Long time behaviour of cooperatively branching and coalescing particle systems

Anja Sturm

Universität Göttingen
Institut für Mathematische Stochastik

Joint work with Jan Swart (UTIA Prague) and Tibor Mach (Uni Göttingen)

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Outline

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Interacting particle system - definition

"Countable system of locally interacting Markov processes"

The state space of the system

- **Lattice** Countable space Λ with some notion of distance.
- **Local states** Usually a finite set S.
- **State space of the system** $E = S^\Lambda$
  Each point of the lattice is in one of the local states.

Example: $\Lambda = \mathbb{Z}^d, S = \{0, 1\}, E = \{0, 1\}^{\mathbb{Z}^d}$

Interacting particle system

Change the local state at one point (finitely many points) in the lattice with a rate that depends on the surrounding local states.

General references: Liggett ('85, '99), Swart ('15)

Interacting particle system - Short review

- Interacting particle systems are **toy models** for stochastic systems with a **spatial structure and simple local rules**.
- They lead to surprisingly realistic and interesting behavior on a large space time scale: **macroscopic behavior**.
- **Universality classes:** Often, it turns out that more detailed and realistic local rules lead to the same kind of macroscopic behavior.

Central questions: Longtime and macroscopic behavior, phase transitions, behavior at the phase transitions ...

Applications: Population dynamics, spread of disease or rainwater particle motion, ferromagnetism, traffic flow, social network dynamics...

Interacting particle system - classical examples

Contact process on $\mathbb{Z}^d$:

- Continuous time Markov process with $E = \{0, 1\}^{\mathbb{Z}^d}$.
- Interpretation: "1" as particle and "0" as an empty site.
- At some rate $q(|i - j|)$ a particle at site i produces a particle at site j (if empty).
- Each particle dies at rate 1.

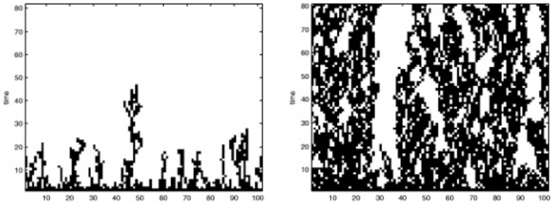

Figure : **Directed percolation model:** Analogous model in discrete time. Simulation on 100 sites by Allhoff and Eckhardt for different nearest neighbor birth rates.

Contact process on $\mathbb{Z}^d$: $X^x = (X^x_t)_{t\geq 0}$ with $X^x_0 = x$

- Spatial version of a binary branching process with local carrying capacity.
- **Longtime behavior: Survival** For $|x| := \sum_{i\in\mathbb{Z}^d} x(i) < \infty$

$$\theta = \theta^x = \mathbb{P}\big[X^x_t \neq \underline{0}\ \forall t \geq 0\big] > 0?$$

- **Longtime behavior: Complete convergence**

$$\mathcal{L}(X^x_t) \Rightarrow \theta^x \bar{\nu} + (1-\theta^x)\delta_{\underline{0}}$$

- The **upper invariant law** $\bar{\nu}$ is the limit for $x = \underline{1}$. Nontrivial if $\bar{\nu} \neq \delta_{\underline{0}}$.

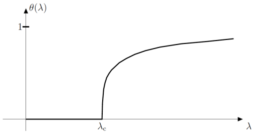

Figure : Phase transition for survival in a one dimensional nearest neighbour contact process with branch rate λ and $|x| = 1$.

Interacting particle system - dualities

Duality: $\mathbb{E}\big[1_{\{|X_t^x \cdot y|=0\}}\big] = \mathbb{E}\big[1_{\{|x \cdot Y_t^y|=0\}}\big], t \geq 0$
$|x \cdot y| = \sum_i x(i)y(i)$.

For the **contact process** $X \sim Y$ (self-dual).

