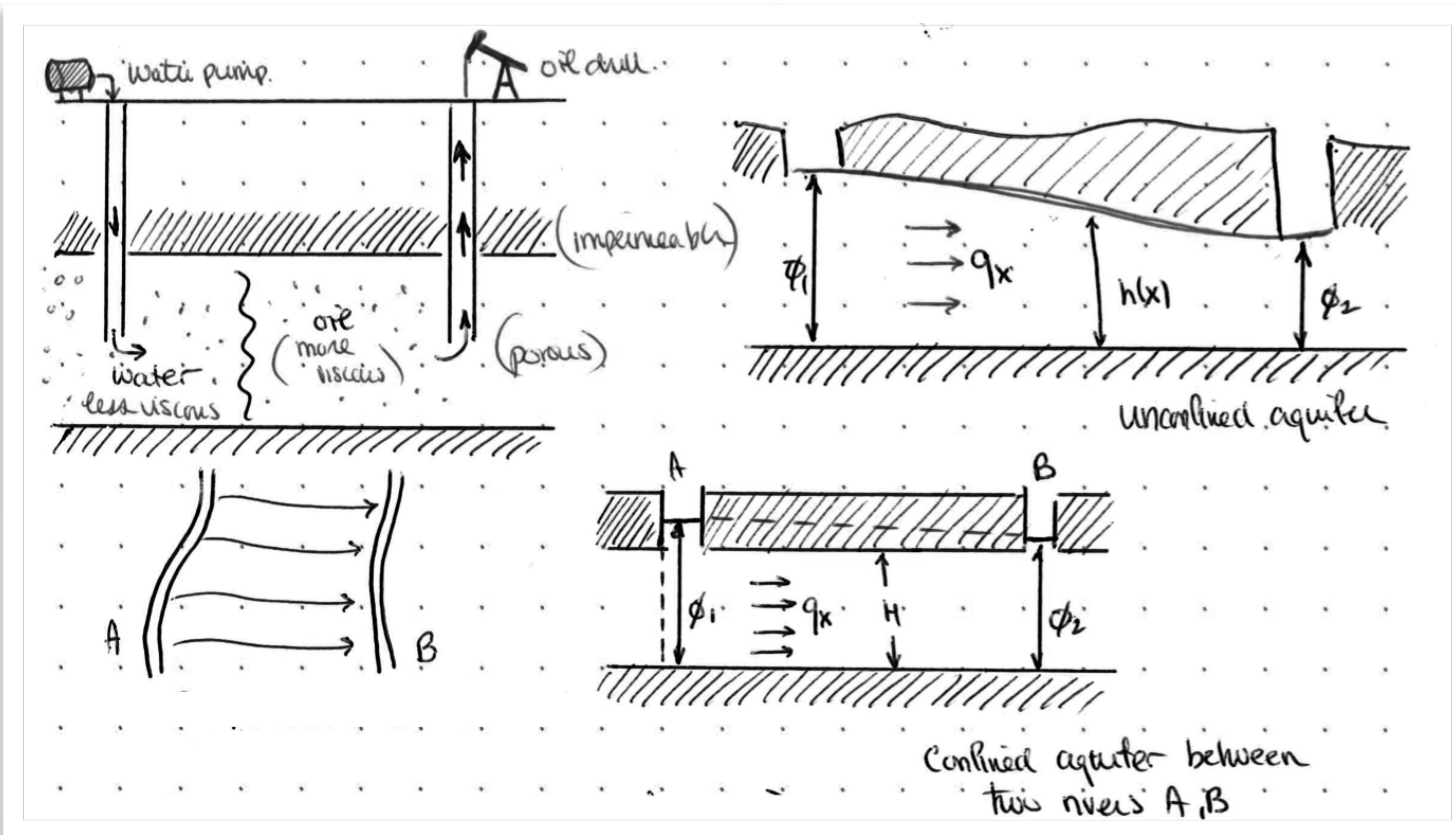
AN INTRODUCTION TO FLOW THROUGH POROUS MEDIA AND GROUNDWATER FLOW

with bonus discussion on rainoff model

A presentation for the Jan 18′ ITT by Phil Trinh (Bath)



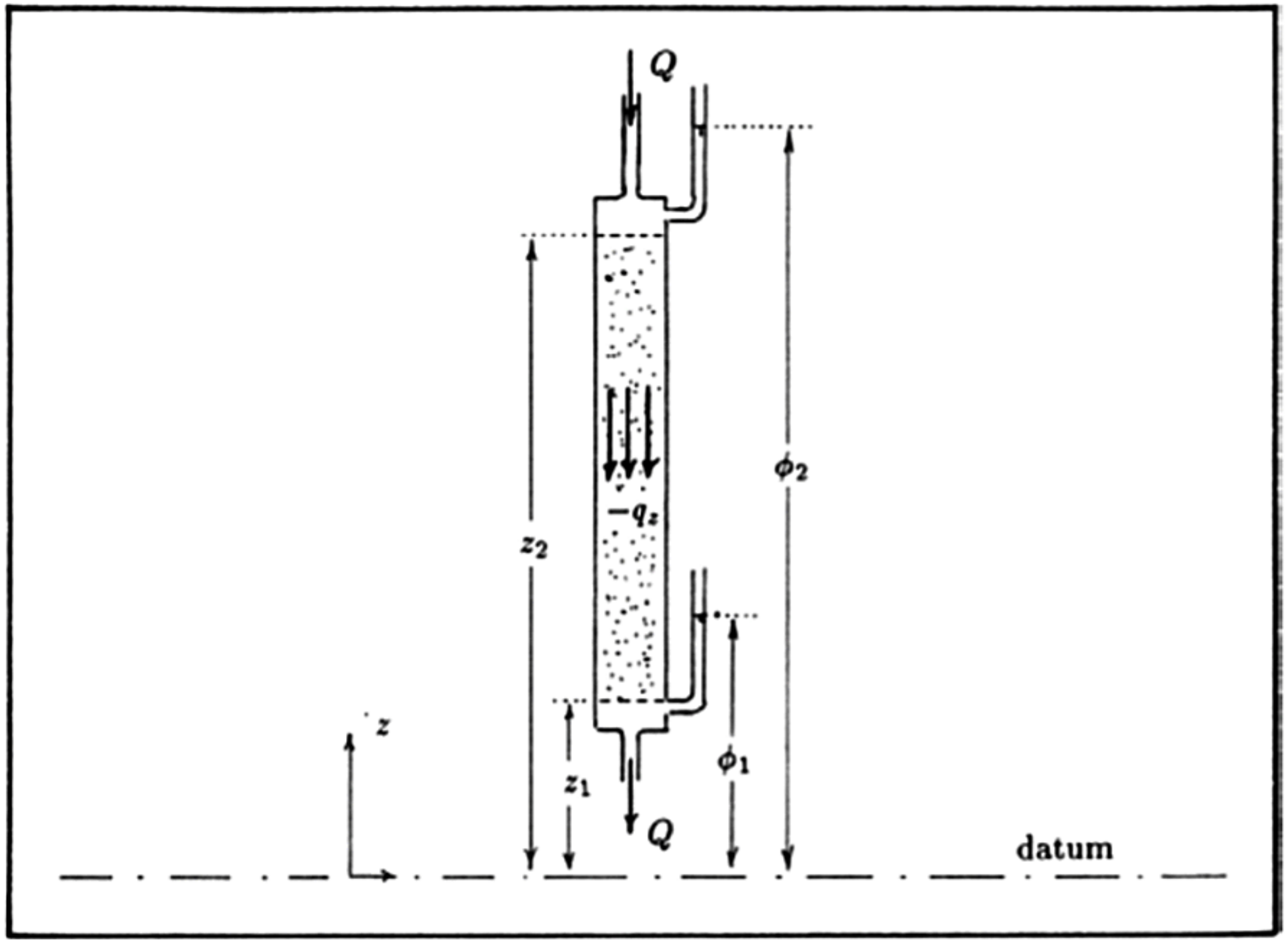
OUTLINE

1. How do we model flow through porous media?
2. What can be done analytically?
3. *Rainfall-Runoff method from the FEH book (flood risk modeling)*

References:

