# Condensation in reinforced branching processes

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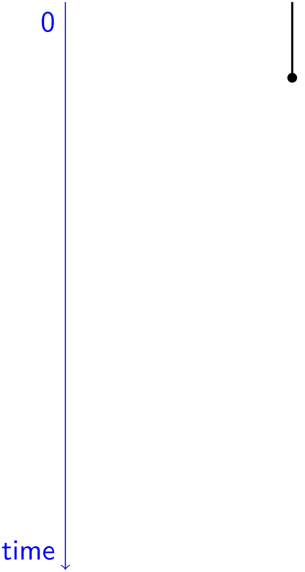
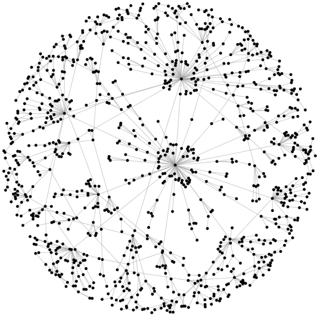
# Overview

# Preferential Attachment Tree: Barabasi and Albert



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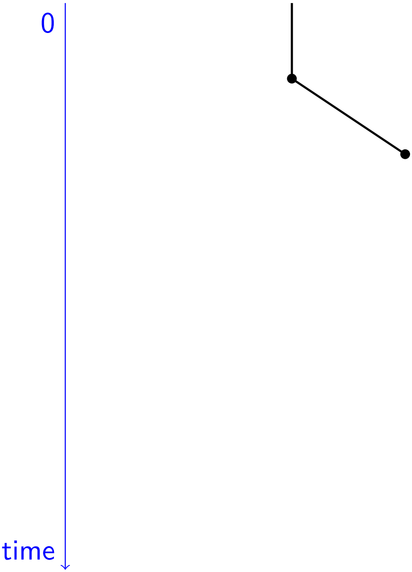


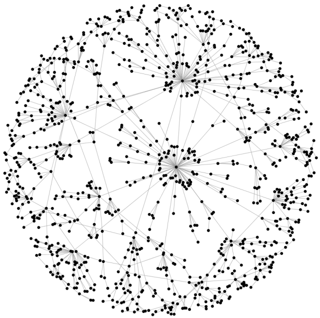

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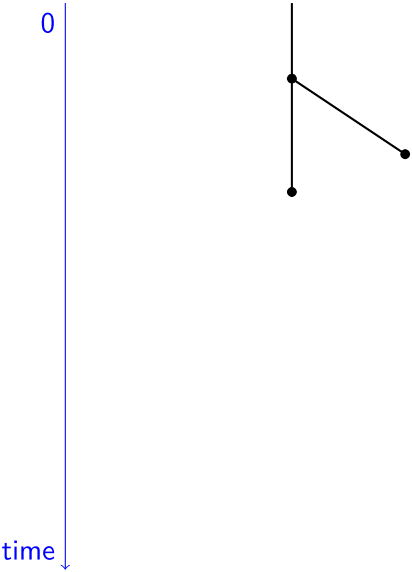


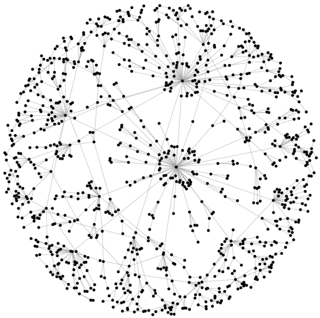

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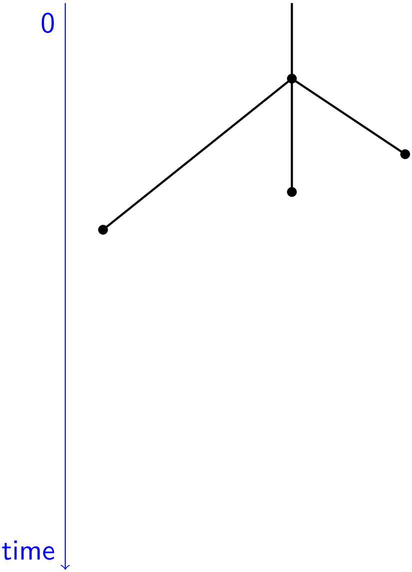


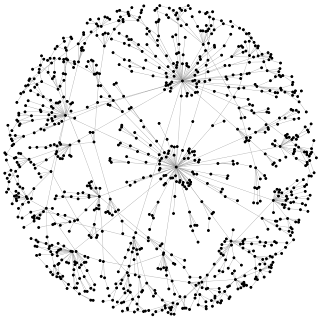

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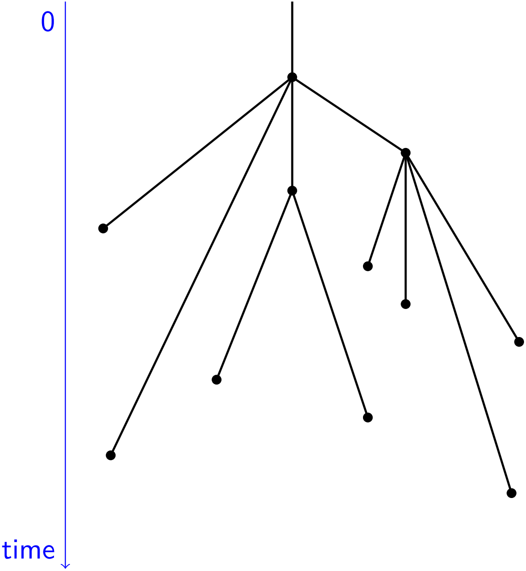


