Deep Learning for Image Reconstruction

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Inverse Problems

$$y = \mathcal{T}(x_{\text{true}}) + \delta y.$$

$y \in Y$	Data
$x_{\text{true}} \in X$	Image
$\mathcal{T} : X \to Y$	Forward operator
$\delta y \in Y$	Noise

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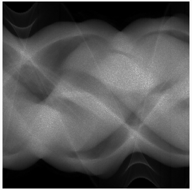

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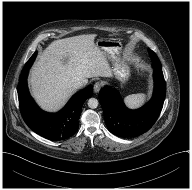

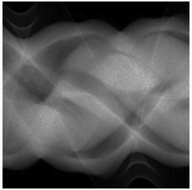

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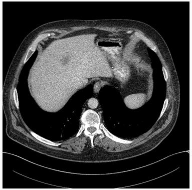

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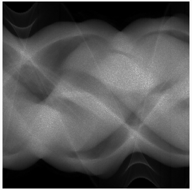

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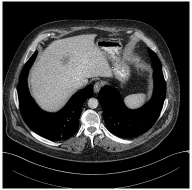

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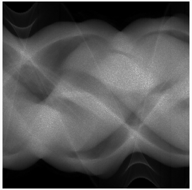

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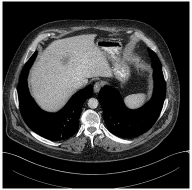

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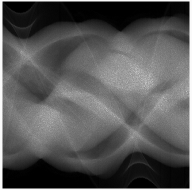

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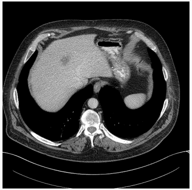

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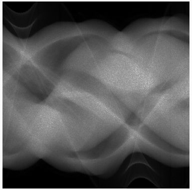

The problem is ill-posed: non-uniqueness, instability

Model-driven inversion

- Assume that we know $\mathrm{P}(x)$ and $\mathrm{P}(y|x)$ and use Bayes' law

$$\mathrm{P}(x|y) = \frac{\mathrm{P}(x)\mathrm{P}(y|x)}{\mathrm{P}(y)}$$

Maximum a-posteriori (MAP) reconstruction

$$\mathcal{T}^\dagger(y) = \arg\max_x \mathrm{P}(x|y) = \arg\min_x \left[\log \mathrm{P}(y|x) + \log \mathrm{P}(x)\right]$$

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- Major complications:
  - How do we pick $P(x)$?
  - How do we solve minimization?

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Standard approach: Gibbs-style priors $p(x) = e^{-R(x)}$

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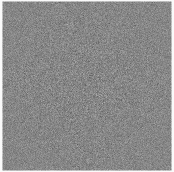
$\|x\|_2^2$



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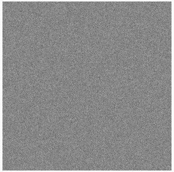

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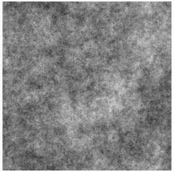

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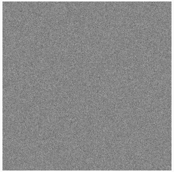

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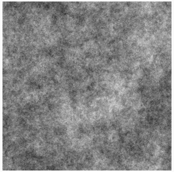

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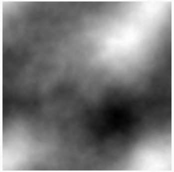

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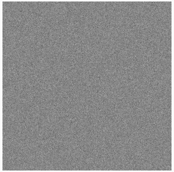

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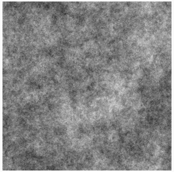

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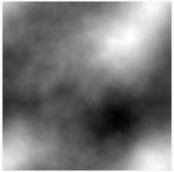

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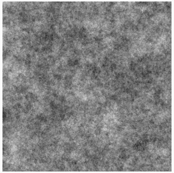