1. Haitjema, H. M. *Analytic element modeling of groundwater flow*
2. Anderson, M. P., Woessner, W. W. and Hunt, R. J. *Applied groundwater modeling*
3. Kjeldsen, T. R. *The revitalised FSR/FEH rainfall-runoff method,* Flood Estimation Handbook, Supplementary I

DARCY FLOW



Q = flow rate $[L^3/T^2]$

q = specific discharge ("flux") $[L/T]$

ϕ = head (hydraulic) $[L]$

z = height $[L]$

k = hydraulic conductivity, $[L/T]$

$$Q = kA\frac{\phi_2 - \phi_1}{z_2 - z_1} \Rightarrow q_z = -k\frac{\mathrm{d}\phi}{\mathrm{d}z}$$

Darcy's Law can then be generalized, firstly: $\boldsymbol{q} = -k\nabla\phi$

There are other ways of writing the conductivity:

$$k = \frac{\kappa\gamma}{\mu}$$

$\kappa =$ permeability $[L^2]$

$\gamma = \rho g$ unit weight of pore fluid $[M/L^2T^2]$

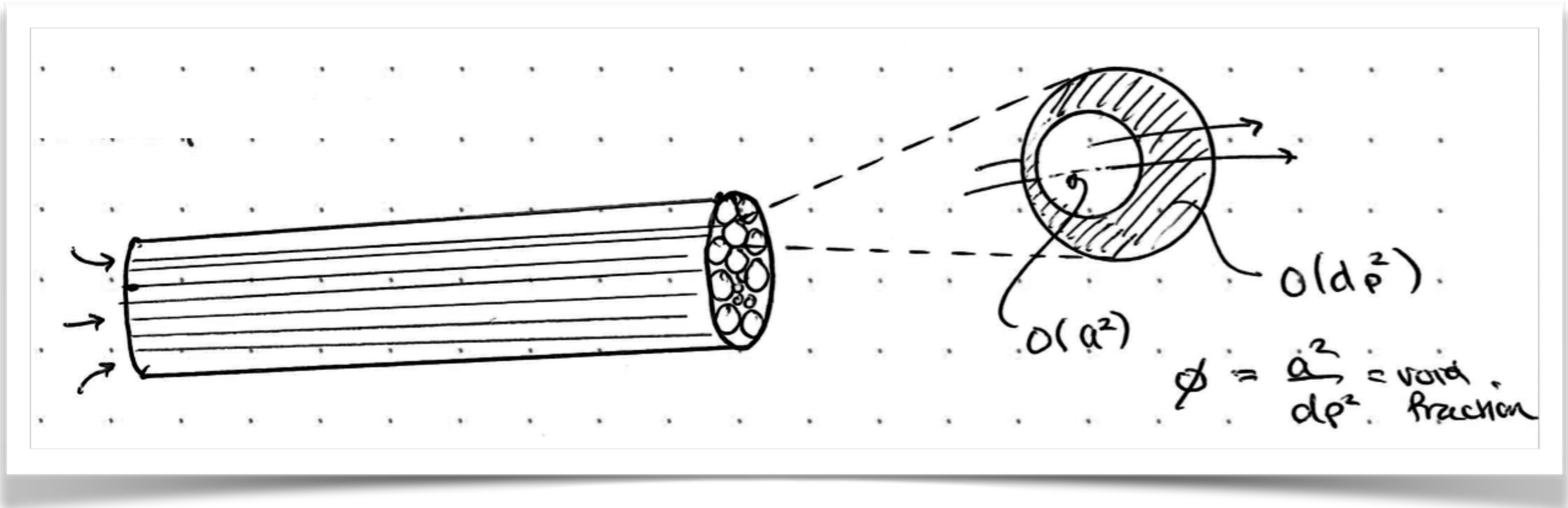
$\mu =$ dynamic viscosity of pore fluid, $[M/LT]$

Here are some typical values of k

soil type	k in ft/day
gravel	> 500
sand and gravel	200 - 500
coarse sand	50 - 200
medium sand	20 - 50
fine sand	1 - 20
silty sand	0.1 - 1
silt	0.01 - 0.1
peat	0.001 - 0.01
sandy clay	0.0001 - 0.001
clay	< 0.0001

Take care as discharge is not the same as velocity. Moreover head is not the same as (pore) pressure.

Darcy's law can be alternatively understood via a 'homogenization' argument':

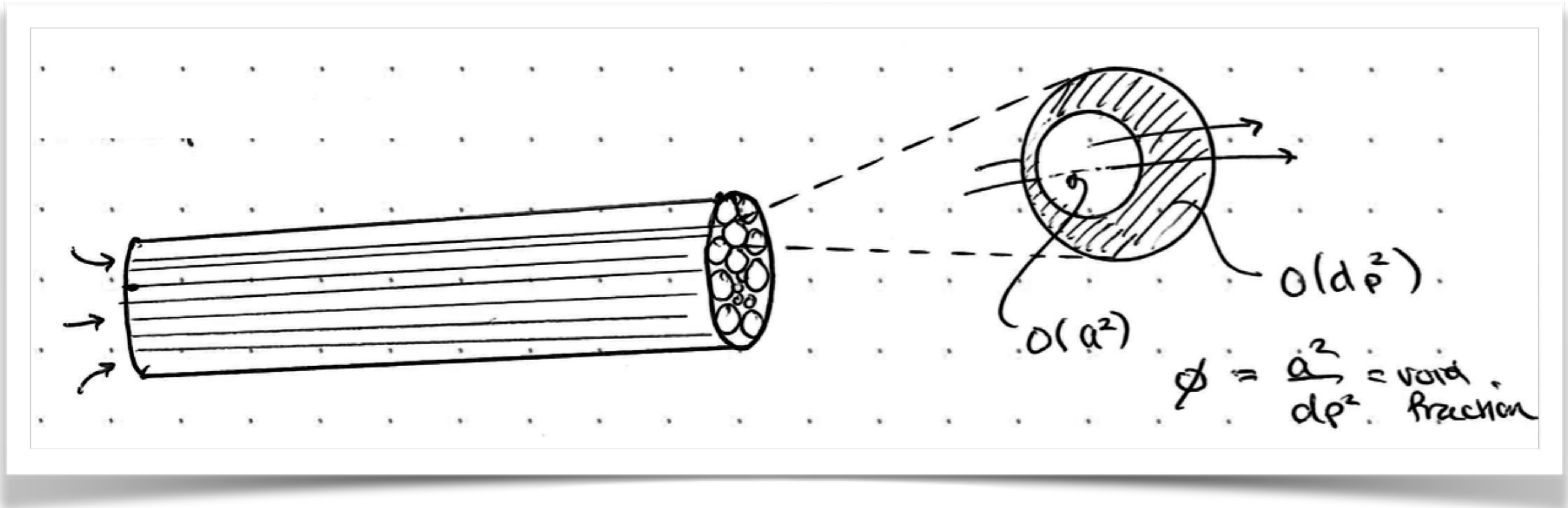


Assume the pores are tubular in shape, and are of radius *a*. Then by viscous pipe flow (Poiseuille),

$$\text{flux in one pipe} = \boldsymbol{q}_{\text{tube}} \approx -\frac{a^4}{\mu X}\nabla(p + \rho g z)$$

where X >~ 1 is a factor that accounts for the arrangement of the tubes (in Poiseuille flow—related to the pressure drop across the tube).

