamma}{2}} \rho = 0 \qquad \text{in } (0,\infty) \times \mathbb{R}^3,$$
$$\rho(0,\cdot) = \rho_0 \qquad \text{on } \mathbb{R}^3,$$

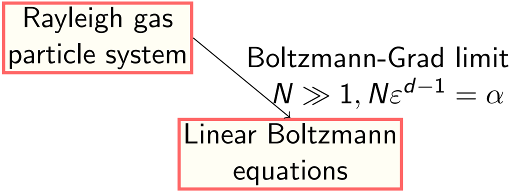
where κ is given by

$$\kappa = \frac{\kappa_0 \nu_0}{1-\beta} \int_{\mathbb{R}^3} \frac{w_1^2}{\nu_0^2 + w_1^2} \frac{1}{|w|^{3+\gamma}} dw.$$

- $\rho = \rho(t,x)$: the density $\int_{\mathbb{R}^3} f(v) dv$
- The fractional operator: $(-\Delta_x)^{\frac{\gamma}{2}} \rho := \mathcal{F}^{-1}(|k|^\gamma \mathcal{F}(\rho)(k))$, with $\mathcal{F}$ the Fourier transform in the space variable
- Difference with Laplace operator: It is nonlocal for $0 < \gamma < 2$

See, e.g., *Fractional Diffusion Equations and Anomalous Diffusion, L. R. Evangelista and E. K. Lenzi '18*

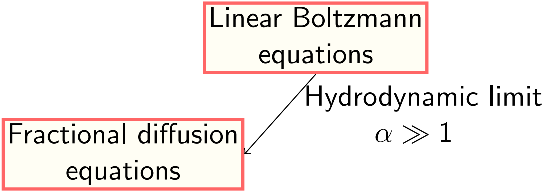
Microscopic to Mesoscopic



Sketch of the proof: Follow the steps for the hard sphere case:

- Consider the Empirical equation: It is connected to the particle dynamics
- Consider the Idealised equation: It is equivalent of the linear Boltzmann equation but on the set of collision histories
- Estimate of the probability of the set of not well controlled histories
- Estimate of the difference between the empirical distribution on collision histories $\hat{P}_t$ and the solution P_t of the idealized equation

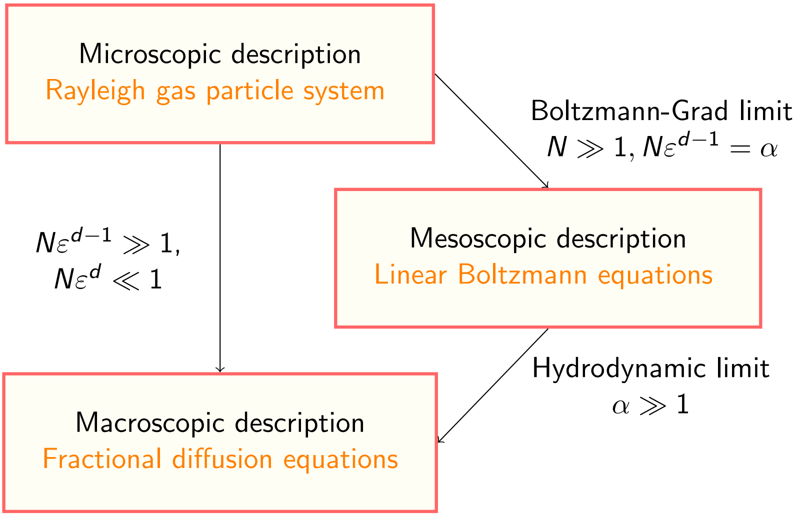
Mesoscopic to Macroscopic



Plan: Follow techniques of Mellet, Mischler, Mouhot '11

- Convergence of the solution f of the rescaled linear Boltzmann equation, to $\rho(t,x)F(v)$ with ρ the solution of a fractional diffusion equation in the $L^\infty(0,T;L^2(\mathbb{R}^3 \times \mathbb{R}^3))$
- F is the equilibrium distribution of the linear Boltzmann equation, i.e. $Q(F) = 0$
- Use radial, fat-tailed background distribution g

Conclusion


Microscopic description
Rayleigh gas particle system
Boltzmann-Grad limit
$N \gg 1, N\varepsilon^{d-1} = \alpha$
$N\varepsilon^{d-1} \gg 1$,
$N\varepsilon^{d} \ll 1$
Mesoscopic description
Linear Boltzmann equations
Hydrodynamic limit
$\alpha \gg 1$
Macroscopic description
Fractional diffusion equations

Thank you

Thank you!