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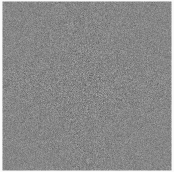

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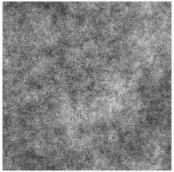

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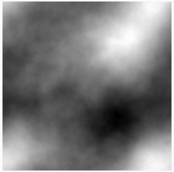

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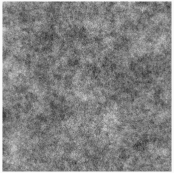
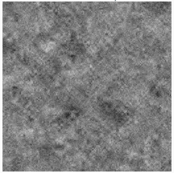


$\|x\|_{B^1_{1,1}}$



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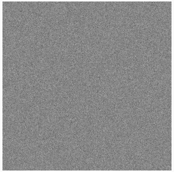

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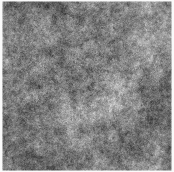

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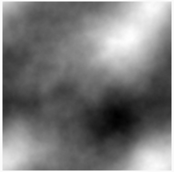

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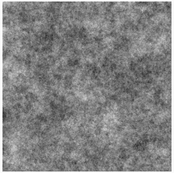

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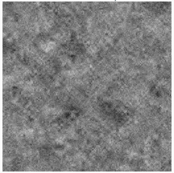

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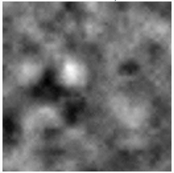

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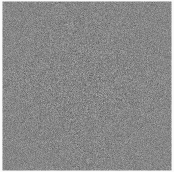

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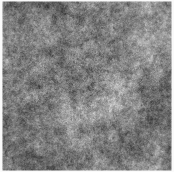

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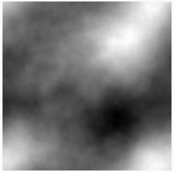

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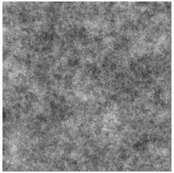

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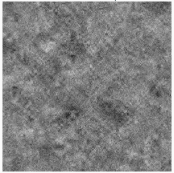

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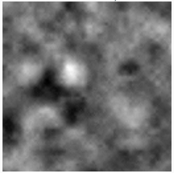
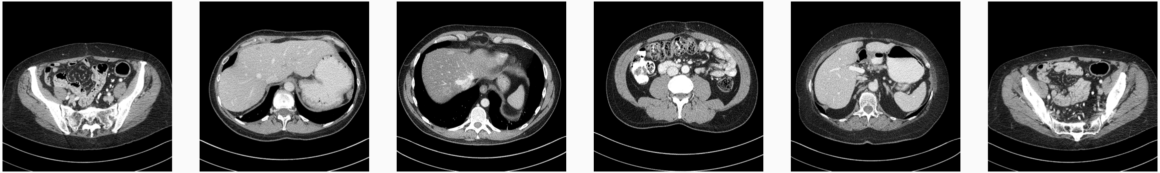


Actual humans:



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- Most successful approaches rely on dictionary learning, but we still need to solve an optimization problem to find the MAP estimator.

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- Parametrized by θ which we learn
- Selected by optimization of a *loss* function $\mathrm{L}(\theta)$

$$\theta^* = \arg\min_\theta \mathrm{L}(\theta)$$

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$$\mathbb{E}(x \mid y)$$

- Characterization:

$$\mathbb{E}(x \mid \cdot\,) = \operatorname*{arg\,min}_{h\colon Y \to X} \mathbb{E}\left[\left\| h(y) - x \right\|_X^2\right].$$

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- To approximate the conditional expectation, we pick

$$\mathrm{L}(\theta) = \mathbb{E}\left[\left\| \mathcal{T}_\theta^\dagger(\mathrm{y}) - \mathrm{x} \right\|_X^2\right].$$

which gives $\mathcal{T}_\theta^\dagger(y) \approx \mathbb{E}(\mathrm{x} \mid y)$

Learned inversion methods

Architecture: Specification of the class of operators $\{\mathcal{T}_\theta^\dagger\}_{\theta \in \Theta}$.