Darcy's law can be alternatively understood via a 'homogenization' argument':



Then since we have,

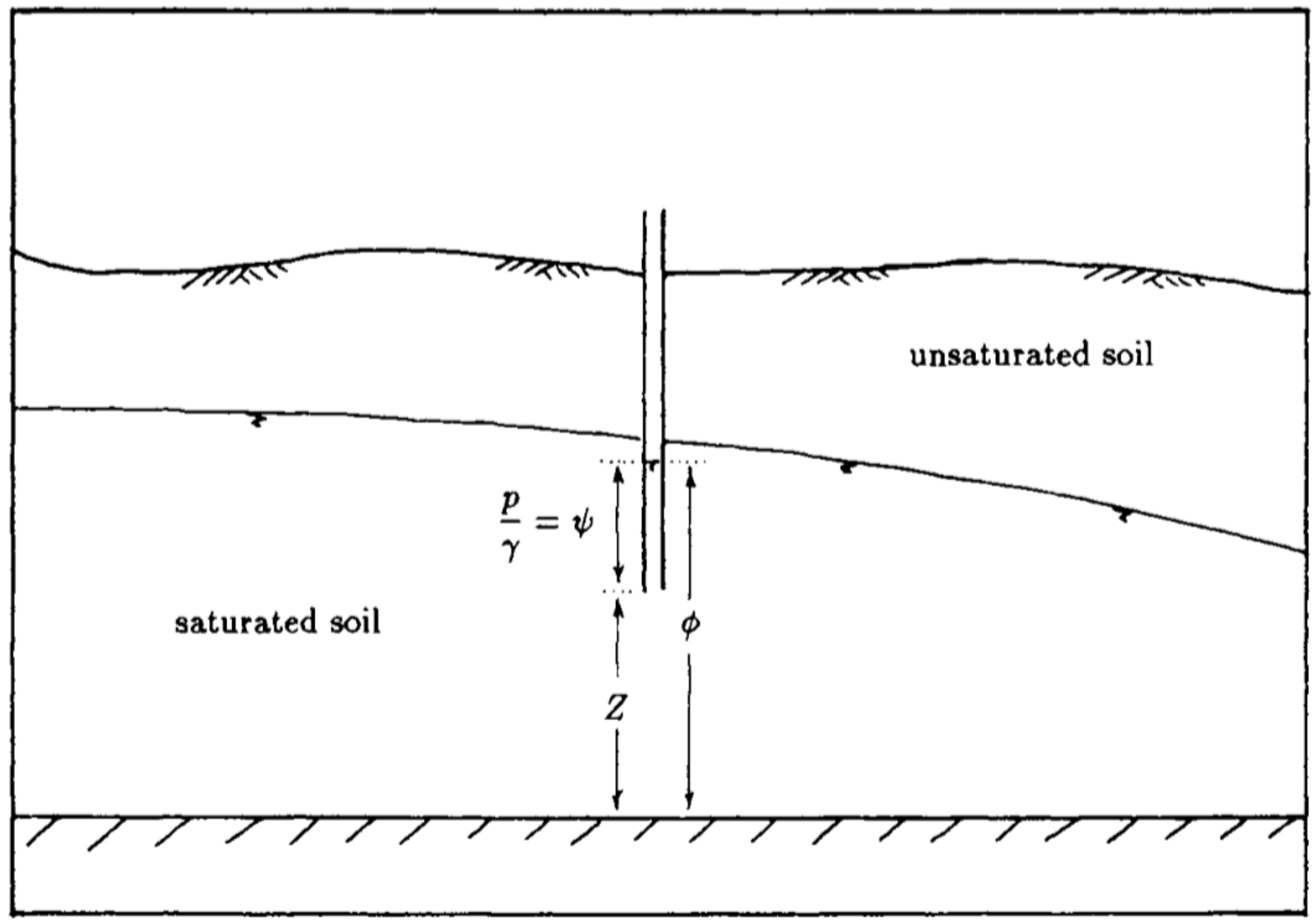
$$\begin{aligned} \pi_p &= a^2/d_p = \text{void/pore area fraction (porosity)} \\ d_p &= \text{representative grain/particle size (radius)} \end{aligned}$$

The flux (flow rate per unit cross-sectional area) is then

$$\boldsymbol{q} \text{ (per unit cross-sectional area of material)} = \frac{\boldsymbol{q}_{\text{tube}}}{d_p^2} = -\frac{\overbrace{(\phi^2 d_p^2/X)}^{\kappa}}{\mu}\nabla(p+\rho g z)$$

where κ is the permeability of the material.

NB: In groundwater flow, it's more typical to work with quantities involving the head instead of the pressure.



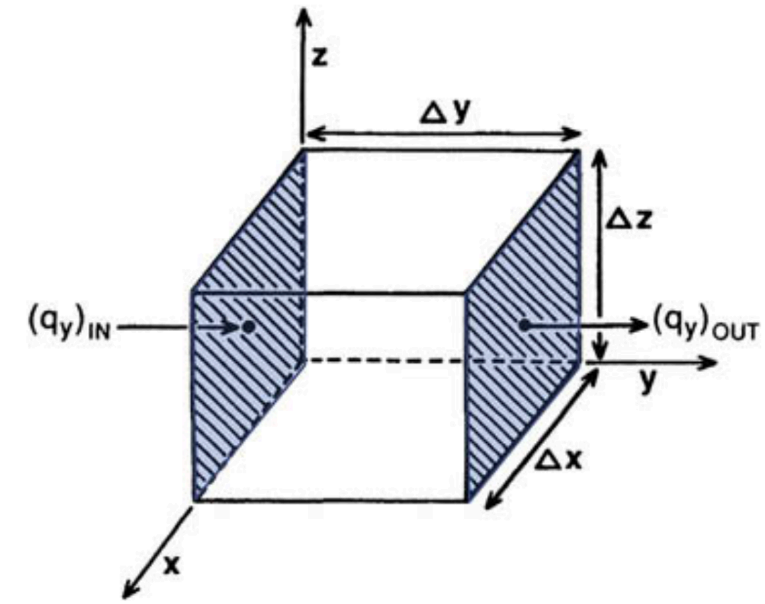
For the piezometer

$$\phi \text{ (hydraulic head)} = \psi \text{ (pressure head)} + Z$$

Thus, the height Z,

$$\phi = \frac{p}{\rho g} + Z \Rightarrow p = \rho g(\phi - Z)$$

Conservation of mass:



$$\text{outflow - inflow + creation due to sources} = \frac{\Delta V}{\Delta t}$$

$$\Rightarrow \left(\frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + \frac{\partial q_z}{\partial z} - W^*\right)\Delta x \Delta y \Delta z = \frac{\Delta V}{\Delta t}$$

We assume that the change of volume is approximated by the change in (local) head. Thus we write:

$$\frac{\Delta V}{\Delta t} = -S_s \frac{\Delta \phi}{\Delta t}\Delta x \Delta y \Delta z$$

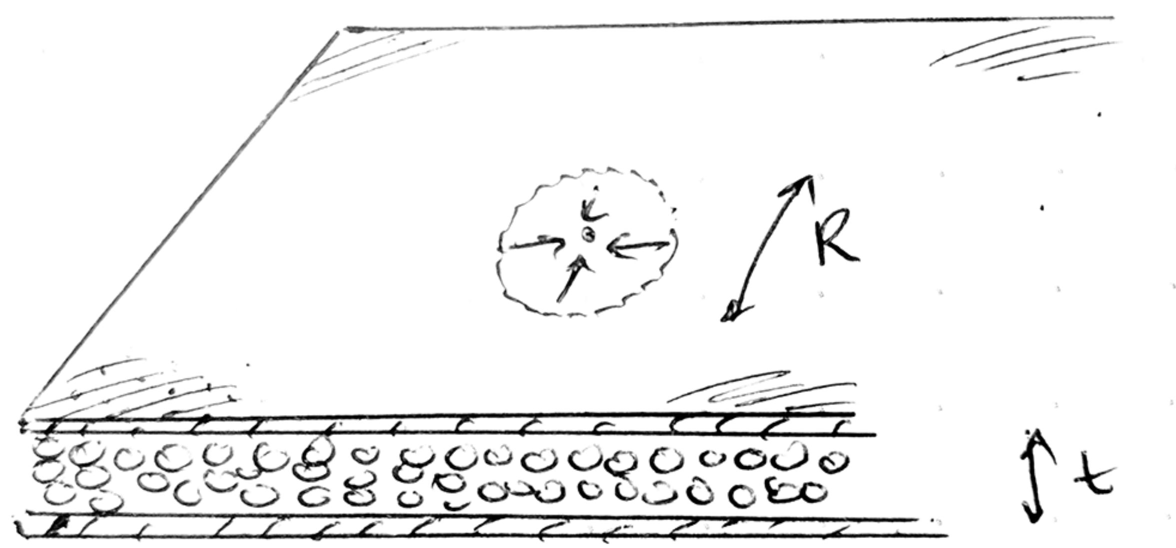
where $S_s = \left(-\frac{\Delta V}{\Delta \phi \Delta x \Delta y \Delta z}\right)$ is the specific storage (i.e. volume of water released per unit change in head per unit change in volume)

