Regularization of Inverse Problems

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What is an Inverse Problem?

- $A : \mathcal{U} \to \mathcal{V}$ mapping between Hilbert spaces $\mathcal{U}, \mathcal{V}$, $A \in L(\mathcal{U}, \mathcal{V})$
- physical model A, **cause** u and **effect** $A(u) = Au$.

Direct / Forward Problem: given u, calculate Au.

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- Example 1: ray transform (used in CT, PET, ...)

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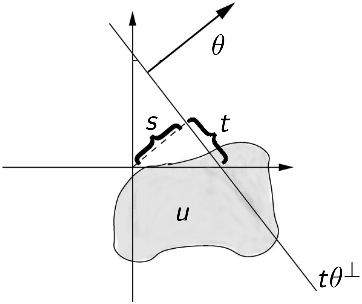

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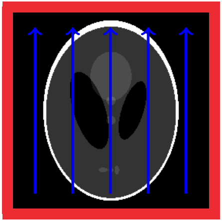

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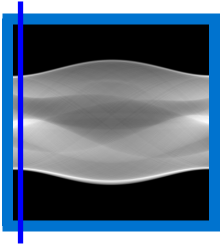

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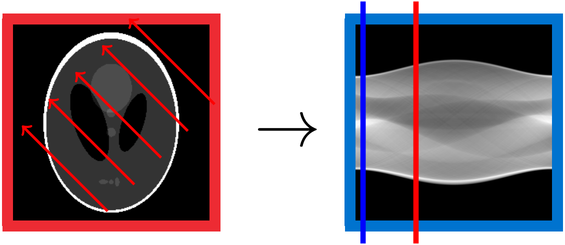

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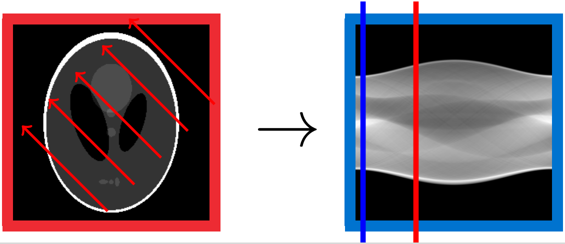

Inverse Problem: Given v, calculate u with $Au = v$.

Infer from the **effect** the **cause**.

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Examples

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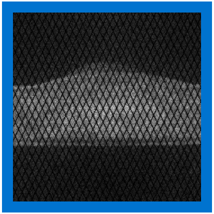
- **not exist**. $Au = 0, v \neq 0$
- **not be unique**. $Au = 0, v = 0$
- **be sensitive to noise**.
  - Positron Emission Tomography (PET)
  - Data: PET scanner in London
  - Model: ray transform, $Au(L) = \int_L u(r)dr$
  - Find u such that $Au = v$



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Examples

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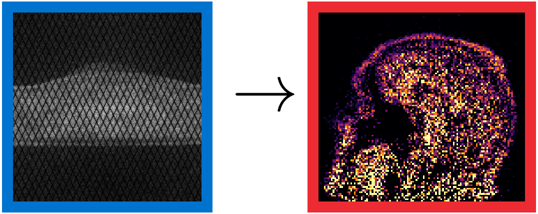

What is the problem with Inverse Problems?

Definition (Jacques Hadamard, 1865-1963):
An Inverse Problem "$Au = v$" is called **well-posed**, if the solution

(1) **exists**.

(2) is **unique**.

(3) depends **continuously** on the data.
"Small errors in v lead to small errors in u."

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Almost all interesting inverse problems are ill-posed.

Generalized Solutions

Definition: Let $v \in \mathcal{V}$. The set of all **approximate solutions** of "$Au = v$" is

$$\mathcal{L} := \left\{ u \in \mathcal{U} \,\middle|\, \|Au - v\| \leq \|Az - v\| \quad \forall z \in \mathcal{U} \right\} .$$

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Definition: An approximate solution $\overline{u} \in \mathcal{L}$ is called **minimal-norm-solution**, if

$$\|\overline{u}\| \leq \|u\| \quad \forall u \in \mathcal{L} .$$

Properties of Minimal-Norm-Solutions

Recall:

- Range / image of A: $\mathcal{R}_A := \{v \in \mathcal{V} \mid \exists u \in \mathcal{U} \; Au = v\}$
- Orthogonal complement: $\mathcal{A}^\perp := \{v \in \mathcal{V} \mid \langle v, z \rangle = 0 \; \forall z \in \mathcal{A}\}$
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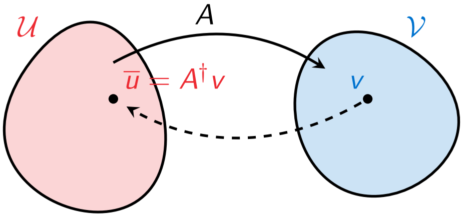
Theorem: If $\mathcal{R}_A$ is **not closed**, then $\overline{u}$ **does not depend continuously** on v, i.e. $A^\dagger$ is not continuous.