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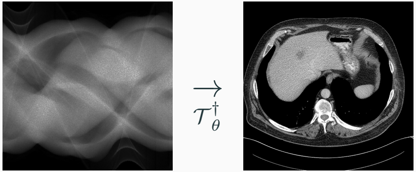

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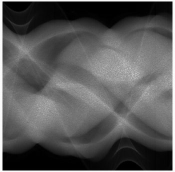
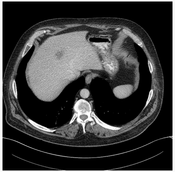


$$\xrightarrow[\mathcal{T}_\theta^\dagger]{}$$



- Fully learned
- Learned post-processing
- Learned iterative schemes

Fully learned reconstruction

Goal: Learn "the whole" mapping from data to signal

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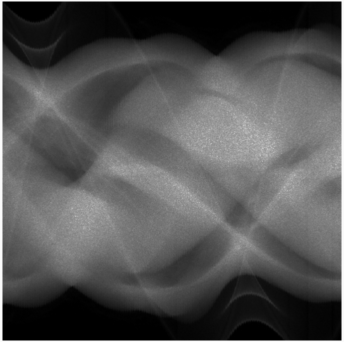

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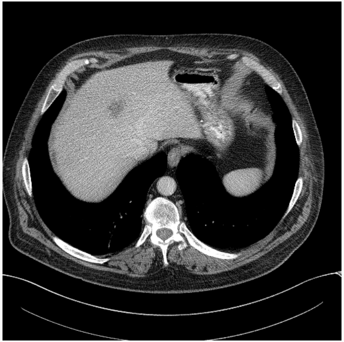

Fully learned reconstruction

Several works:

- *Tomographic image reconstruction using artificial neural networks.*
  Paschalis et. al. Nucl Instrum Methods Phys Res A 2004
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Problem: $\mathcal{T}$ typically has symmetries, but the network has to learn them.
Example: 3D CBCT, data: 10^8 pixels and 10^8 voxels $\implies$ 10^{16} connections!

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Learned post-processing

Use deep learning to improve the result of another reconstruction

$$\mathcal{T}_\theta^\dagger = \Lambda_\theta \circ \mathcal{T}^\dagger$$

where $\mathcal{T}^\dagger$ is some reconstruction (FBP, TV, ...) and Λ_θ is a learned post-processing.

Learned post-processing

Allows *separation of inversion and learning*, data can be seen as $(\underbrace{\mathcal{T}^\dagger(\mathrm{y})}_{\in X}, \underbrace{\mathrm{x}}_{\in X})$.

The problem becomes an image processing problem $\implies$ easy to solve.

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Won AAPM Low-Dose CT Grand Challenge:

A deep convolutional neural network using directional wavelets for low-dose X-ray CT reconstruction
Kang et. al. 2016

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  How to include data in each iteration?
- Inspiration from iterative optimization methods

$$x^* = \arg\min_x \frac{1}{2}||\mathcal{T}(x) - y||_Y^2$$

Algorithm 1 Generic iterative optimization algorithm

1: **for** $i = 1, \dots$ **do**
2: $\quad x_{i+1} \leftarrow \text{Update}(x_i)$

Gradient descent:

$$\text{Update}(x_i) = f_i - \alpha \nabla f(x_i)$$

Learned iterative reconstruction

- With $f(x) = -\log \mathrm{P}(y \mid x)$ (maximum likelihood)

$$\mathrm{Update}\big(x_i\big) = f_i + \alpha \nabla \log \mathrm{P}(y \mid x_i)$$

Learned iterative reconstruction

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$$\mathrm{Update}(x_i) = f_i + \alpha \nabla \log \mathrm{P}(y \mid x_i)$$