$$\nabla \cdot \boldsymbol{q} - W^* = -S_s \frac{\partial \phi}{\partial t}$$

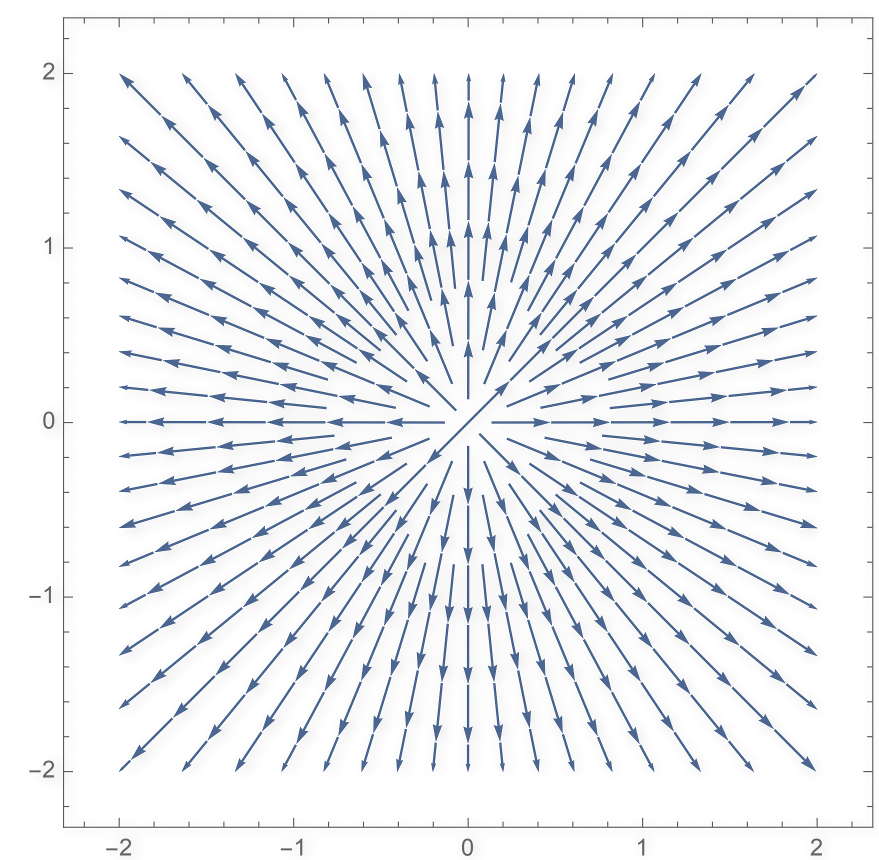
Using Darcy, this leads to the governing equation (Anderson, Woessner, Randall):

$$\boxed{\frac{\partial}{\partial x}\left(K_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(K_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(K_z \frac{\partial \phi}{\partial z}\right) = S_s \frac{\partial \phi}{\partial t} + W^*}$$

Example 1: (2D sink/source, **steady**) Consider pumping oil at a rate |Q| out of a hole of radius R in a thin layer of porous media between two plates of separation height, H



Solving Laplace's equation on an axi-symmetric 2D domain



$$\nabla^2\phi = 0 \Rightarrow \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{r}\frac{\mathrm{d}p}{\mathrm{d}r}\right) = 0 \Rightarrow \phi = \phi_0 + C\log r$$

Hence imposing the flux condition on the boundary:

$$-|Q| = \left(\int_0^{2\pi} (-k\nabla\phi\cdot\boldsymbol{n})\ \mathrm{d}\theta\right) H \Rightarrow C = \frac{|Q|}{2\pi k H}$$

Example 2: (Dupuit-Forchheimer approximation) The DF approximation is analogous to the thin-film limit of viscous flows, and allows a reduction in the dimension of the problem.