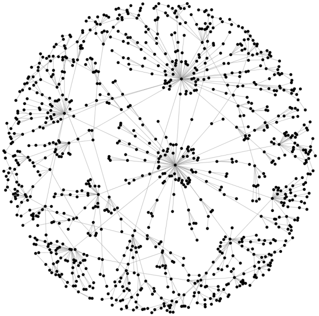

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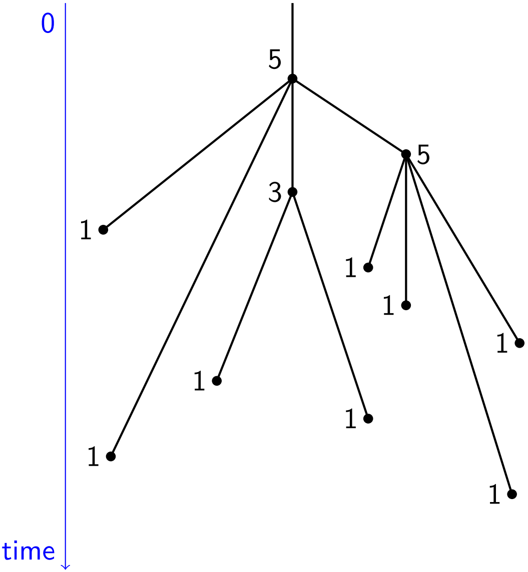


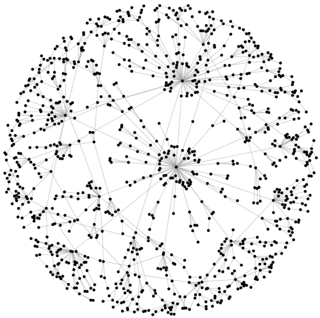

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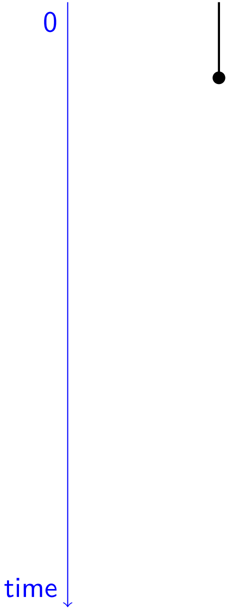
# Preferential Attachment Tree: Bianconi and Barabasi



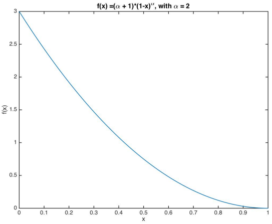




Figure: Probability density function of a given distribution $\mu$.

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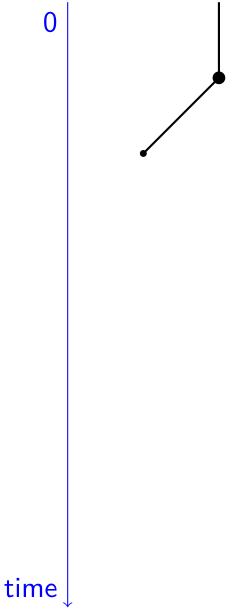


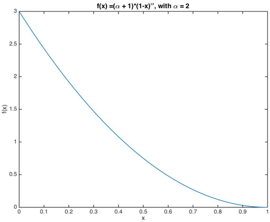


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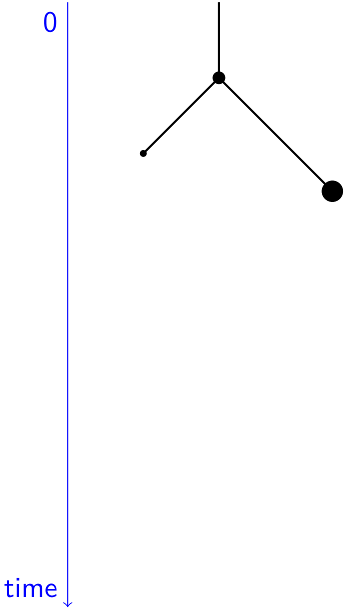


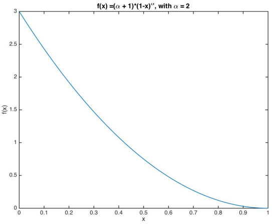


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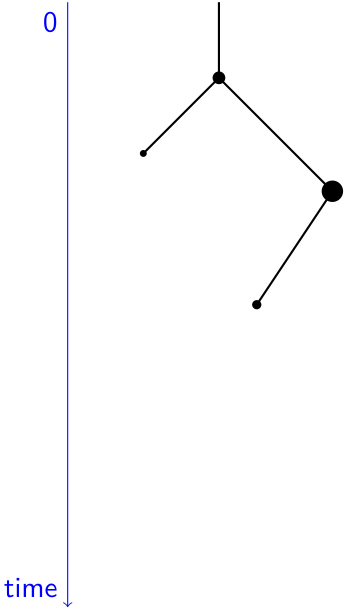