- With prior $f(x) = -\log \mathrm{P}(x \mid y)$ (maximum a-posteriori), using Bayes:

$$\mathrm{Update}(x_i) = f_i + \alpha \left(\nabla \log \mathrm{P}(y \mid x_i) + \nabla \log \mathrm{P}(x_i) \right)$$

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- But $P(x)$ is unknown! Learn its "gradient" $\Lambda_\theta \approx \nabla \log P(x)$:

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- Learn everything except gradient of data likelihood:

$$\text{Update}\big(x_i\big) = \Lambda_\theta\big(x_i, \nabla \log \mathrm{P}(y \mid x_i)\big)$$

Learned gradient descent

- Set a stopping criteria (fixed number of steps, I)

Learned gradient descent

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- Pick a noise model (here, Gaussian noise)

$$-\nabla \log \mathrm{P}(y \mid x) = \mathcal{T}^*(\mathcal{T}(x) - y)$$

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Algorithm 2 Learned gradient descent

1: **for** $i = 1, \ldots, I$ **do**

2: $\quad x_{i+1} \leftarrow \Lambda_\theta\big(x_i, \mathcal{T}^*(\mathcal{T}(x_i) - y)\big)$

3: $\mathcal{T}_\theta^\dagger(g) \leftarrow x_I$

Learned gradient descent

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We separate problem dependent (and possibly global) components into $\mathcal{T}^*(\mathcal{T}(x_i) - y)$, and prior dependent (local) components into Λ_θ!

References

- *ADMM-Net: A Deep Learning Approach for Compressive Sensing MRI*
  Yang et. al. NIPS 2016
- *Recurrent inference machines for solving inverse problems*
  Putzky and Welling, arXiv 2017
- *Solving ill-posed inverse problems using iterative deep neural networks*
  A and Öktem, Inverse Problems 2017
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Results

Results for CT with *Human data*

- Inverse problem:

$$y = \mathcal{P}(x) + \delta y$$

- Geometry: fan beam 1000 angles
- Noise: Poisson noise (low dose CT)
- Training data: 2000 512×512 pixel slices

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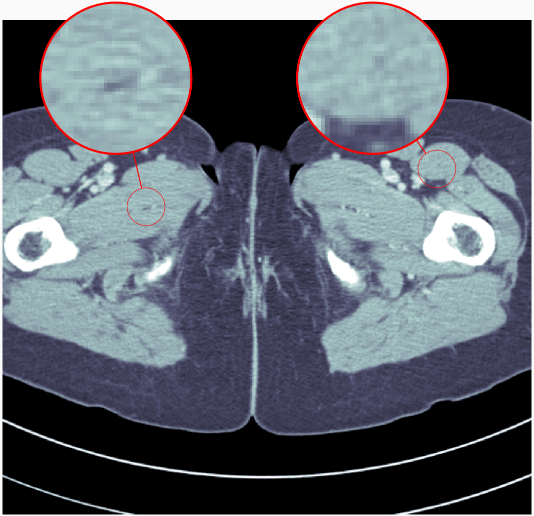
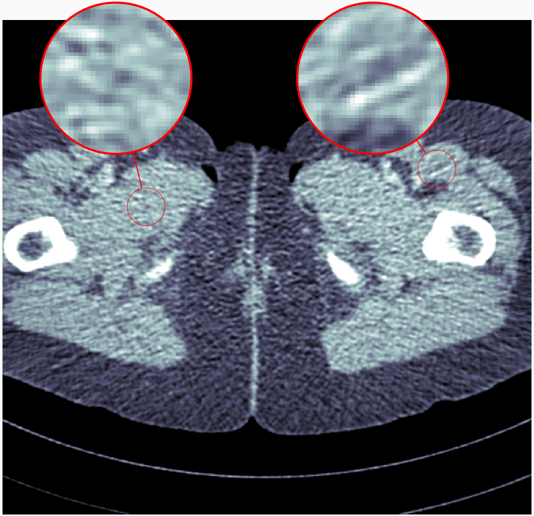
Compare to:

- Analytic Pseudo-Inverse (FBP)
- Variational methods (TV-regularization)
- Post-processing deep learning by U-Net

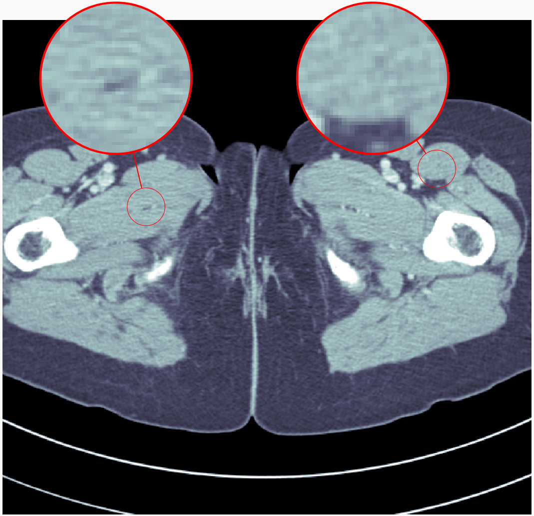


Phantom



FBP

PSNR 33.65 dB, SSIM 0.830, 423 ms



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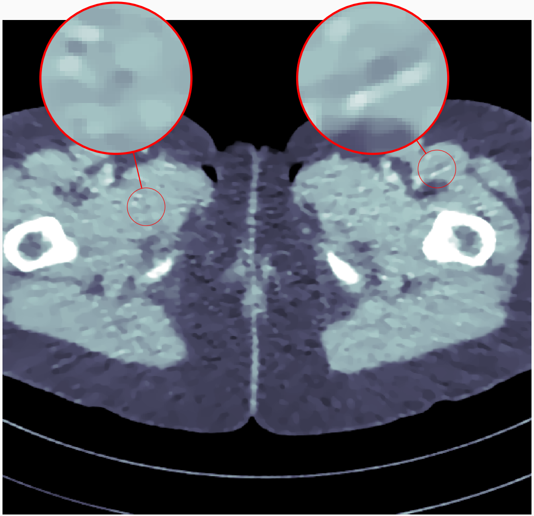
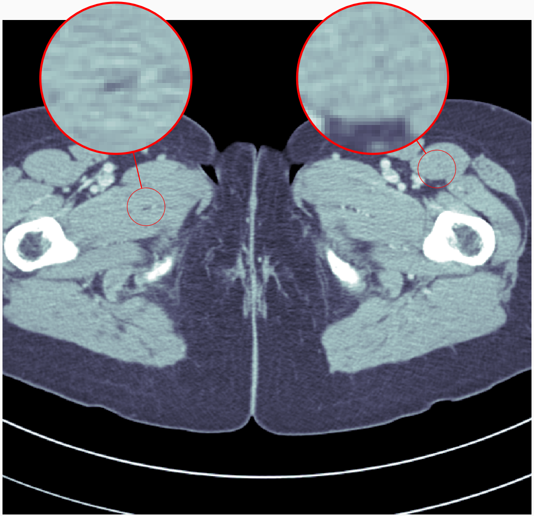
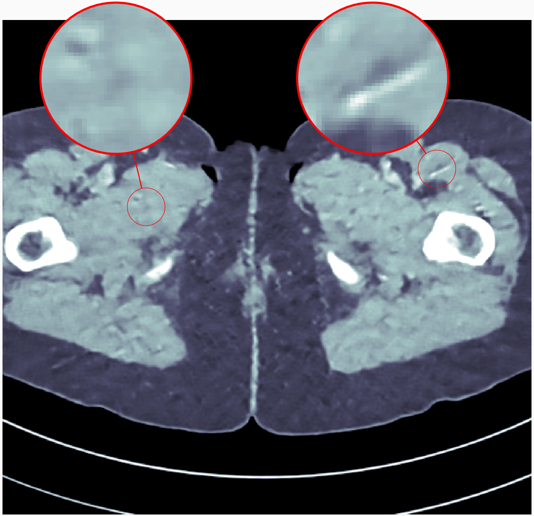