The above duality relates survival $\theta_X^{\delta_i} > 0$ with nontriviality of $\bar{\nu}_Y$:

$$\begin{aligned} \mathbb{E}\big[1_{\{|X_t^x \cdot \underline{1}|=0\}}\big] &= \mathbb{E}\big[1_{\{|x \cdot Y_t^{\underline{1}}|=0\}}\big] \\ \Leftrightarrow \mathbb{P}\big[|X_t^x \cdot \underline{1}| \neq 0\big] &= \mathbb{P}\big[|x \cdot Y_t^{\underline{1}}| \neq 0\big] \\ \Leftrightarrow \mathbb{P}\big[X_t^x \neq \underline{0}\big] &= \mathbb{P}\big[|x \cdot Y_t^{\underline{1}}| \neq 0\big] \end{aligned}$$

With $t \to \infty$ and $x = \delta_i$

$$\theta_X^{\delta_i} = \mathbb{P}\big[Y_\infty^{\underline{1}}(i) \neq 0\big] = \int \bar{\nu}_Y(\mathrm{d}y) 1_{\{y(i)=1\}}.$$

Voter model on $\mathbb{Z}^d$:

- Continuous time Markov process with $E = \{0,1\}^{\mathbb{Z}^d}$.
- Interpretation: Particle at each site of type either 0 or 1.
- At some rate $q(|i-j|)$ site i adopts the local state of site j.

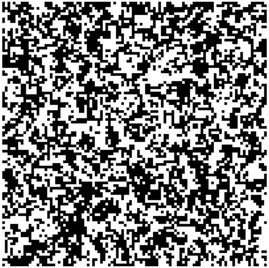

Figure : Sequential snapshots of the nearest neighbour voter model produced with an online simulator by Bryan Gillespie (Berkeley) on a 100x100 grid.

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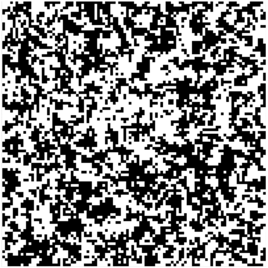

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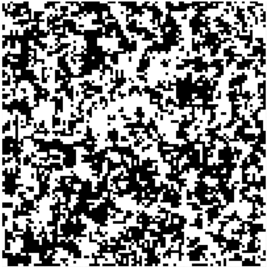

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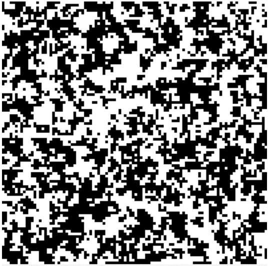

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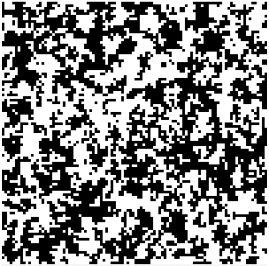

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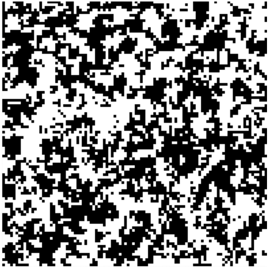

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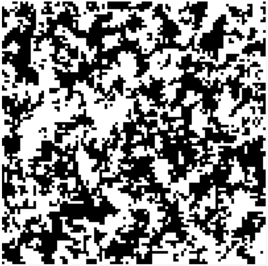

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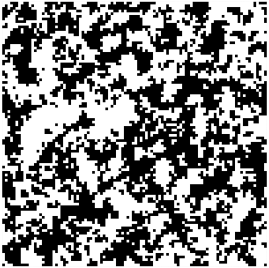

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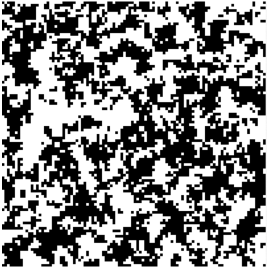

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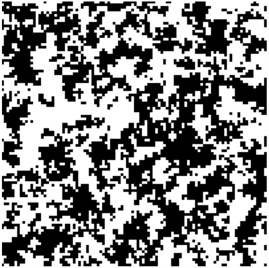

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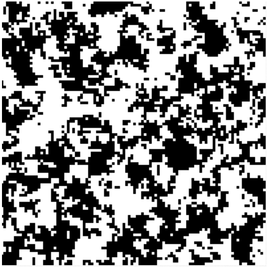