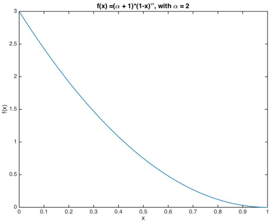


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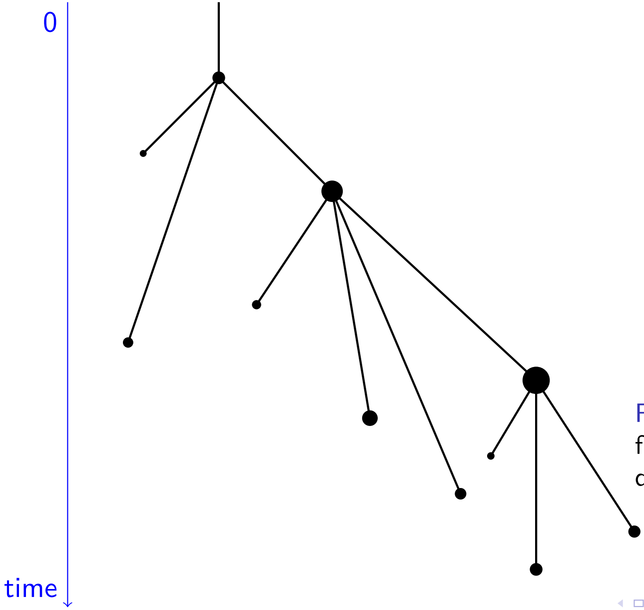


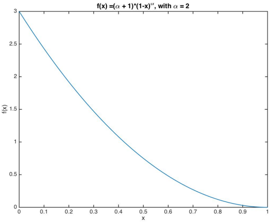


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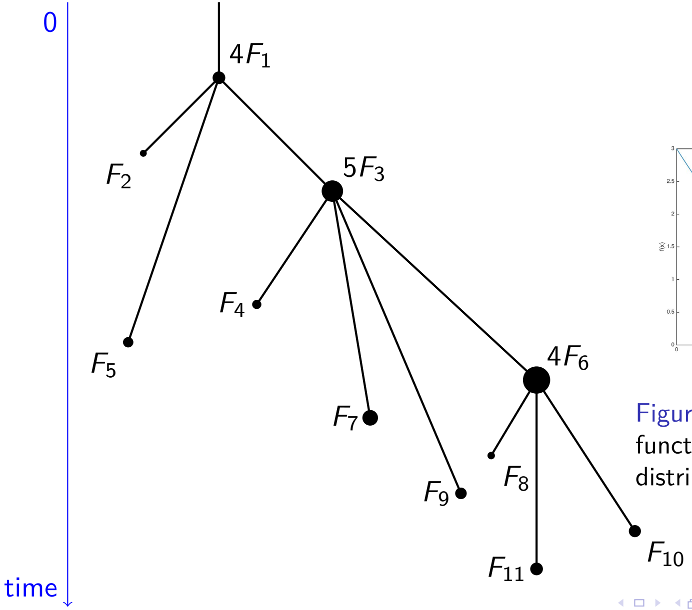


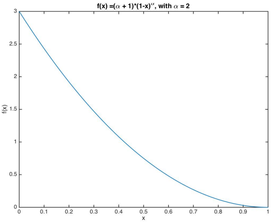


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# Model Definition

At time $t$ we have

- $N(t)$ particles ($=$ half-edges);
- $M(t)$ families ($=$ set of particles sharing the same fitness $=$ nodes);
- $Z_n(t)$ the size of the $n^{\text{th}}$ family ($=$ degree);
- $F_n$ fitness of the $n^{\text{th}}$ family;
- $\tau_n$ the time of the foundation of the $n^{\text{th}}$ family.

At time $t$, each family reproduces at rate $F_n Z_n(t)$.

# Model Parameters

## Model Parameters

- $0 \leq \beta, \gamma \leq 1$ mutation and selection probability;
- $\mu$ the fitness distribution on $(0, 1)$;

## Mutation/Selection probability

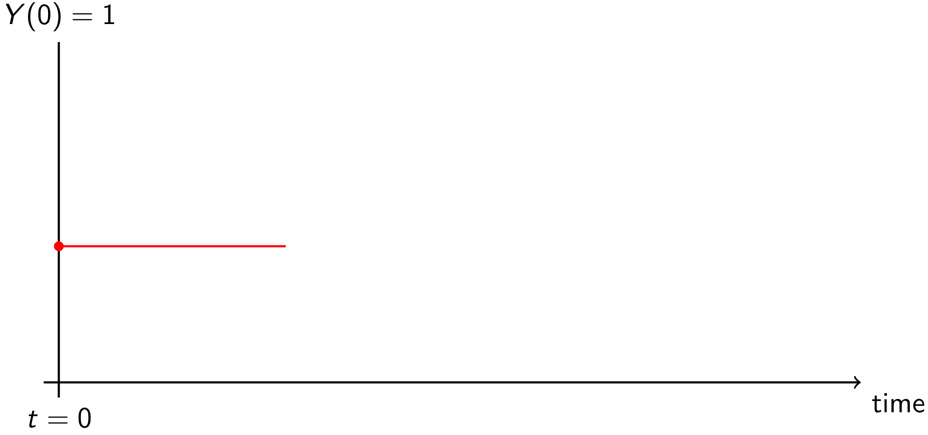
When a birth event happens in a family $n$

- with probability $\gamma$ a new particle is added to family $n$;
- with probability $\beta$ a mutant having fitness $F_{M(t)+1}$ is born.

## Specific models

Bianconi and Barabasi model: $\beta = 1 = \gamma$.
Kingman model : $\gamma = 1 - \beta$.

# Yule process with rate $\eta$ (= Growth of nth family)




The size of a family with fitness $F_n$ grows like a *Yule process*, $Y(t)$, with rate $\gamma F_n$. So that $Y(t) \sim_{t\to\infty} e^{\eta t}\xi$, where $\xi$ is an exponentially distributed random variable.