TV

PSNR 37.48 dB, SSIM 0.946, 64 371 ms

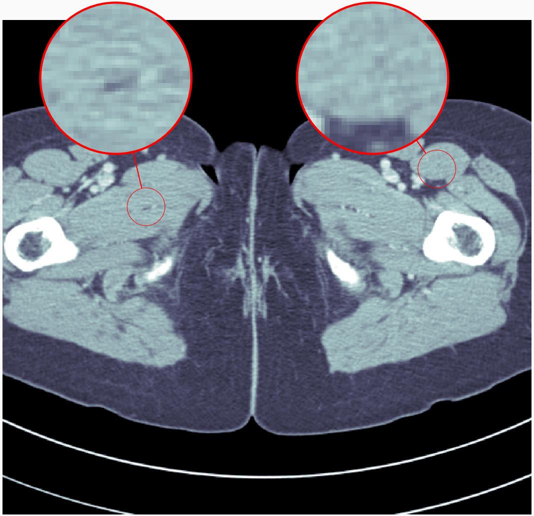


Phantom

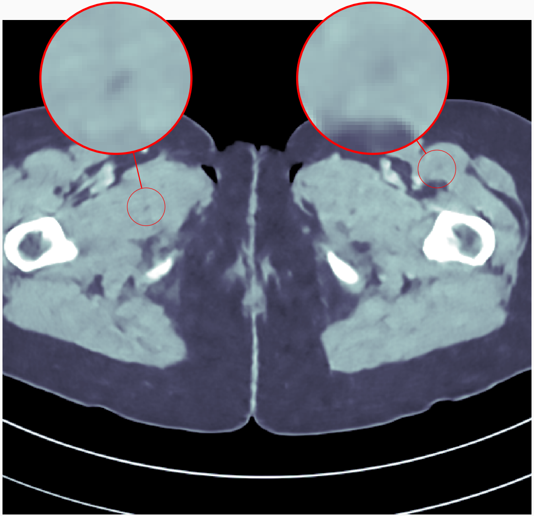


Learned post-processing

PSNR 41.92 dB, SSIM 0.941, 463 ms



Phantom



Learned Iterative

PSNR 44.11 dB, SSIM 0.969, 620 ms

Comments

- Very large quantitative improvement

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- Noticeable visual improvement

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- Noticeable visual improvement
- Very short run-times

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- Noticeable visual improvement
- Very short run-times
- Looks oversmoothed

Oversmoothing

- Optimal reconstruction operator given by conditional expectation

$$\mathcal{T}_\theta^\dagger(y) \approx \mathbb{E}\left(\mathsf{x} \mid y\right) = \int x \mathrm{dP}(x \mid y)$$

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- This is a pointwise-average
- Small scale variations (texture, edges) are lost
- Is there some better estimator? That depends on what you want.
- The only truly general answer is the whole posterior, $\mathrm{P}(x \mid y)$.

Deep Bayesian Inversion

- Model: Assume that the reconstruction $\mathcal{T}_\theta^\dagger(y)$ is a random variable.

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- Loss: Define the best reconstruction to be as close to the posterior as possible

$$\theta^* \in \arg\inf_{\theta \in \Theta} \mathbb{E}_{y \sim \mathrm{y}}\Big[d\big(\mathcal{T}_\theta^\dagger(y), (\mathrm{x} \mid \mathrm{y} = y)\big)\Big].$$

In the above, d is some distance function, measuring the distance between the random variables $\mathcal{T}_\theta^\dagger(y)$ and $(\mathrm{x} \mid \mathrm{y} = y)$.

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- Possible options: Kullback-Leibler, Jensen-Shannon, etc.
- Those are not (a.e.) differentiable and finite! Prefer Wasserstein distance:

$$\mathcal{W}(p, q) := \inf_{\mu \in \Pi(p,q)} \mathbb{E}_{(\mathrm{x},\mathrm{x}') \sim \mu} \big[\|\mathrm{x} - \mathrm{x}'\|_X \big]$$

where the minimization is taken over all probability distributions on $X \times X$.