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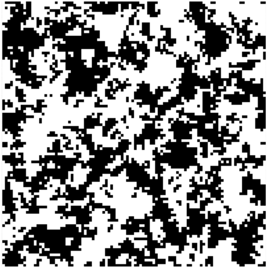

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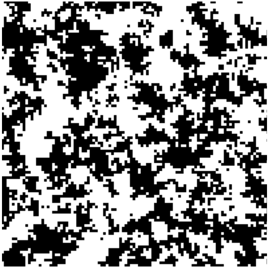

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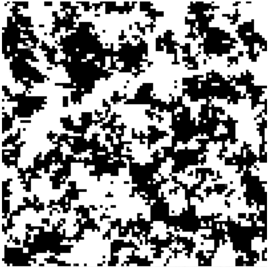

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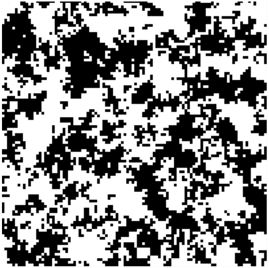

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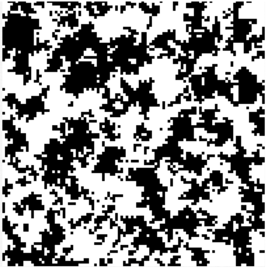

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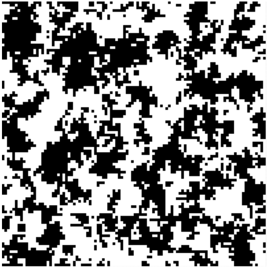

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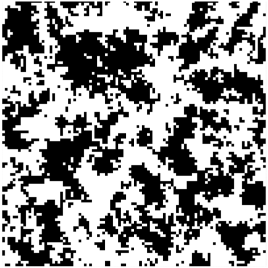


Figure : Sequential snapshots of the nearest neighbour voter model produced with an online simulator by Bryan Gillespie (Berkeley) on a 100x100 grid. **Clustering occurs! Longtime coexistence?**

Outline

Cooperative branching coalescent (CBC)

Sturm and Swart: Annals of Applied Probability 2014

- Continuous time Markov process with state space $\{0,1\}^{\mathbb{Z}}$:

$$X = (X_t)_{t\geq 0}$$

- "1" represents a particle, "0" an unoccupied site.
- **Symmetric random walk with coalescence:** particles on adjacent sites merge at rate 1.
- **Adjacent pairs of particles produce a new particle:** particle is placed on a (randomly chosen) neighbouring site at cooperative branching rate λ.

Motivation

As a model in the biological context:

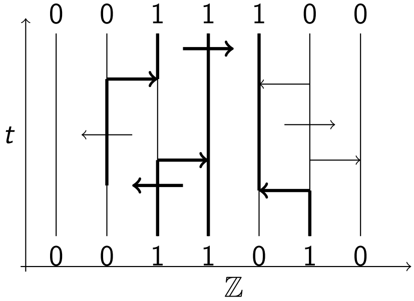
1. **Pair reproduction with migration and competition:**
   "1" is a site occupied by an individual, "0" is an empty site.
   Cooperative branching: pairs of individuals reproduce.
   Coalescing random walk: death due to competition.

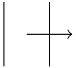
2. **Interface model of a multi type voter model:**
   "1" is an interface between different "types".
   Cooperative branching: singletons give birth to a new type.
   Coalescing random walk: voter dynamics and disappearance of types.

As a mathematical toy model:

Tractable one dimensional model with interesting properties.

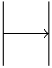
The graphical representation





$t \in \vec{\omega}(i)$

cooperative branching



$t \in \vec{\omega}(i - \frac{1}{2})$

coalescing jump

For $i \in \mathbb{Z}$

$$\vec{\omega}(i), \overleftarrow{\omega}(i) \text{ as well as } \vec{\omega}(i - \tfrac{1}{2}), \overleftarrow{\omega}(i - \tfrac{1}{2})$$

are Poisson processes with rate $\frac{1}{2}\lambda$ and $\frac{1}{2}$.

