Systems and control theory — a very brief introduction

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SAMBa ITT 8

11th June 2018

Systems & Control

- Control theory or Control engineering focusses on the design and synthesis of controllers (feedback, optimal or otherwise) in causal dynamical systems to achieve a desired outcome.

- Systems theory is the mathematical framework for (inter)connecting dynamical objects.

- Combined systems & control theory is *the* mathematical language for describing and abstracting feedback.

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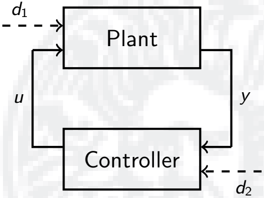

Connections to mathematical biology

1. Many biological systems are described by dynamical objects and are studied with a view to affecting a change in their behaviour

   Examples: SIR models. Crop–pest–bio-control dynamics.

   Control theory offers solutions for how to affect these changes

2. Many biological systems are themselves the arrangement, combination and interconnection of (sub)systems

   Examples: Human body — cells, tissues, organs. Ecosystems — organisation by species, trophic level, functional trait.

   Systems theory offers both descriptions and explanations of (complex) biological phenomena

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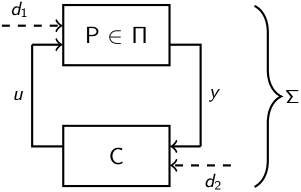
Competing objectives within control theory



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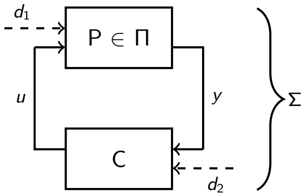

- Optimal control — choose C so that desired dynamic behaviour of Σ is achieved *and* a prescribed cost functional is minimised
- Robust control — P belongs to class Π (uncertainty set)
  - Robust stability if Σ stable for all $P \in \Pi$
  - Robust performance if performance objective satisfied for all $P \in \Pi$

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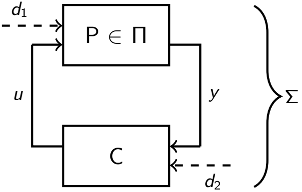

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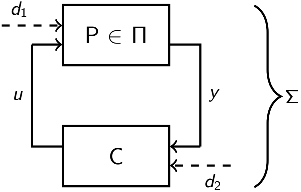

- At the current level of generality these problems are too broad — some assumptions are required
- Some are technical — such as the modelling framework — discrete- vs. continuous- time, finite- vs. infinite- dimensional, linear- vs. nonlinear- ...
- Some are more philosophical — such as a input-state-output or behavioral framework

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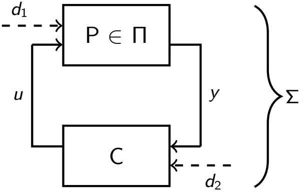

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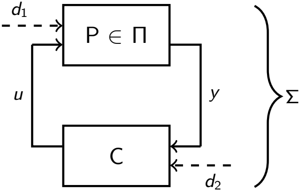

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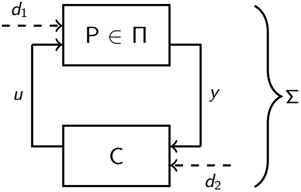

- In an input-state-output setting, inputs u are available (chosen by the modeller), which give rise to states x and outputs y. Outputs are typically known, states may not be. Causal relationship. Think applied force causes a velocity.

- In a behavioral setting, there are no inputs, states or outputs, just trajectories (typically the solutions of some ODE etc). Think currents and voltages.

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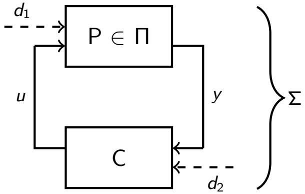

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Some fundamental concepts within systems theory

- Controllability — to what extent is it possible to reach a desired state? What states are reachable by control?
- Stabilisability — given an unstable equilibrium x_*, is it possible to choose a control such that $\lim_{t\to\infty} x(t) = x_*$?
- Observability — given a sequence of measurements y, is it possible to reconstruct the state x?
- Detectability — given a sequence of measurements y, is it possible to construct z such that $\lim_{t\to\infty}(z(t) - x(t)) = 0$?

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Consider a linear, discrete-time control system

$$x(t+1) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x^0\,, \quad t \in \mathbb{N}_0\,. \quad (1)$$

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- For given input u, the solution x of (1) is given by

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- With $x^0 = 0$ we may write

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- By the Cayley-Hamilton Theorem, the rank of

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- So if $\hat{x}$ reachable, it is reachable in at most n time-steps

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- For nonlinear systems, often a more bespoke approach is required

Control of diseases

Consider a simple SIR model for the transmission of an infectious disease

$$\left.\begin{aligned} S' &= -\beta S I + \sigma(1-S), & S(0) &= S^0 \\ I' &= \beta S I - \sigma I - \gamma I, & I(0) &= I^0 \\ R' &= \gamma I - \sigma R, & R(0) &= R^0 \end{aligned}\right\} \tag{2}$$

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- Here S, I and R denote the susceptible, infectious and removed populations, respectively.
- Further, σ is equal to the death (=birth) rate, β is the transmission rate and γ is the natural recovery rate. All are positive.
- Note that with $N = S + I + R$ we have $N' = -\sigma N + \sigma$, so that $\lim_{t\to\infty} N(t) = 1$.
- If $N(0) = 1$, with $0 \leq S(0), I(0), R(0) \leq 1$, then $N(t) = 1$ for all $t \geq 0$.

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- Here $R_0 = \beta/(\gamma+\sigma)$. If $R_0 < 1$, then $\xi_0 := (1,0,0)$ is the only equilibrium, which is globally exponentially stable. If $R_0 > 1$, then

$$\xi_* := \left(\frac{1}{R_0}, (R_0-1)\frac{\sigma}{\beta}, (R_0-1)\frac{\gamma}{\beta}\right),$$

is the so-called endemic equilibrium. It is stable and attracts all solutions which do not start at ξ_0.

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$$S' = -\beta S I + \sigma(1-S) - \mu S\,,$$
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- Note that these are (linear) feedback controls
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- Two competing objectives are optimality and robustness
- Key concepts in systems theory are: controllability, stabilisability, observability, detectability
- Linear and nonlinear (SIR model) examples discussed
- Please feel free to ask any questions over the week. Thank you for listening

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