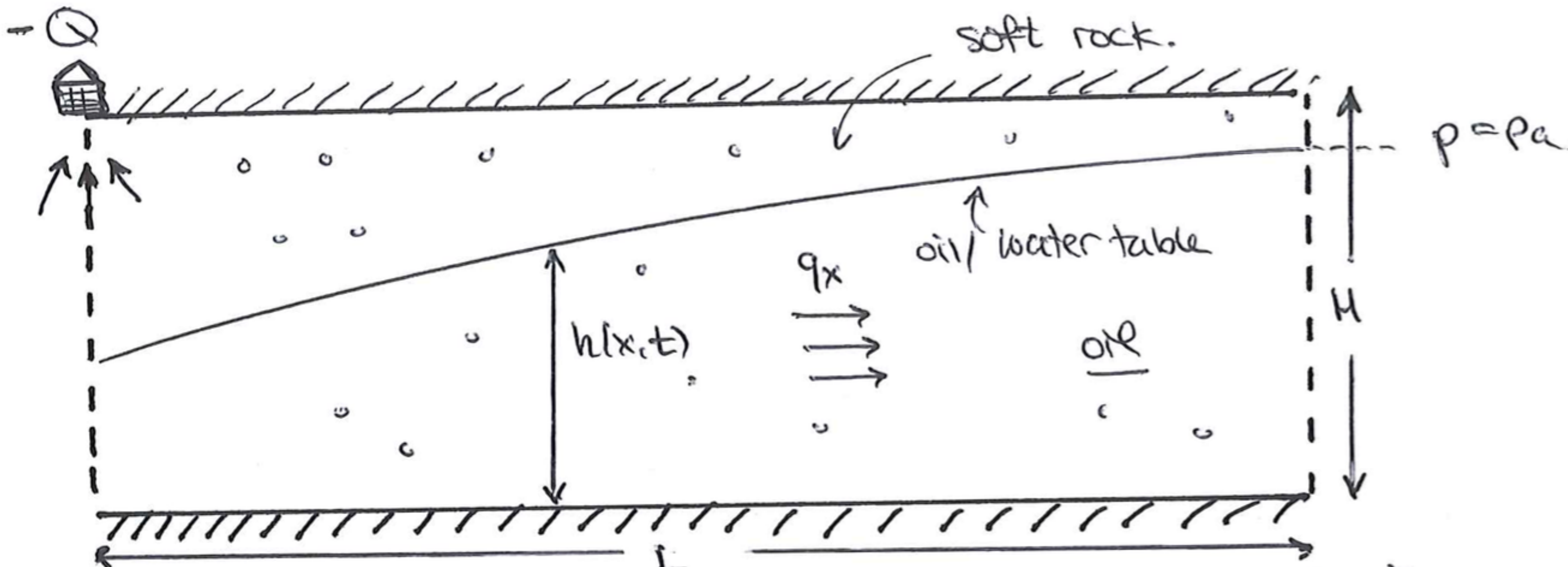


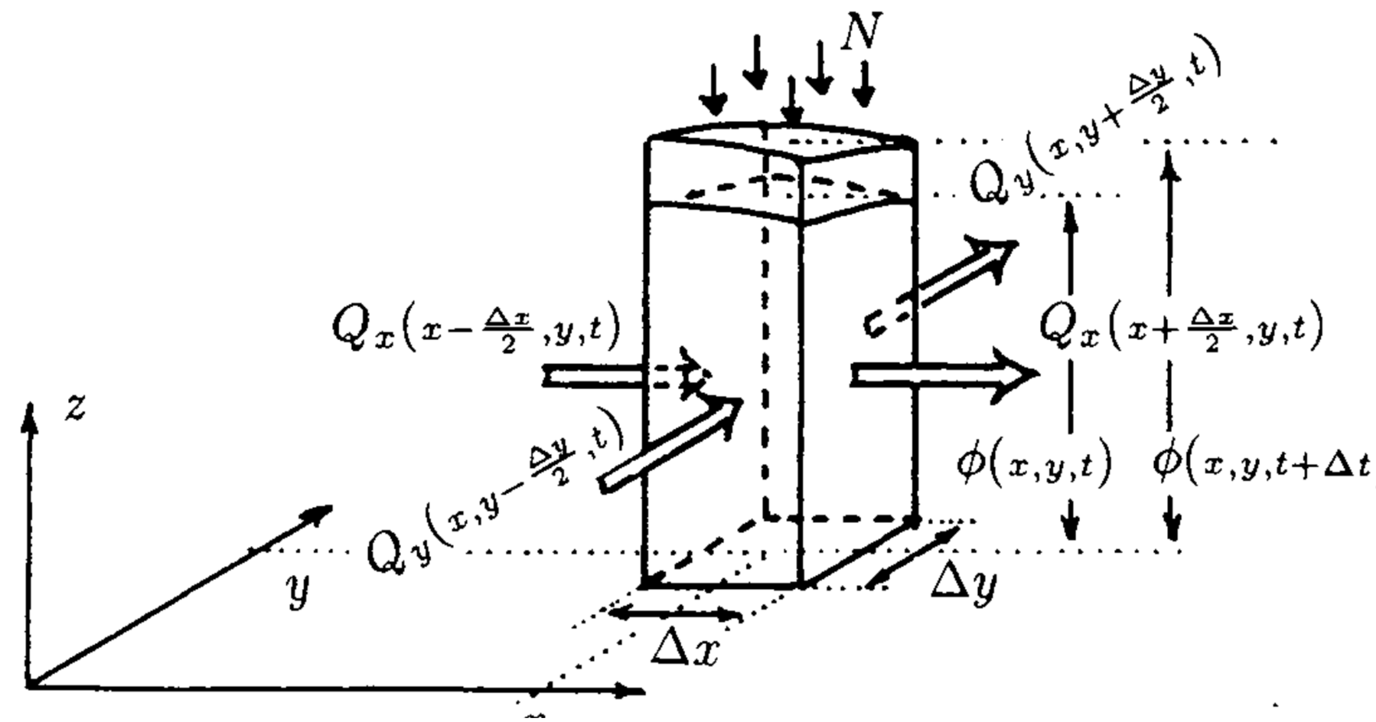
FIG: AN UNCONFINED AQUIFER OR OIL RESERVOIR

The DF approximation assumes that the **flowlines are large in comparison with the geometry thickness**. Hence we assume that

$$q_z = 0 \Rightarrow \phi_z = 0 \Rightarrow \phi = \phi(x, y, t)$$

Conservation of mass argument is then re-done in order to embed the column transport (rather than cell).

Conservation of mass argument is then re-done in order to embed the column mass (rather than cell). Discharge over the height of the aquifer is given by



$$Q_x = hq_x,\, Q_y = hq_y \quad \Rightarrow \quad \boldsymbol{Q} = h\boldsymbol{q}$$

Hence conservation of mass on the column leads to $$S\frac{\partial h}{\partial t} = \left(\frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y}\right)$$

The head is measured so that $\phi = h(x, y, t)$, i.e. the head and the aquifer thickness are the same. It follows from Darcy that

$$S\frac{\partial h}{\partial t} = -\frac{k}{2}\left(\frac{\partial h^2}{\partial x} + \frac{\partial h^2}{\partial y}\right)$$

Which is now a nonlinear equation for the water table, S = storage coefficient

Example 2: Solve for the water table in the below situation at steady-state.

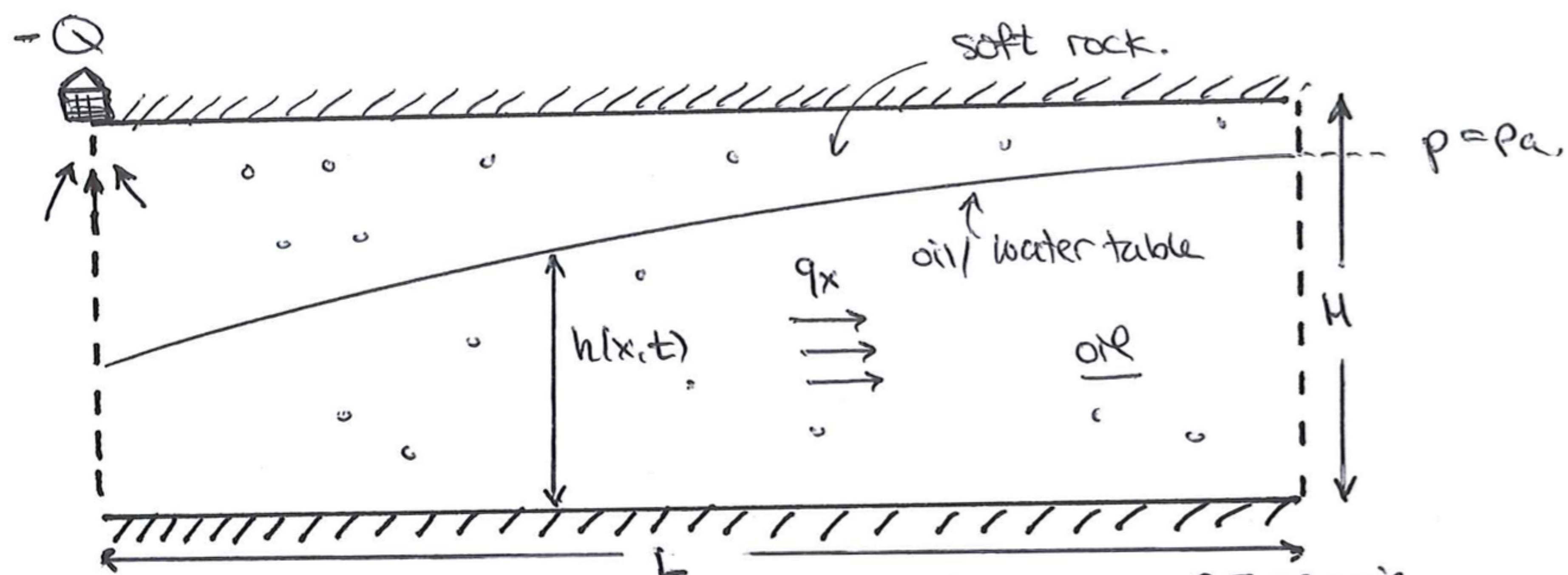
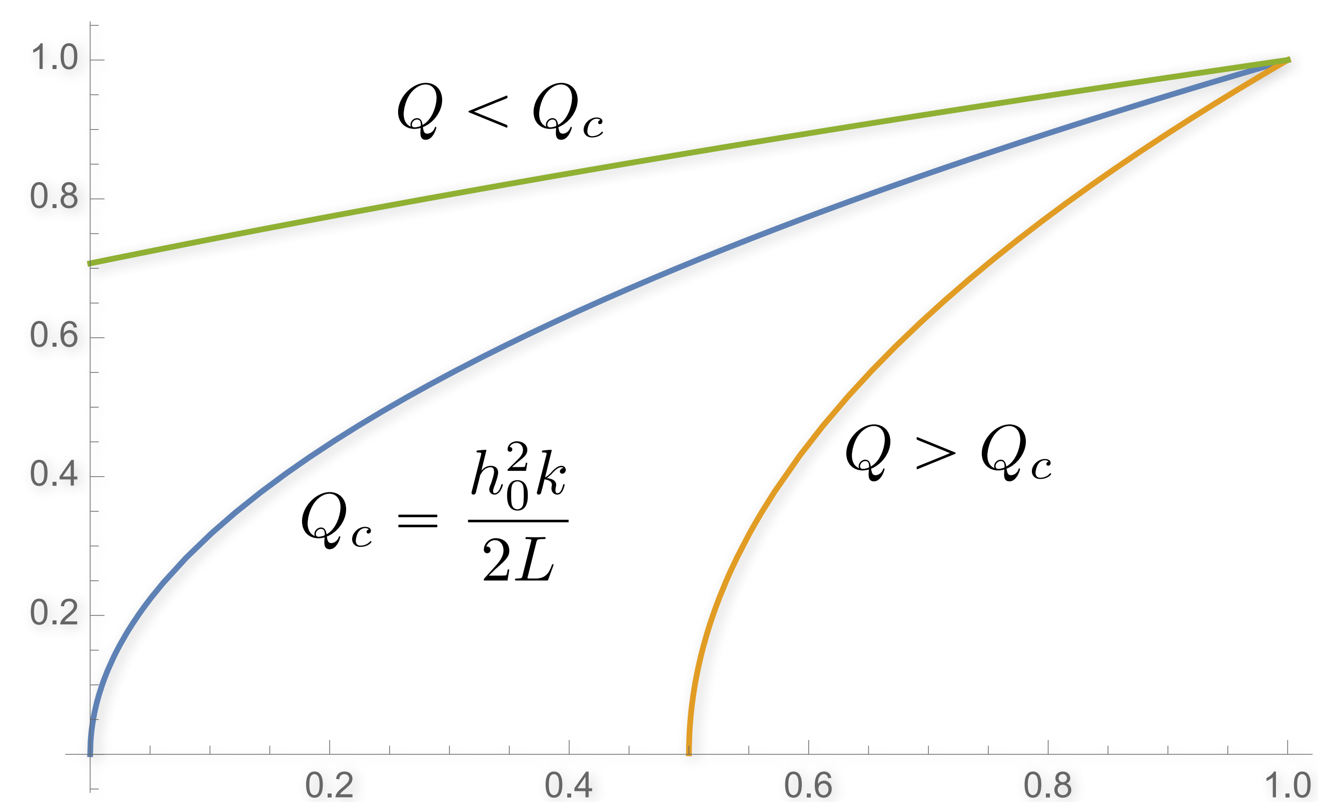