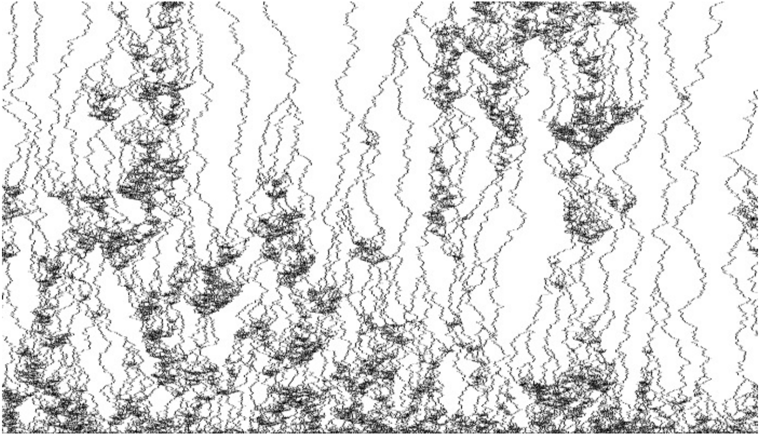
Useful basic properties

The graphical representation provides a "coupling" of processes with different initial states and parameters.

- **Monotonicity**
  If $x \leq y$ (componentwise) then the processes can be coupled such that
  $$X_t^x \leq X_t^y \text{ for all } t \geq 0.$$
  $\Rightarrow$ Monotoniciy in the initial states.

  We also have monotonicity in λ.

Simulation of the model



Simulation of a near-critical cooperative branching-coalescent with $\lambda = 2\frac{1}{3}$ on a lattice of 700 sites with periodic boundary conditions, started from the fully occupied initial state.

Long time behavior

- From monotonicity

$$\mathbb{P}\big[X_t^1 \in \cdot\,\big] \underset{t\to\infty}{\Longrightarrow} \overline{\nu},$$

where $\overline{\nu}$ is the **upper invariant law.**
Probability under $\overline{\nu}$ of finding a particle in the origin:

$$\overline{\theta}(\lambda) := \int \overline{\nu}_\lambda(\mathrm{d}x) 1_{\{x(0)=1\}}$$

$\overline{\nu}_\lambda$ is **nontrivial** if $\overline{\theta}(\lambda) > 0$.

- Survival probability of pairs - "staying active":

$$\theta(\lambda) := \mathbb{P}\big[|X_t^{\delta_0+\delta_1}| \geq 2 \ \forall t \geq 0\big]$$

The process **survives** if $\theta(\lambda) > 0$.

Existence of phase transitions

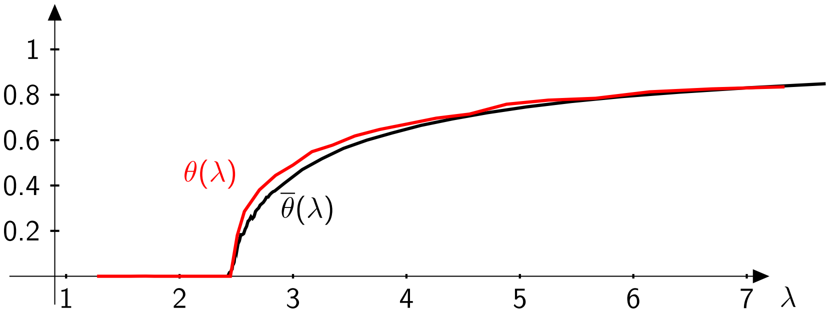
There exist phase transitions for the triviality/nontriviality of the upper invariant law as well for survival/extinction.

Theorem:
Phase transitions for upper invariant law and survival

(a) There exists a $1 \leq \lambda_{\rm c} < \infty$ such that
$\overline{\nu}_\lambda = \delta_{\underline{0}}$ for $\lambda < \lambda_{\rm c}$ but $\overline{\nu}_\lambda$ is nontrivial for $\lambda > \lambda_{\rm c}$.

(b) There exists a $1 \leq \lambda'_{\rm c} < \infty$ such that
the process dies out for $\lambda < \lambda'_{\rm c}$ and survives for $\lambda > \lambda'_{\rm c}$.

