Spectral analysis of Toeplitz matrices

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ITT7 (SAMBa Integrative Think Tank) Group 12

What is the problem?

Extended data fitting model:

$$\mathbf{y} = C\mathbf{a} + \mathbf{e} + \varepsilon, \quad \mathbf{e} \in N(\mathbf{0}, V_0), \quad \varepsilon \in N(\mathbf{0}, \sigma^2 I)$$

$$\mathbf{e} = L_0 d, \quad V_0 = L_0 L_0^T.$$

Eigenvalues give an idea of number of parameters.

Covariance matrices

We have a correlation function which depends only on the distance:

$$\rho(x, x') = \rho(|x - x'|).$$

We grid the domain of the function to obtain a correlation matrix:

$$T_{ij} = \rho(|x_i - x_j|).$$

If the grid is equidistant, the correlation matrix is a Toeplitz matrix e.g.

$$T = \begin{pmatrix} \rho(0) & \rho(h) & \rho(2h) \\ \rho(h) & \rho(0) & \rho(h) \\ \rho(2h) & \rho(h) & \rho(0) \end{pmatrix}.$$

Equidistant spacing

A Toeplitz matrix can be embedded in a circulant matrix

$$C = \begin{pmatrix} t_0 & t_1 & t_2 & t_1 & t_2 & t_1 \\ t_1 & t_0 & t_1 & t_2 & t_1 & t_2 \\ t_2 & t_1 & t_0 & t_1 & t_2 & t_1 \\ t_1 & t_2 & t_1 & t_0 & t_1 & t_2 \\ t_2 & t_1 & t_2 & t_1 & t_0 & t_1 \\ t_1 & t_2 & t_1 & t_2 & t_1 & t_0 \end{pmatrix}.$$

[Graham '17] conjectures an upper bound on the eigenvalues of this, giving an upper bound on the Toeplitz matrix

Non-equidistant spacing

Spacing of the grid - eigenvalues

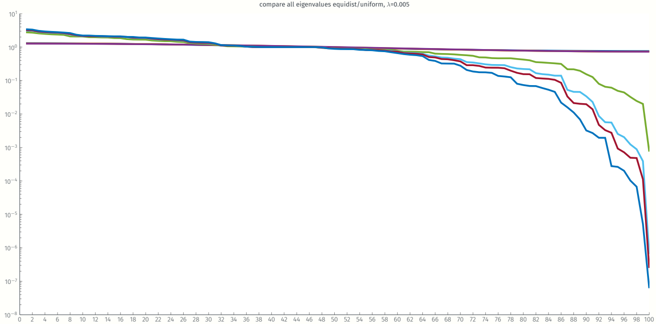

Figure 1: Different Matern covariances, not / equidistant spacing.

Eigenvectors

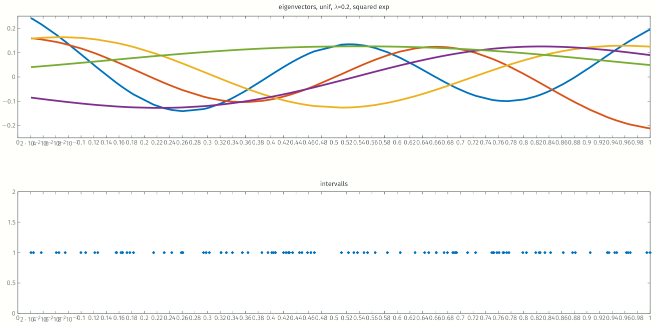

Figure 2: Exponential squared covariance, not / equidistant spacing, $\lambda = 0.2$.

$$T_{ij} = exp((i-j)^2/(2 \cdot \lambda^2)), n = 100$$

Eigenvectors

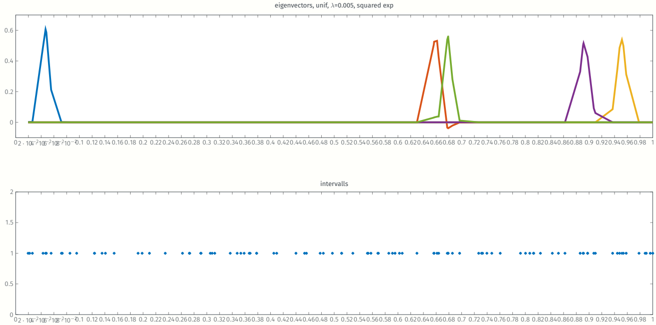

Figure 2: Exponential squared covariance, not / equidistant spacing, $\lambda = 0.005$.

$$T_{ij} = exp((i - j)^2/(2 \cdot \lambda^2)), n = 100$$

How can we approximate T?

Circulant matrices

Approximate a Toeplitz matrix with a circulant matrix

$$\begin{pmatrix} t_0 & t_1 + t_2 & t_2 + t_1 \\ t_1 + t_2 & t_0 & t_1 + t_2 \\ t_2 + t_1 & t_1 + t_2 & t_0 \end{pmatrix}$$

Compute a truncated SVD of the circulant matrix to approximate T.

Did it work?

Short answer: no.

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Short answer: no.