FIG: AN UNCONFINED AQUIFER OR OIL RESERVOIR

We have

$$\boxed{\begin{gathered}\frac{\mathrm{d}^2(h^2)}{\mathrm{d}x^2} = 0 \\ h = h_0 \quad \text{at } x = L \\ -|Q| = -k h_x h \quad \text{at } x = 0\end{gathered}}$$

$$\Rightarrow h(x) = \sqrt{\frac{2|Q|}{k}(x - L) + h_0^2}$$

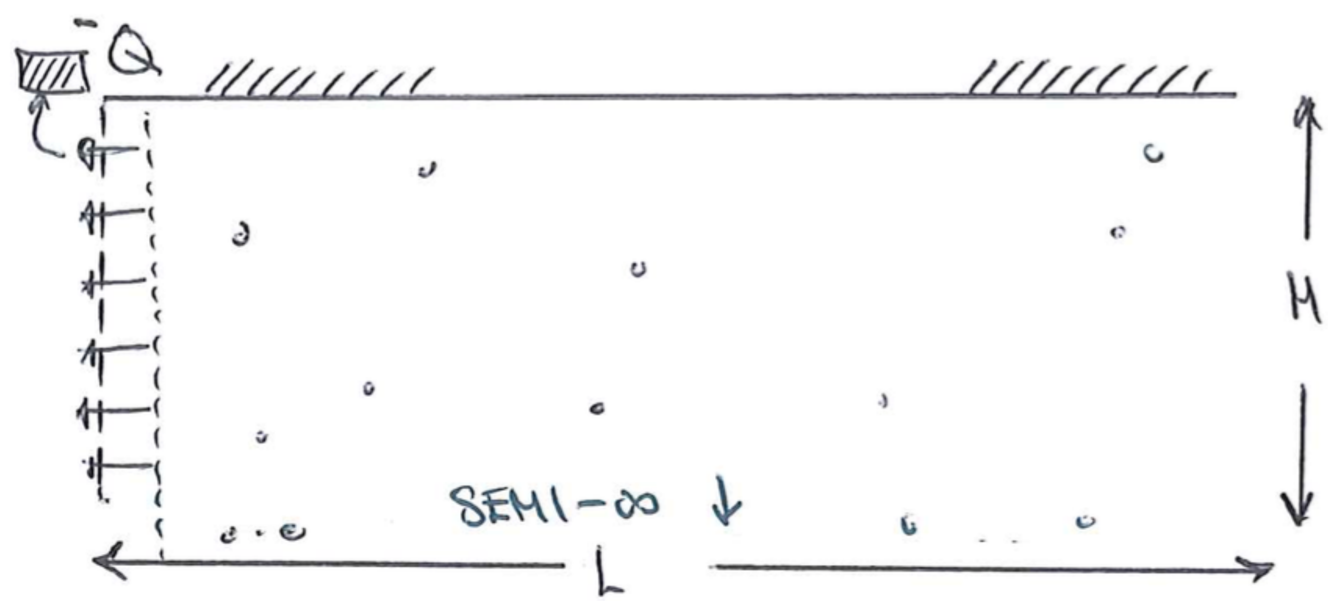


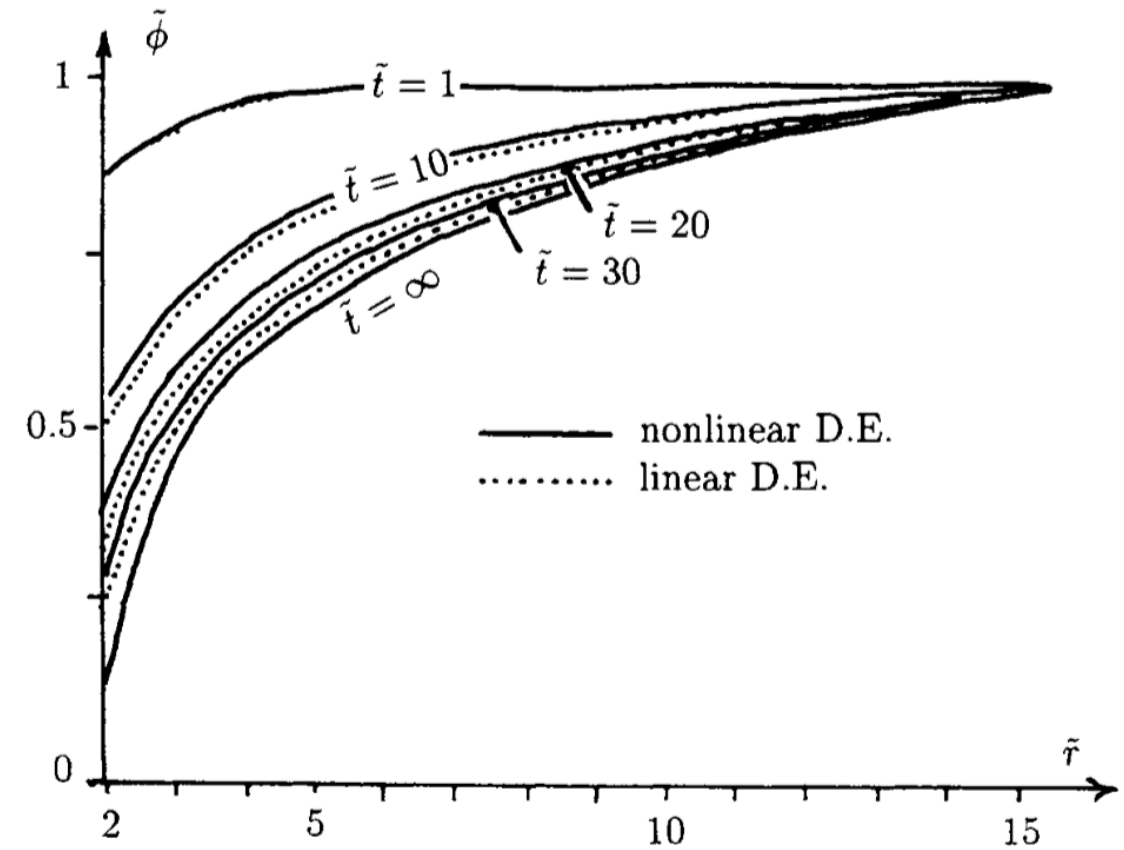
Example 3: (Theis' solution) A common approximation of the nonlinear diffusion equation involves a linearization. Introduce

$$\Phi = \frac{k}{2}\phi^2 \Rightarrow \frac{S}{\phi k}\frac{\partial \Phi}{\partial t} = \nabla^2 \Phi$$

On the assumption that $S/(\phi k) \approx S/(\overline{\phi} k) = S_s/k = \text{const.}$

We are led to the linear diffusion equation: $\dfrac{\partial \Phi}{\partial t} = D\nabla^2\Phi, \qquad D = \dfrac{k}{S_s}$

