Regularization of Inverse Problems

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What is an Inverse Problem?

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- physical model A, **cause** x and **effect** $A(x)$.

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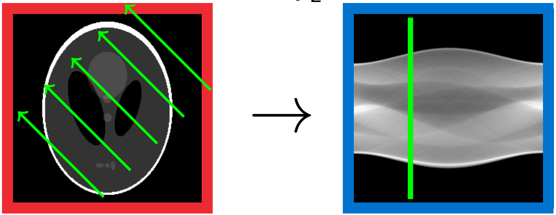
- Example: Positron Emission Tomography (PET)
  Model: X-ray Transformation

$$Ax : L \mapsto \int_L x(r)dr$$



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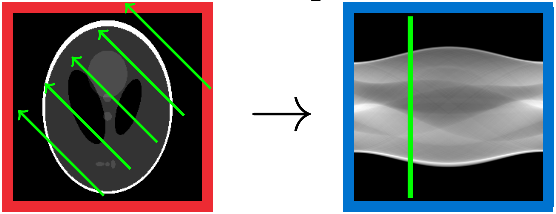

Inverse Problem: Given y, calculate x with $A(x) = y$.

Infer from the **effect** the **cause**.

What is the problem with Inverse Problems?

Examples

- 1. board

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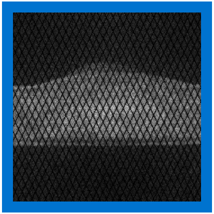
  Data: PET scanner in London, model: $A{\color{red}x} : L \mapsto \int_L {\color{red}x}(r)dr$



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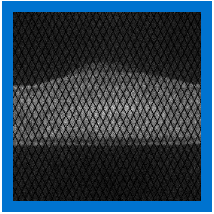

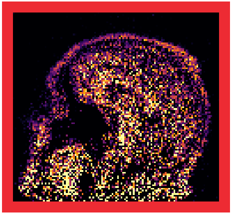

What is the problem with Inverse Problems?

Definition (Jacques Hadamard, 1902):
An Inverse Problem "$A(x) = y$" is called **well-posed**, if the solution

(1) **exists**.

(2) is **unique**.

(3) depends **continuously** on the data.
"Small errors in y lead to small erros in x."

Otherwise, we call it **ill-posed**.

Generalized Solutions

Definition: Let $y \in Y$. The set of all **approximate solutions** of "$A(x) = y$" is

$$L := \left\{ x \in X \,\middle|\, \|A(x) - y\| \leq \|A(z) - y\| \quad \forall z \in X \right\} .$$

If a solution $z \in X$ exists, $\|A(z) - y\| = 0$, then

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Definition: An approximate solution $\overline{x} \in L$ is called **minimal-norm-solution**, if

$$\|\overline{x}\| \leq \|x\| \quad \forall x \in L .$$

Properties of Minimal-Norm-Solutions

Assume: $A \in L(X, Y)$
Recall:

- Range of A: $R(A) := \{y \in Y \mid \exists x \in X : Ax = y\} \subset Y$
- Orthogonal complement: $U \subset Y$,

$$U^{\perp} := \{y \in Y \mid \langle y, u \rangle = 0 \quad \forall u \in U\}$$

- Minkowski sum: $U + V := \{u + v \mid u \in U, v \in V\}$

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- If the range $R(A)$ is closed, then $R(A) + R(A)^\perp = Y$, otherwise $R(A) + R(A)^\perp \subsetneq Y$.
  Example: $A : \ell^2 \to \ell^2, (Ax)_j = \frac{x_j}{j}$. Range **not** closed.

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Theorem: If $R(A)$ is **not closed**, then $\overline{x}$ **does not depend continuously** on y. I.e. $A^\dagger$ is not continuous.

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Definition: A family $\{R_\alpha\}_{\alpha>0}$ is called **regularization** of $A^\dagger$, if for all

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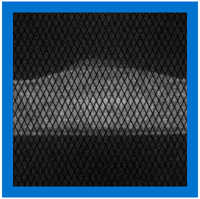

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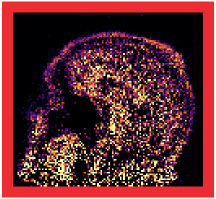

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