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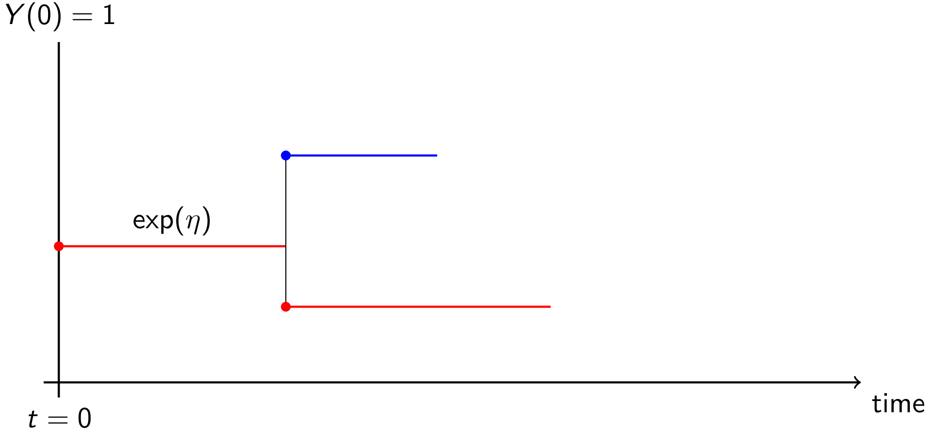


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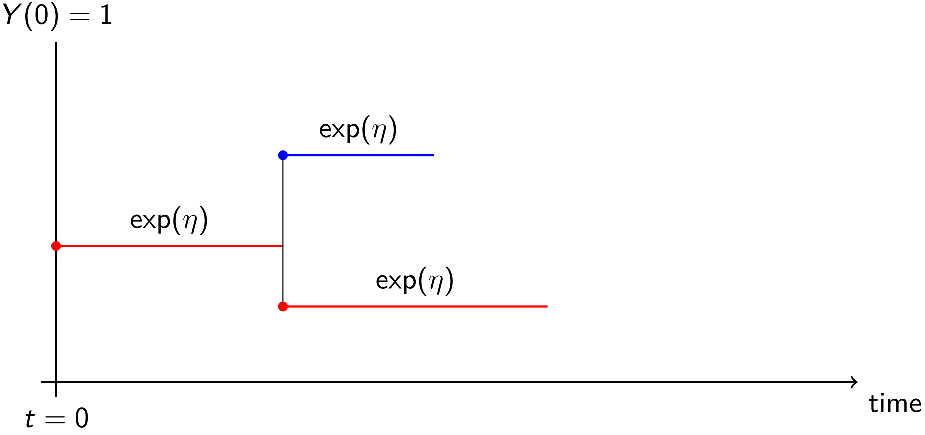


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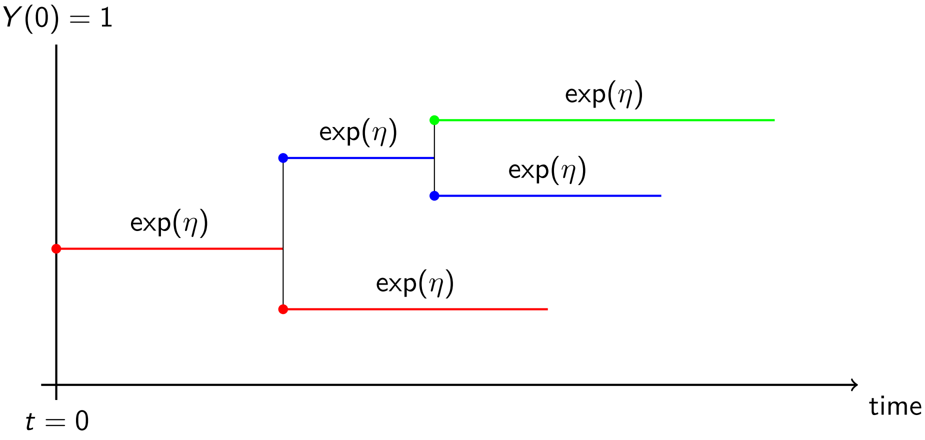


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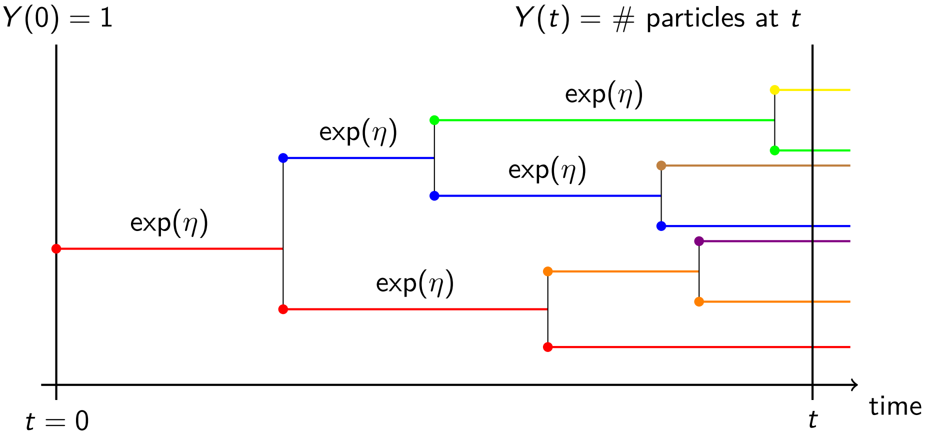