Existence of phase transitions



Simulation of the density $\overline{\theta}(\lambda)$ of the upper invariant law and survival probability $\theta(\lambda)$ (plotted in black and red, respectively) rate suggesting

$$\lambda_c \approx \lambda_c' \approx 2.47 \pm 0.02,$$

Proof ideas: Existence of phase transitions

Monotonicity implies the existence of λ_c and λ'_c if we can show

(a) $\overline{\nu} = \delta_{\underline{0}}$ for $\lambda \leq 1$ and $\overline{\nu} \neq \delta_{\underline{0}}$ for large λ.

(b) The process dies out for $\lambda \leq 1$ and survives for large λ.

Proof ideas: Existence of phase transitions

Triviality of the upper invariant law for $\lambda \leq 1$: $\overline{\nu} = \delta_\emptyset$

If $\lambda > 0$ and the process is started translation invariant let
$p_t(1) = \mathbb{P}(X_t(i) = 1), \quad p_t(11) = \mathbb{P}(X_t(i) = 1, X_t(i+1) = 1), \ldots.$

$$\begin{aligned}\frac{\partial}{\partial t} p_t(1) &= -p_t(1) + \tfrac{1}{2} p_t(10) + \tfrac{1}{2} p_t(01) + \tfrac{1}{2}\lambda p_t(110) + \tfrac{1}{2}\lambda p_t(011) \\ &= -p_t(11) + \lambda\big(p_t(11) - p_t(111)\big) \\ &= (\lambda - 1) p_t(11) - \lambda p_t(111),\end{aligned}$$

If the process is furthermore started from an invariant law

$$0 = \frac{\partial}{\partial t} p_t(1) \leq -\lambda p_t(111) \Rightarrow p_t(111) = 0.$$

As $p_t(1) > 0$ would imply $p_t(111) > 0$ we are done.
(Case $\lambda = 0$ similar.)

Proof ideas: Existence of phase transitions

Extinction for $\lambda \leq 1$: $\mathbb{P}\big[\exists T < \infty \text{ s.t. } |X_t^x| = 1 \; \forall t \geq T\big] = 1.$

With similar calculations

$$\begin{aligned}\frac{\partial}{\partial t}\mathbb{E}\big[|X_t^x|\big] \qquad &= (\lambda - 1)\sum_{i\in\mathbb{Z}} \mathbb{P}[X_t^x(i) = X_t^x(i+1) = 1] \\ &\quad -\lambda \sum_{i\in\mathbb{Z}} \mathbb{P}[X_t^x(i) = X_t^x(i+1) = X_t^x(i+2) = 1]\end{aligned}$$

So $|X_t^x|$ is a supermartingale for $\lambda \leq 1$: $|X_t^x| \underset{t\to\infty}{\longrightarrow} N \quad \text{a.s.}$
$\Rightarrow N = 1 \; a.s.$
since if there were more particles left they would meet (a.s. due to recurrence) and interact (through branching or coalescence).

Proof ideas: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :
"Coupling" with another process: Pairs of adjacent particles are coupled with a contact process variant.

The contact process with double deaths $Y = (Y_t)_{t \geq 0}$

- Sites infect any neighbor at rate $\frac{1}{2}\lambda$.
- Any particles on two neighboring sites die at rate 1.

Graphical representation with Poisson processes:

$$\overleftarrow{\pi}(i - \frac{1}{2}), \overrightarrow{\pi}(i - \frac{1}{2}), \quad \text{and } \pi^*(i - \frac{1}{2}), \quad i \in \mathbb{Z}.$$

Proof ideas: Existence of phase transitions

Nontrivial upper invariant law and survival for large λ :

Comparison of X with the contact process with double deaths Y

Let

$$X_t^{(2)}(i) := 1 \Leftrightarrow X_t^x(i) = X_t^x(i+1) = 1 \qquad t \geq 0$$

denotes the locations of pairs of neighbouring particles in X_t. Then $(X_t^{(2)})_{t\geq 0}$ and $(Y_t)_{t\geq 0}$ can be coupled such that

$$Y_0 \leq X_0^{(2)} \quad \text{implies} \quad Y_t \leq X_t^{(2)} \qquad t \geq 0.$$