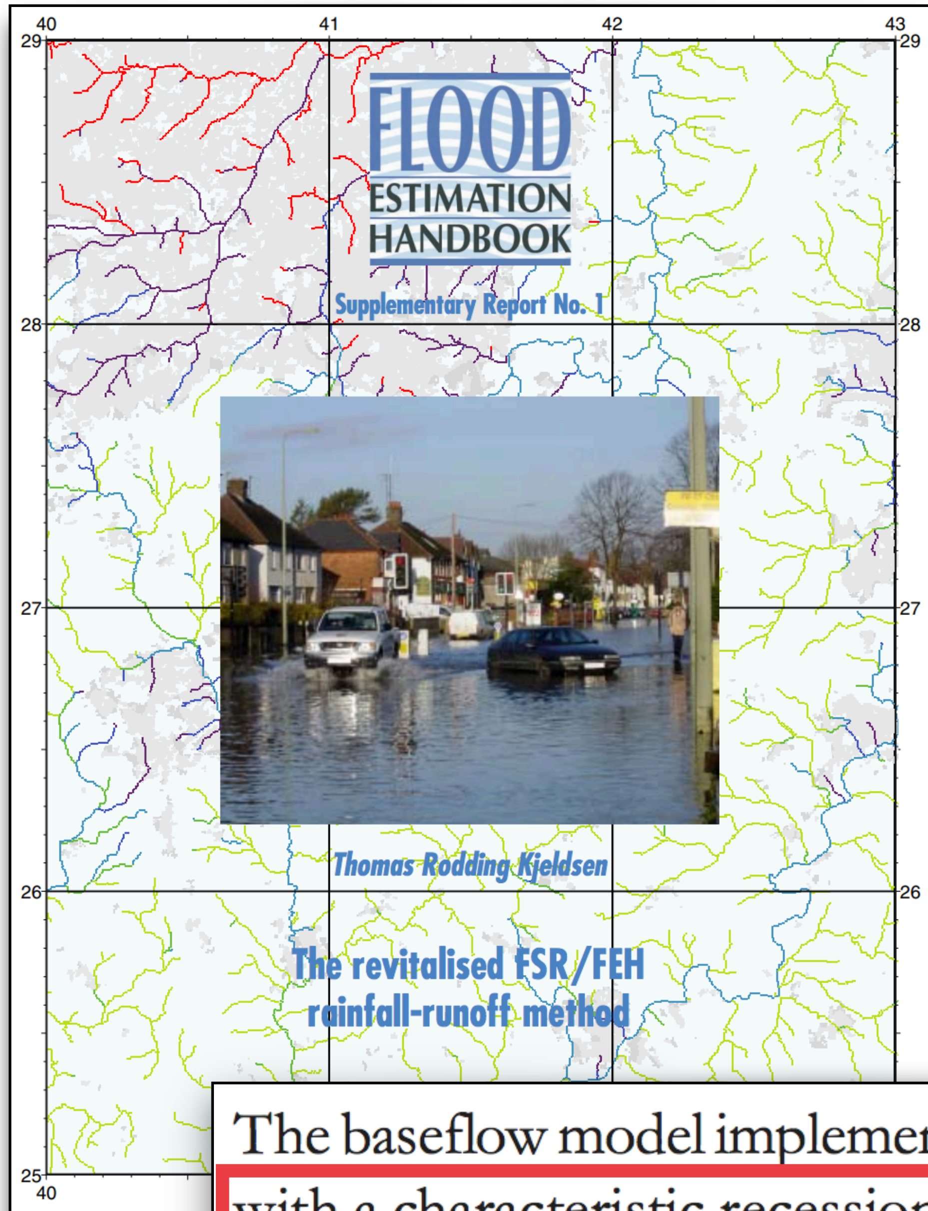




$$\Phi(r,t) = -\frac{Q}{4\pi}E_1(u) + \Phi_0, \qquad t > t_0$$

$$u = \frac{r^2}{4D(t-t_0)}$$

RAINFALL-RUNOFF METHOD



The baseflow model implemented in the ReFH model is based on the linear reservoir concept, with a characteristic recession defined as an exponential decay. The approach, discussed by Appleby (1974) allows the separation of total flow in baseflow and surface flow without knowing the rainfall input. It is based on the contributing area concept, and assumes that the

Appleby, F.V. 1974. Unpublished notes on baseflow modelling, Imperial College, London, UK.
Ashfaq, A. and Webster, P. 2002. Evaluation of the FEH rainfall-runoff method for catchments in the UK. *J. CIWEM*, **16**(3), 223-228.