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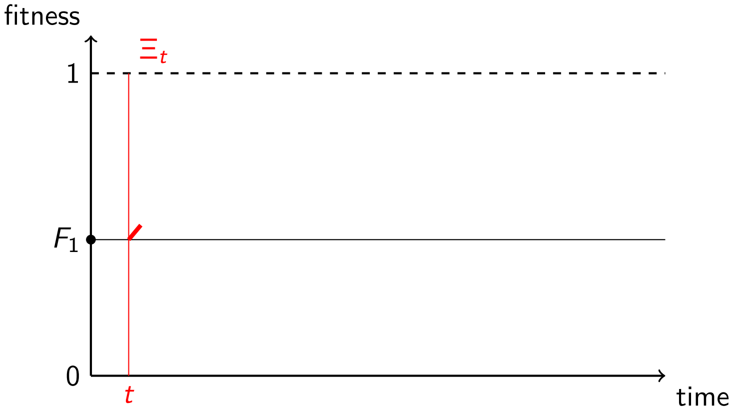


Empirical Fitness Distribution at time $t$: $\Xi_t = \frac{1}{N(t)} \sum_{n=1}^{M(t)} Z_n(t) \delta_{F_n}$.

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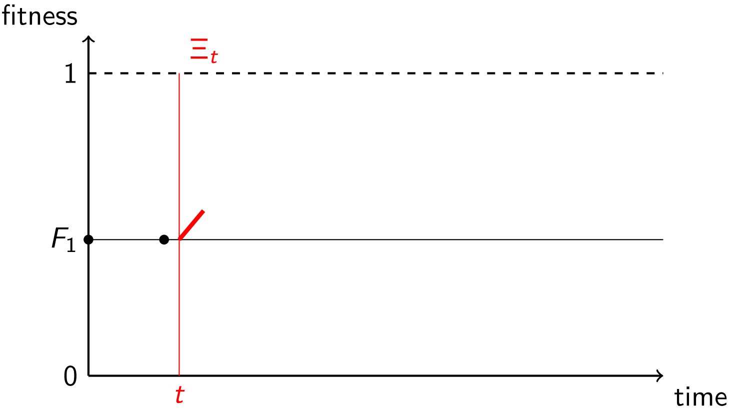


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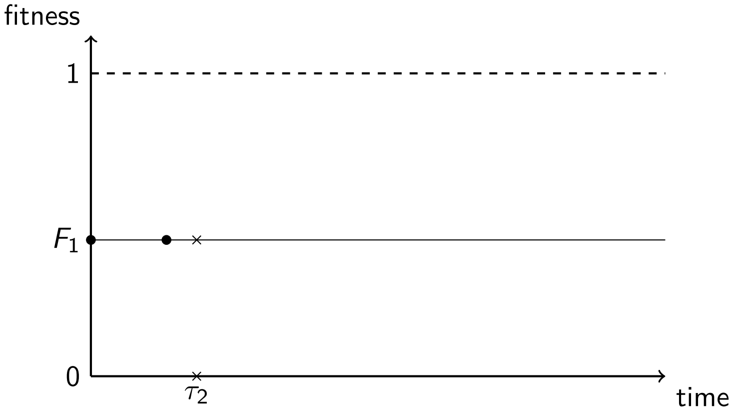


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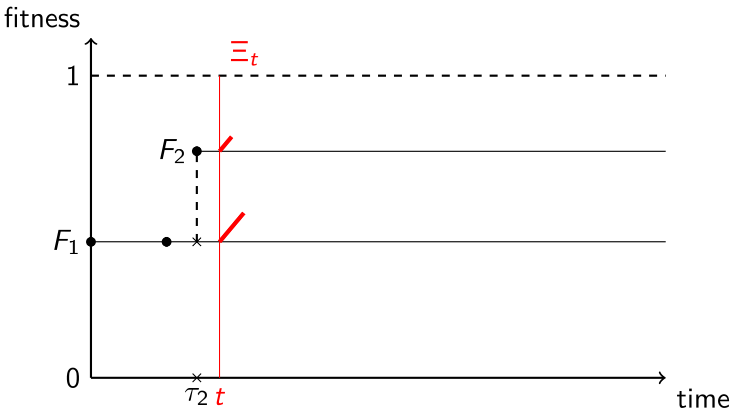


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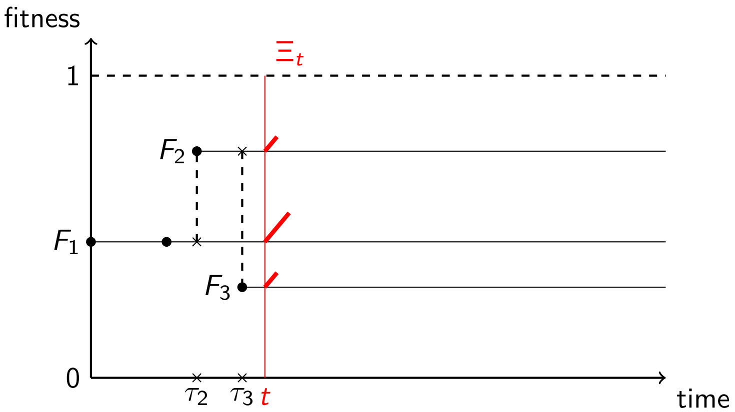