Coupling:

$$\overleftarrow{\pi}(i-\frac{1}{2}) := \overleftarrow{\omega}(i), \quad \overrightarrow{\pi}(i-\frac{1}{2}) := \overrightarrow{\omega}(i), \quad \pi^*(i-\frac{1}{2}) := \overleftarrow{\omega}(i-\frac{1}{2}) \cup \overrightarrow{\omega}(i+\frac{1}{2})$$

Proof ideas: Existence of phase transitions

Comparison with oriented percolation

- By considering large times blocks we can can bound the contact process with double deaths from below by **oriented percolation** with arbitrarily large p for large enough λ.
- For large enough p the oriented percolation process has a nontrivial upper invariant law and survives completing the proof.

Decay rate in the subcritical regime

Theorem:
Decay rates of the survival probability and the density

(a) There exists a constant $c > 0$ such that for all $\lambda \geq 0$,

$$\mathbb{P}\big[|X_t^{\delta_0+\delta_1}| \geq 2\big] \geq ct^{-1/2} \text{ and } \mathbb{P}[X_t^{\underline{1}}(0) = 1] \geq ct^{-1/2} \qquad t \geq 0.$$

(b) Moreover, there exists a constant $C < \infty$ such that for each $0 \leq \lambda \leq \frac{1}{2}$,

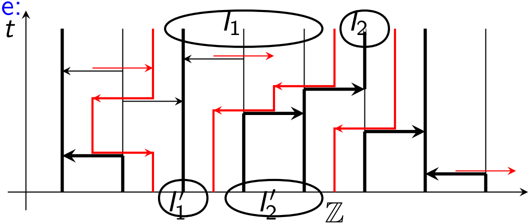
$$\mathbb{P}\big[|X_t^{\delta_0+\delta_1}| \geq 2\big] \leq Ct^{-1/2} \text{ and } \mathbb{P}[X_t^{\underline{1}}(0) = 1] \leq Ct^{-1/2} \qquad t \geq 0.$$

Note: $\frac{1}{2} \leq \lambda_c, \lambda'_c$ (subcritical regime)
Proof technique: Pathwise (super-)duality

Proof ideas: Decay of the survival probability and density

Lower bound Suffices to consider $\lambda = 0$: coalescing random walk

Consider coalescing random walk ξ following reversed arrows in reversed time:



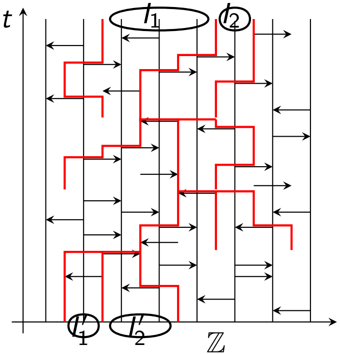
No particles in I_1 if and only if no particles in I_1'.

Pathwise duality to coalescing random walks:

$$\sum_{k=i+\frac{1}{2}}^{j-\frac{1}{2}} X_t^x(k) = 0 \quad \text{if and only if} \quad \sum_{k=\xi_0^{(i,t)}+\frac{1}{2}}^{\xi_0^{(j,t)}-\frac{1}{2}} x(k) = 0 \quad \text{a.s.}$$

Proof ideas: Decay of the survival probability and density

Upper bound A pathwise superdual for $\lambda > 0$ (similar to Gray '86)



Superduality: If there are particles in either I_1 or I_2 then there must exist a "backward 3-path" as drawn such that there are particles in either I'_1 or I'_2. **We can bound the expected number of 3-paths over time t "started" in adjacent sites.**

Extensions of the model

Work in progress with Jan Swart and Tibor Mach:

- Include natural deaths.
  Exponential decay of particle density and survival
- Consider different graphs: $\mathbb{Z}^d$, trees, complete graph.
  Dual process for the mean field model
- Consider different sexes:
  Offspring only produced when parents are of opposite sex.
  Convergence to well mixed sexes and similar behavior to one sex model.

Thank you for your attention!