Taking $T_{ij} = exp((i-j)^2/(2 \cdot 20^2))$, $n = 100$:

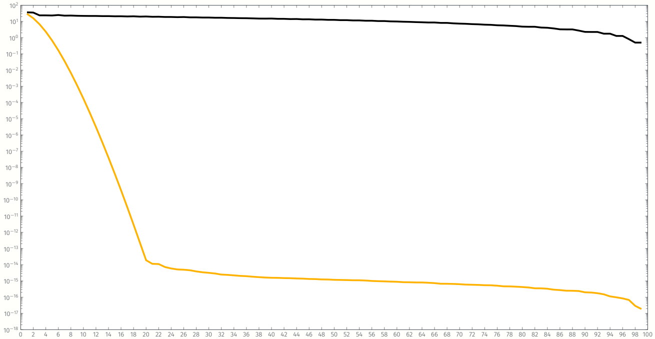

Figure 3: Error of a rank k approximation (2-norm)

Lyapunov equations

Toeplitz matrices in a Lyapunov equation.

$$T - STS^* = Gen,$$

where $S = \begin{bmatrix} 0 & & & \\ 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}$, $Gen = \begin{bmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ t_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ t_{n-1} & 0 & \cdots & 0 \end{bmatrix}$

Toeplitz matrices in a Lyapunov equation.

$$T - STS^* = Gen,$$

$$Gen = GB^T, G = \begin{bmatrix} t_0 & 1 \\ t_1 & 0 \\ \vdots & \vdots \\ t_{n-1} & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 0 & \bar{t}_1 \\ \vdots & \vdots \\ 0 & \bar{t}_{n-1}. \end{bmatrix}$$ [Kressner '17]

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$$Gen = CC^T, C = \begin{bmatrix} \sqrt{t_0} & 0 \\ t_1\sqrt{t_0} & t_1\sqrt{t_0}i \\ \vdots & \vdots \\ t_{n-1}\sqrt{t_0} & t_{n-1}\sqrt{t_0}i \end{bmatrix}$$ [Bereux '05]

Toeplitz matrices in a Lyapunov equation.

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$$Gen = DFD^T, D = \begin{bmatrix} t_0 & t_1 & \cdots & t_{n-1} \\ 0 & t_1 & \cdots & t_{n-1} \end{bmatrix}^T, F = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$ [Gallivan '96]

What’s the idea?

$$T - STS^* = DFD^T,$$

Solve

$$T - STS^* = DD^T,$$

using a Low-Rank ADI Lyapunov Solver [c.f. Patrick Kürschner] to get $Z_k \in \mathbb{R}^{n \times k}$, then

$$\tilde{T} = Z_k F_k Z_k^T,$$

where $F_k = I_{k/2} \otimes F$.

Hopefully $\tilde{T} \approx T$.

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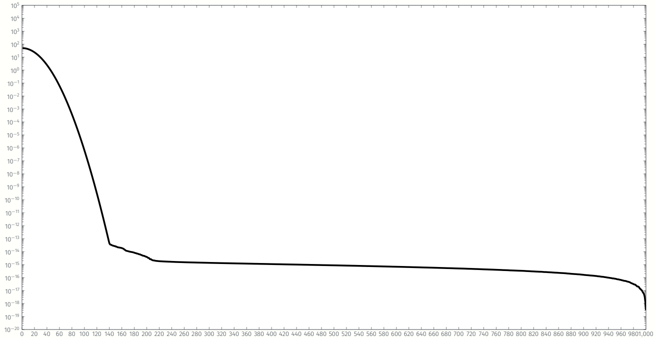

Figure 4: Singular values of T

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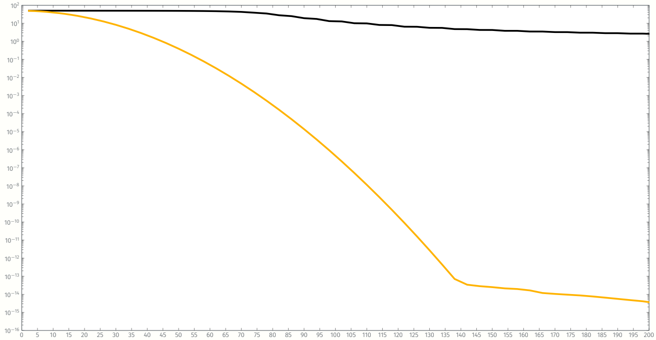

Figure 4: Error of a rank k approximation (2-norm)

More information

D. Kressner, R. Luce: Fast computation of the matrix exponential for a Toeplitz matrix, 2017, arXiv:1607.01733v3.

N. Bereux: Fast direct solvers for some complex symmetric Block Toeplitz linear systems, 2005, Linear Algebra and its Applications 404, 193-202.

K. Gallivan, S. Thirumalai, P. Van Dooren, V. Vermaut: High performance algorithms for Toeplitz and block Toeplitz matrices, 1996, Linear Algebra and its Applications 241-243, 343-388.

I. Graham, F. Kuo, D. Nuyens, R. Scheichl, I. Sloan: Analysis of circulant embedding methods for sampling stationary random fields, 2017, arXiv:1710.00751.

Thank you for listening.
Any Questions?

Exponential spacing

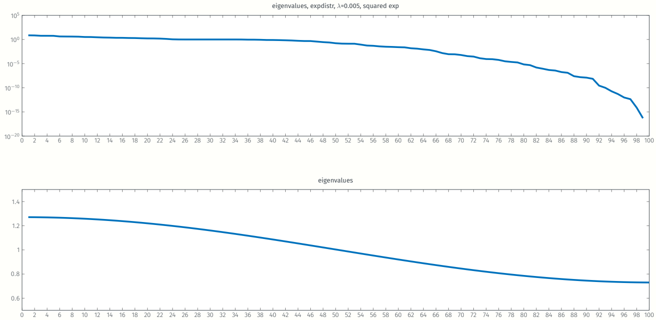

Figure 5: Exponential squared covariance, not / equidistant spacing, $\lambda = 0.005$.

$$T_{ij} = exp((i-j)^2/(2 \cdot \lambda^2)), n = 100$$