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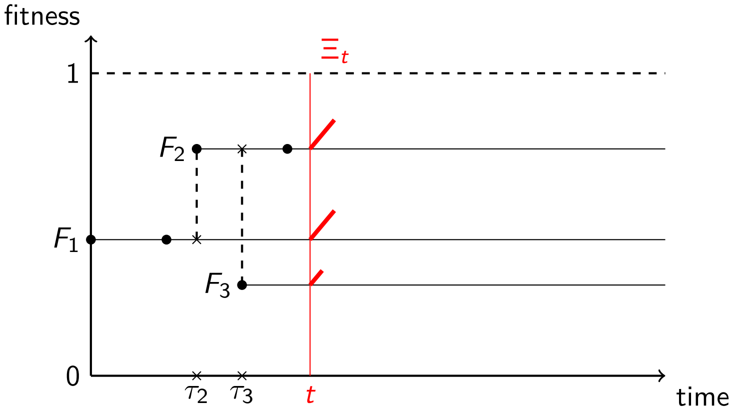


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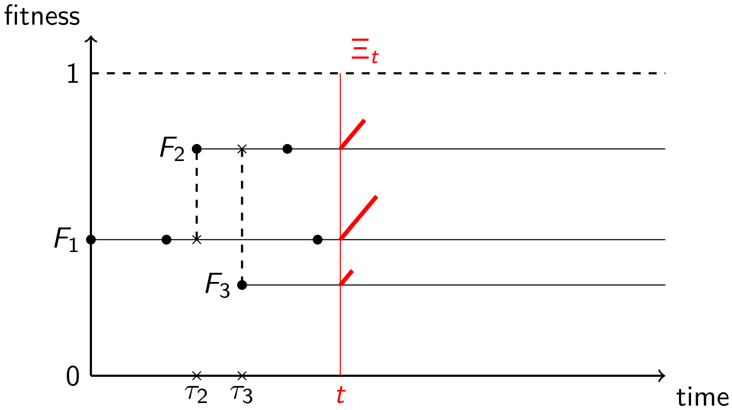


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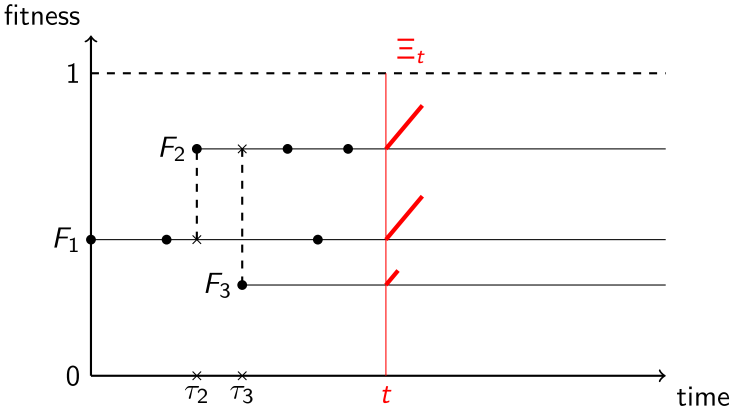


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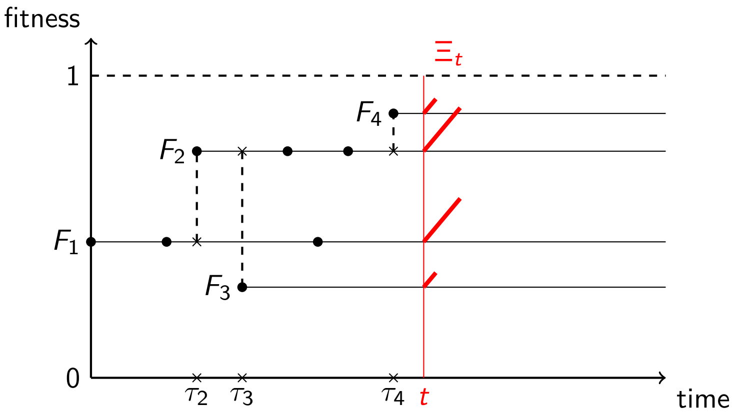


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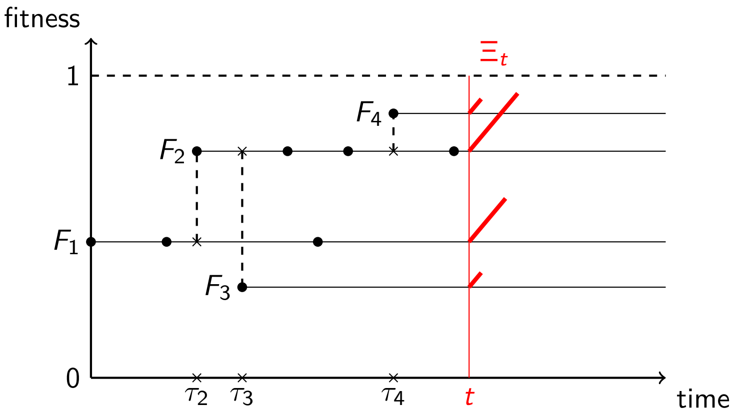