Regularization



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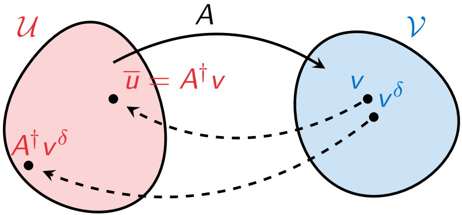

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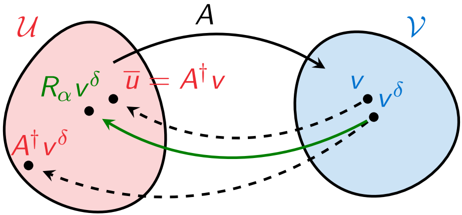

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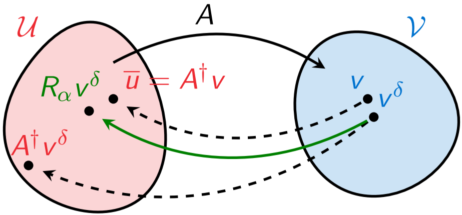

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- for all $\alpha > 0$ the mapping $R_\alpha : \mathcal{V} \to \mathcal{U}$ is continuous.
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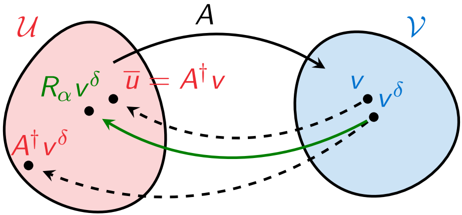

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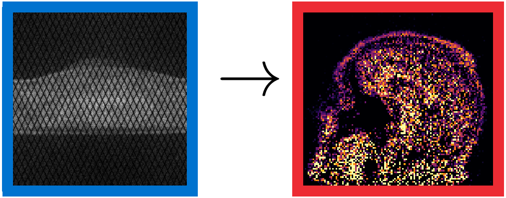

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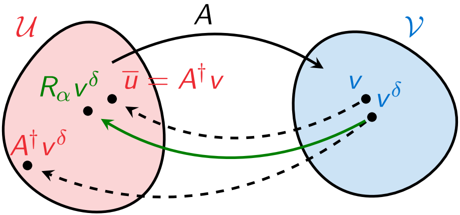

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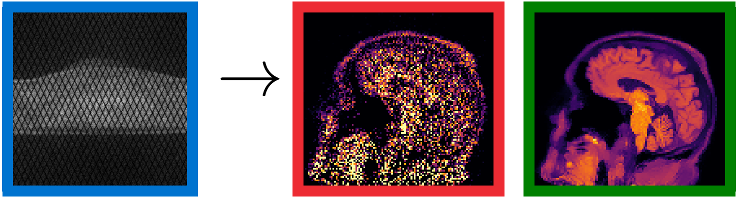

Popular examples of regularization

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(Andrey Tikhonov, 1906-1993)

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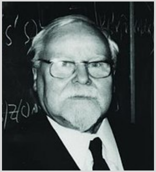

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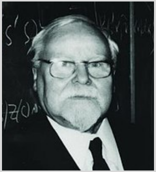

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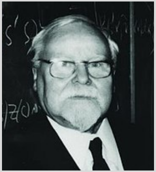

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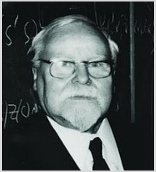

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$$R_\alpha v^\delta = \arg\min_u \left\{ D(Au, v^\delta) + \alpha J(u) \right\}$$

- **data fit** D: "divergence" $D(x, y) \geq 0, D(x, y) = 0$ iff $x = y$
  Examples: $D(x, y) = \|x - y\|^2$, $\|x - y\|_1$, $\int x - y + y \log(y/x)$
- **regularizer** J: Penalizes unwanted features; ensures stability
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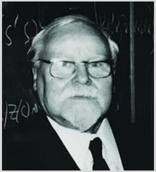

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- decouples solution of inverse problem into **2 steps**:
  1. **Modelling**: choose D, J, A, α.
  2. **Optimization**: connection to statistics, machine learning ...

Summary

- **Inverse problems**
  - forward / direct problem
  - ill-posedness; interesting inverse problems are ill-posed
  - generalized solutions, minimal-norm-solution

- **Reguarization**
  - stable approximation of minimal-norm-solution
  - Tikhonov regularization
  - variational regularization