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- Optimal reconstruction given by:

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- Problems:
  - But we have barely any idea about how $(\mathrm{x} \mid \mathrm{y} = y)$ looks! We have only some samples (x_i, y_i).
  - How can we compute the Wasserstein distance?

Deep Posterior Sampling

- Kantorovich-Rubinstein dual characterisation of the Wasserstein distance:

$$\mathcal{W}\big(\mathcal{T}_\theta^\dagger(y), (\mathsf{x} \mid \mathsf{y} = y)\big) = \sup_{\substack{\mathsf{D}_y \colon X \to \mathbb{R} \\ \mathsf{D}_y \in \mathrm{Lip}(1)}} \mathbb{E}_{x \sim (\mathsf{x} \mid \mathsf{y} = y),\, x' \sim \mathcal{T}_\theta^\dagger(y)} \Big[\mathsf{D}_y(x) - \mathsf{D}_y(x')\Big]$$

where the *discriminator* D_y has Lipschitz constant ≤ 1.

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- The parameters can thus be written as

$$\theta^* \in \operatorname*{arg\,inf}_{\theta \in \Theta} \mathbb{E}_{y \sim \mathsf{y}} \left[\sup_{\substack{\mathrm{D}_y \colon X \to \mathbb{R} \\ \mathrm{D}_y \in \mathrm{Lip}(1)}} \mathbb{E}_{x \sim (\mathsf{x} \mid \mathsf{y}=y), x' \sim \mathcal{T}_\theta^\dagger(y)} \Big[\mathrm{D}_y(x) - \mathrm{D}_y(x')\Big] \right].$$

Deep Posterior Sampling

- Using monotonicity, we can let $\mathrm{D}_y = D(\,\cdot\,, y)$ where $\mathrm{D}\colon X \times Y \to \mathbb{R}$ and reorder

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- Replace expectation by empirical mean
- Assume that reconstruction is of the form $\mathcal{T}_\theta^\dagger(y) \sim \mathrm{G}(y, z)$ where $z \sim \mathcal{N}(0, I)$.

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- Replace expectation by empirical mean
- Assume that reconstruction is of the form $\mathcal{T}_\theta^\dagger(y) \sim \mathrm{G}(y, z)$ where $z \sim \mathcal{N}(0, I)$.
- Use deep convolutional neural network to model the discriminator.

Deep Posterior Sampling

Summary:

- Reconstructing a single point estimate (e.g. mean) does not tell the whole story
- We want the *whole* posterior
- Reconstruct a random variable
- Minimize the (empirical) Wasserstein distance using duality

$$\theta^* \in \operatorname*{arg\,inf}_{\theta\in\Theta} \sup_{\substack{\mathrm{D}\colon X\times Y\to\mathbb{R} \\ \mathrm{D}(\,\cdot\,,y)\in\mathrm{Lip}(1)}} \mathbb{E}_{y\sim \mathsf{y}}\left[\mathbb{E}_{x\sim(\mathsf{x}\mid\mathsf{y}=y),\, x'\sim\mathcal{T}_\theta^\dagger(y)}\Big[\mathrm{D}(x,y)-\mathrm{D}(x',y)\Big]\right].$$

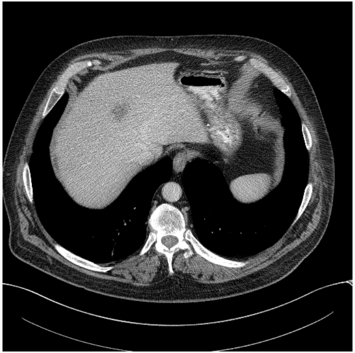
Results

Results for CT with *Human* abdomen scans

- Machine: Siemens SOMATOM Definition AS+
- Geometry: 3D Helical scan
- Noise: Ultra-low dose CT (2% of normal dose)
- Training data from 9 patients.