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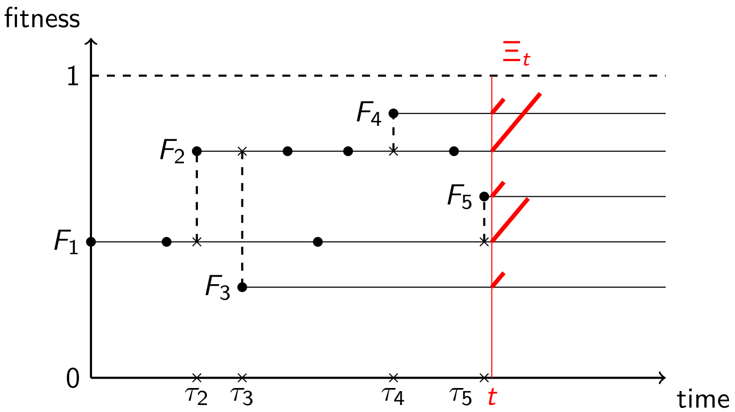


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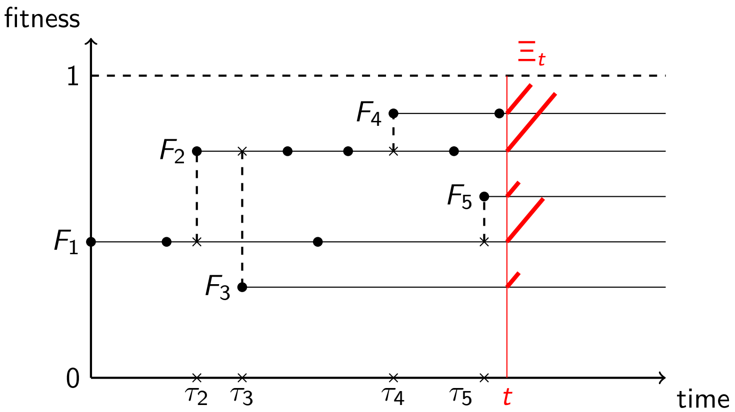


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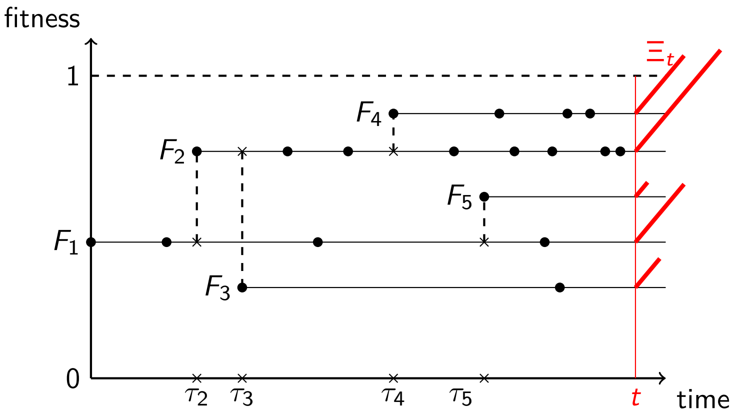


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# Population Growth: possible scenarios

Scenarios of growth of the system:

1. growth driven by *bulk behaviour*;
2. growth driven by *extremal behaviour* (condensation):
   - non-extensive occupancy;
   - macroscopic occupancy.

**Condition for condensation**

$$\frac{\beta}{\beta+\gamma}\int_0^1 \frac{d\mu(x)}{1-x} < 1. \qquad \text{(cond)}$$

**Definition of Macroscopic Occupancy**

$$\liminf_{n\to\infty} \frac{\text{max degree at time n}}{n} > 0.$$


Figure: $\Xi_{t=\infty}$, growth driven by *bulk behaviour.*


Figure: $\Xi_{t=\infty}$, growth driven by *extremal behaviour.*

# Condensation

## Condition for condensation

$$\frac{\beta}{\beta+\gamma}\int_0^1 \frac{d\mu(x)}{1-x} < 1. \tag{cond}$$

## Theorem

If (cond) fails then there exists $\lambda^* \in [\gamma, \beta+\gamma)$ such that $\frac{\beta}{\beta+\gamma}\int_0^1 \frac{\lambda^*}{\lambda^*-\gamma x} d\mu(x) = 1$, otherwise, we let $\lambda^* = \gamma$. In both cases:

- $\int_0^1 x d\Xi_t(x) \to \frac{\lambda^*}{\beta+\gamma}$ almost surely when $t \to \infty$;
- $\Xi_t \to \pi$ almost surely weakly when $t \to \infty$, where

1. if (cond) fails then $d\pi(x) = \frac{\beta}{\beta+\gamma}\frac{\lambda^*}{\lambda^*-\gamma x} d\mu(x)$;
2. if (cond) holds then $d\pi(x) = \frac{\beta}{\beta+\gamma}\frac{d\mu(x)}{1-x} + \varpi(\beta,\gamma)\delta_1$.

# An example without condesnation

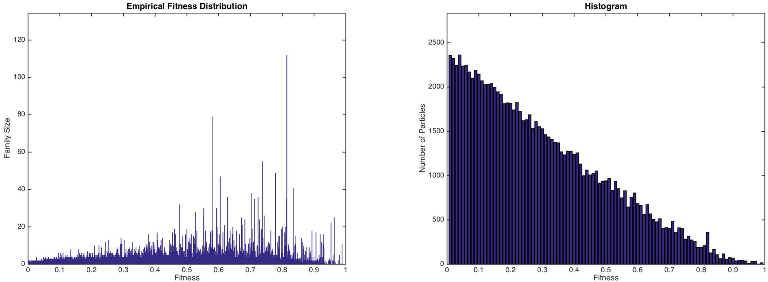



Figure: Empirical fitness distribution, for $\mu(x, 1) = (1 - x)^{\alpha+1}$, for $\alpha = 2$, $\beta = 0.8$, $\gamma = 0.2$.

# An example with condensation

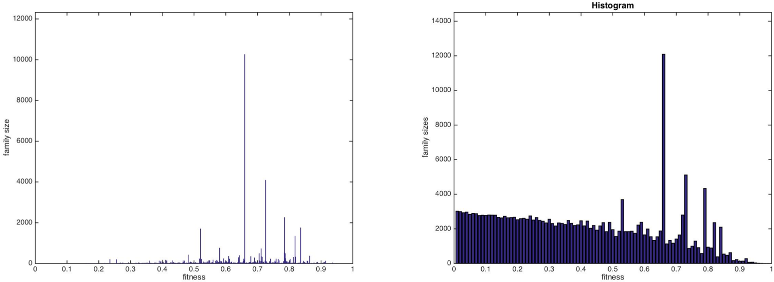
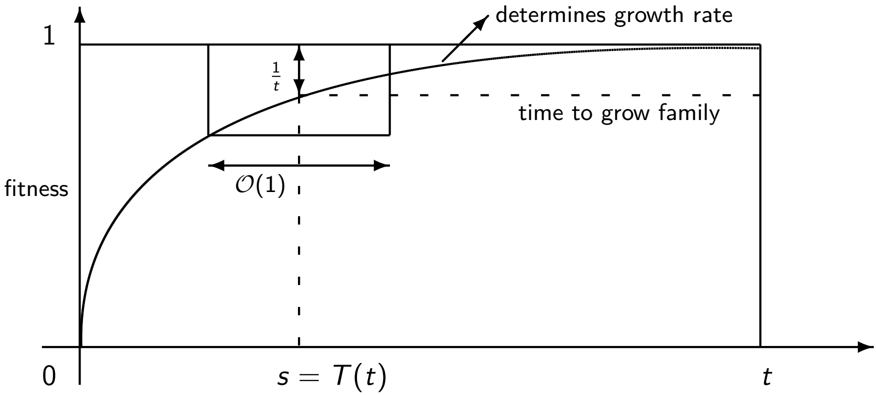



Figure: Empirical fitness distribution, for $\mu(x, 1) = (1 - x)^{\alpha+1}$, for $\alpha = 2$.

# Window for the emergence of the largest family at time $t$

We introduce $n(t) := \left\lceil \frac{1}{\mu(1-\frac{1}{t},1)} \right\rceil \approx t^\alpha$ and the random times

$$\begin{aligned} T(t) &= \inf\{s > 0 : M(s) \geq n(t)\} \approx \log t \\ &\approx \text{ first time when there exists a fitness at least } 1 - 1/t. \end{aligned}$$



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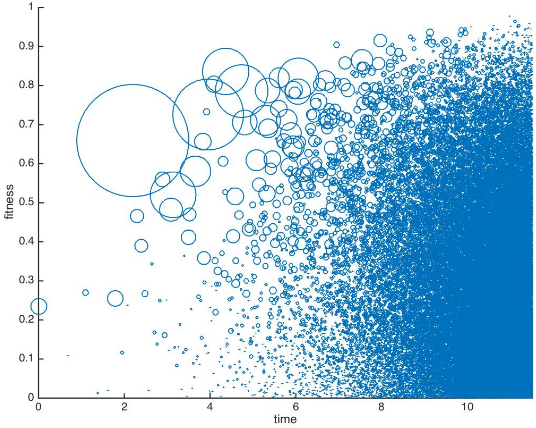


Figure: Time of introduction of nodes of different fitnesses, with a relative degree of a node indicated by the area of the bubble, for $\mu(x,1) = (1-x)^{\alpha+1}$, $\alpha = 2$.

# Results for a Class of Regularly Varying Functions

## Regular variation assumption on $\mu$

$$\frac{\mu(1-x\varepsilon,1)}{\mu(1-\varepsilon,1)} \to x^{\alpha}, \quad \alpha > 1,\ \forall x > 0 \text{ as } \varepsilon \downarrow 0.$$

## Theorem [2]

- Size $S(t)$ of the largest family: $e^{-\lambda^*(t-T(t))}S(t) \Rightarrow \Gamma(\lambda^*,\alpha)$.
- Fitness $V(t)$ of the largest family: $t(1-V(t)) \Rightarrow W$ (explicit).
- Time of birth $\Theta(t)$ of the largest family: $\Theta(t) - T(t) \Rightarrow Z$.

## The winner does not take it all [2]

In probability when $t \to \infty$, $\frac{S(t)}{N(t)} = \frac{\max_{n\in\{1\ldots M(t)\}} Z_n(t)}{N(t)} \to 0$.

# Open Problems

- Precise growth of the system: $\log N(t) = \lambda^* t + o(t)$;
- More general branching, and Bianconi and Barabasi networks;
- Different classes of fitness distributions: whether there exist bounded fitness distributions where we experience condensation by macroscopic occupancy.

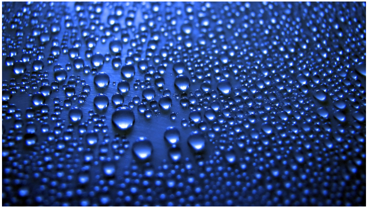

Figure: Not this condensation.

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[2] Dereich, Steffen and Mailler, Cécile, and Mörters, Peter. *Non-extensive condensation in reinforced branching processes*. arXiv:1601.08128 Preprint.

[3] Dereich, Steffen. Preferential attachment with fitness: Unfolding the condensate. *Electronic Journal of Probability*, Vol. 21, 2016.