Samples

Phantom



FBP

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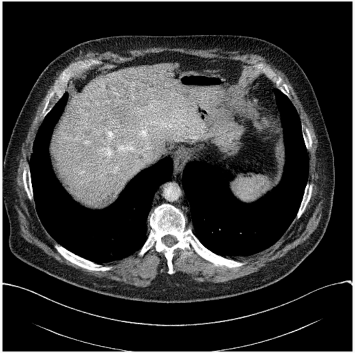
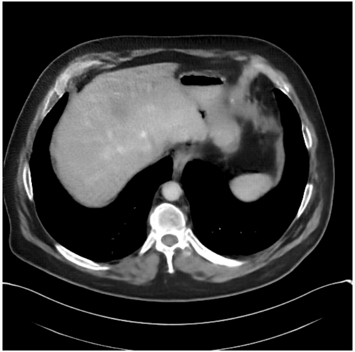
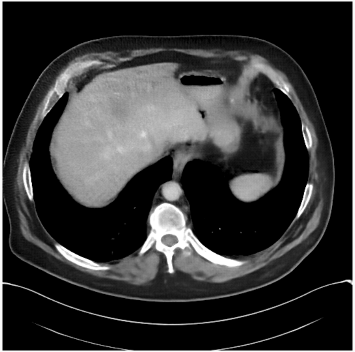


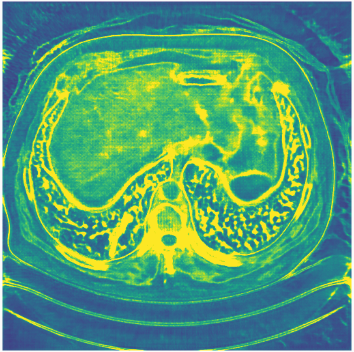
Statistics

Mean



Statistics





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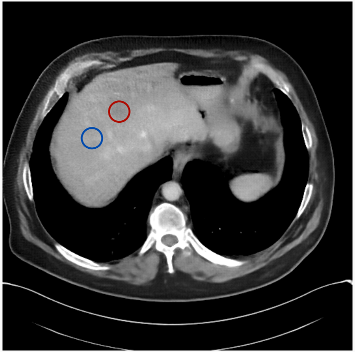
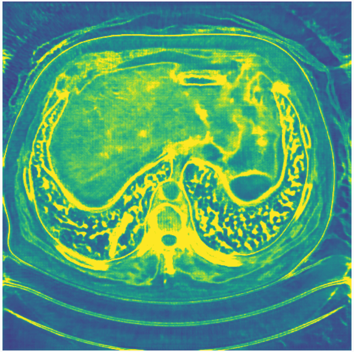


Std



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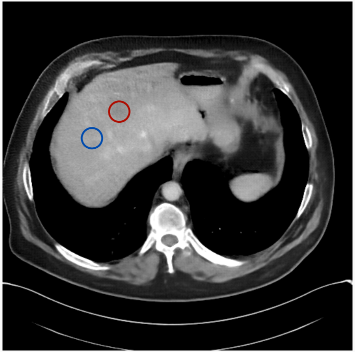

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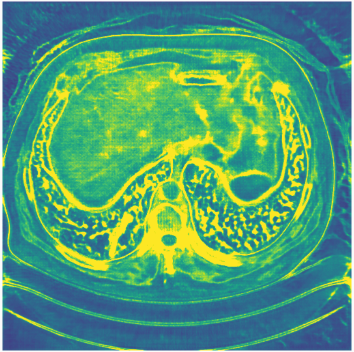

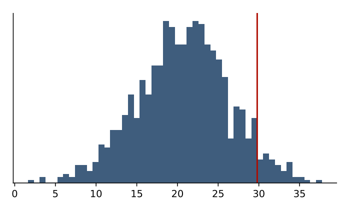

References

Deep Bayesian Inversion
A and Öktem, arXiv 2018

Conclusions

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