The location is modeled as a point,

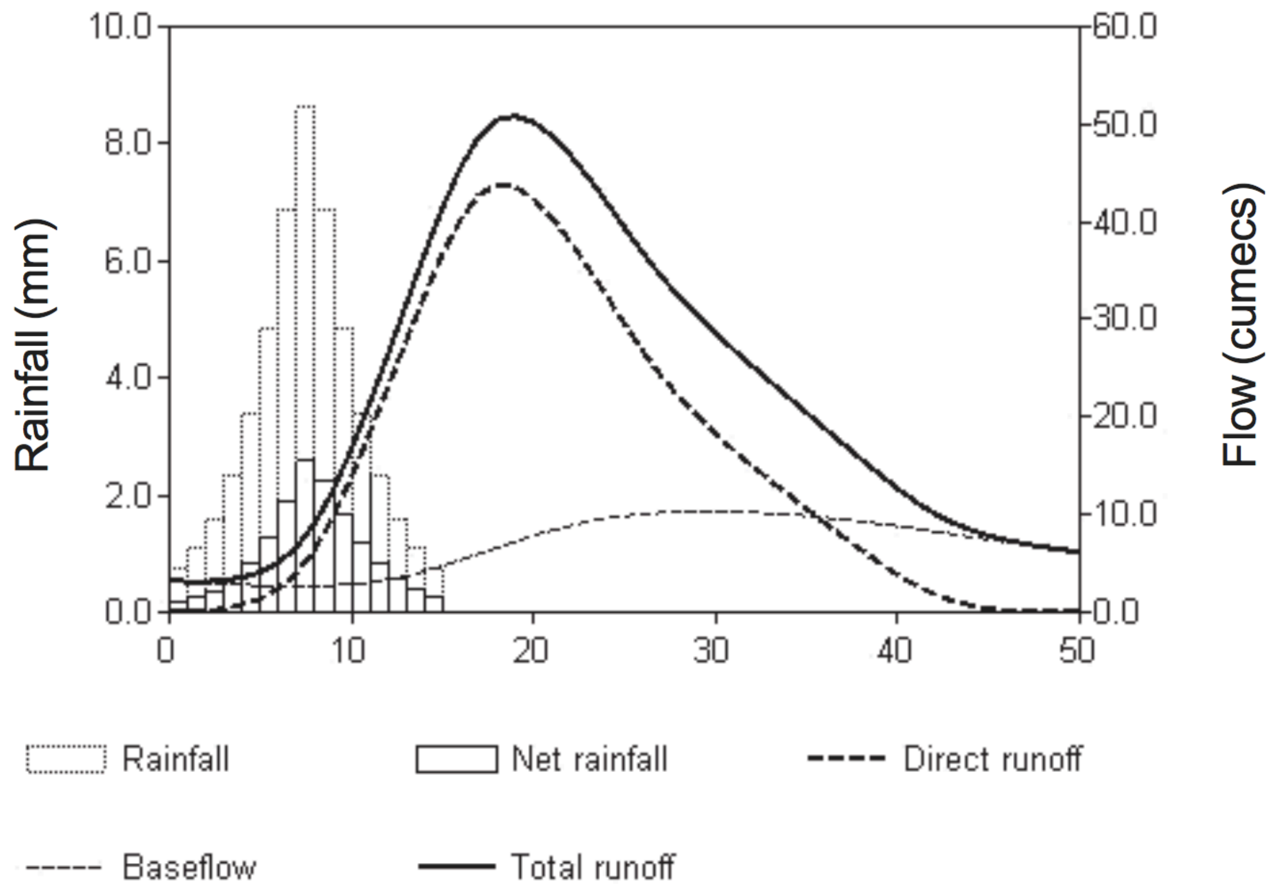


Figure 6.3 *100-year design flood hydrograph for Salwarpe at Harford Mill*

FEH

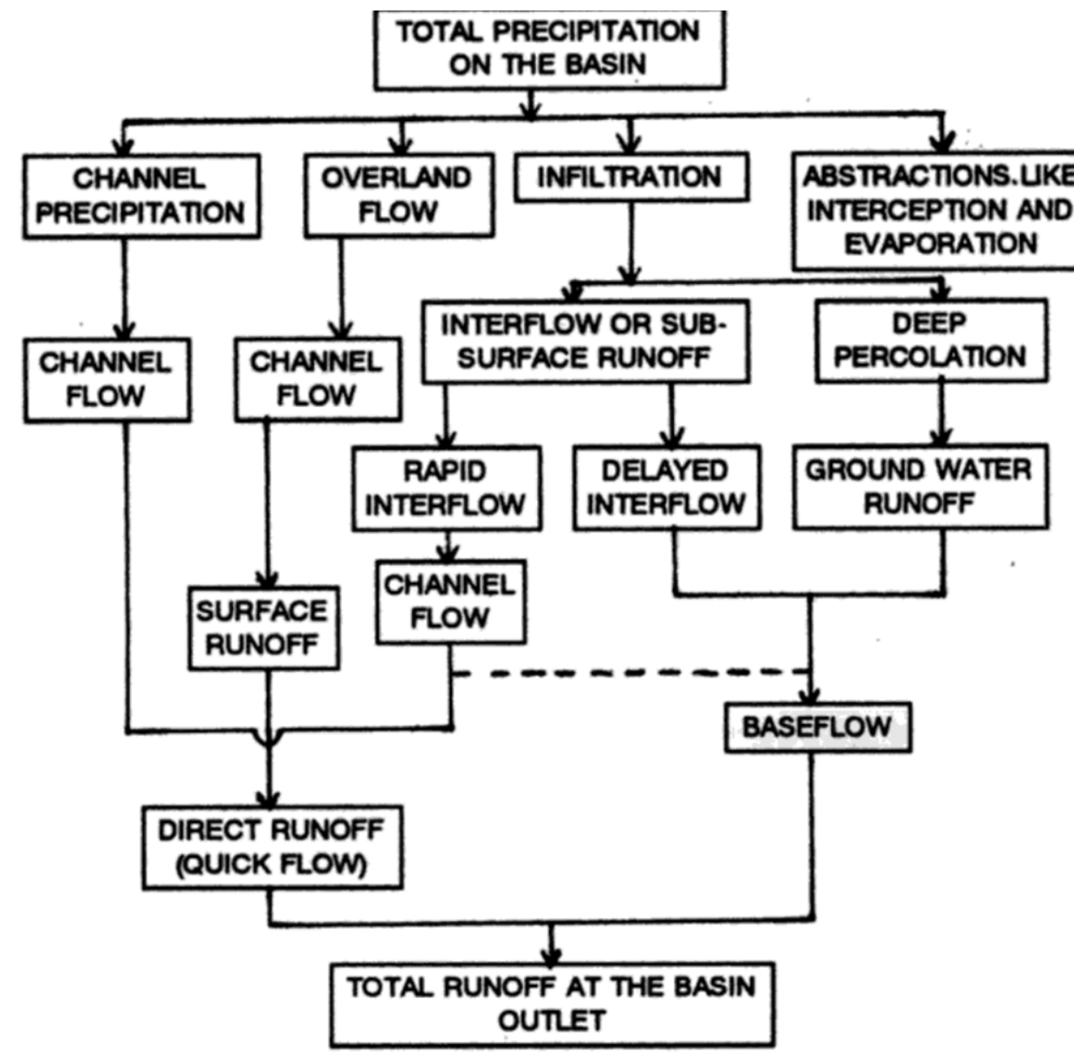


Fig. 10.1. Schematic representation of runoff process.

A Text Book of Hydrology
By P. Jaya Rami Reddy

The basic idea of this model involves a pre-processing step to determine net rainfall given total rainfall. Then two (independent) models [discrete difference equations] are used to determine the baseflow and quick flow.

$$\begin{aligned} \text{base flow} &= z(t) \\ \text{quick flow} &= q(t) \\ \text{total flow/runoff} &= Q(t) = q(t) + z(t) \end{aligned}$$

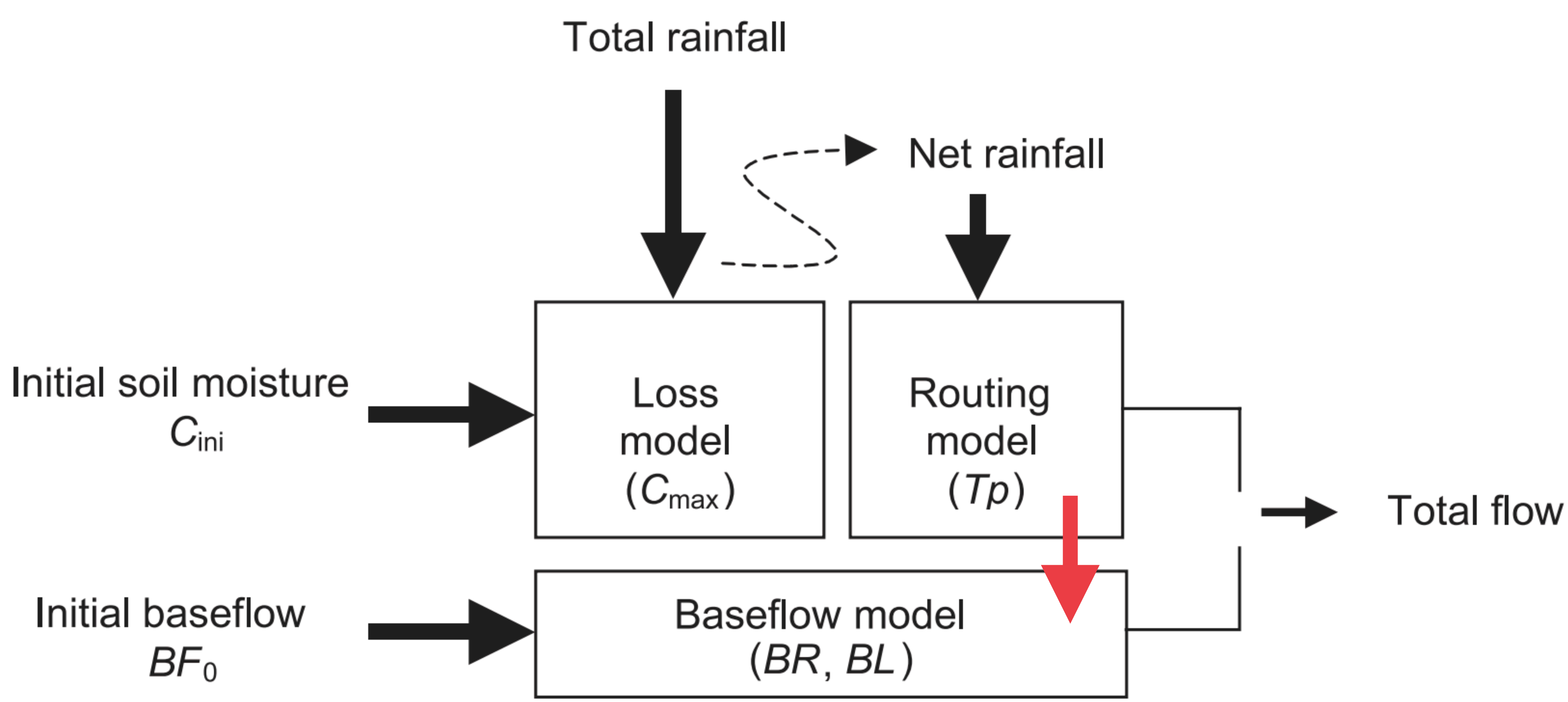


Figure 2.1 *Schematic representation of the ReFH model*

First, using [uniformly distributed] 'soil' properties, they determine what proportion (q/P) of rainfall is then converted to total rain runoff.

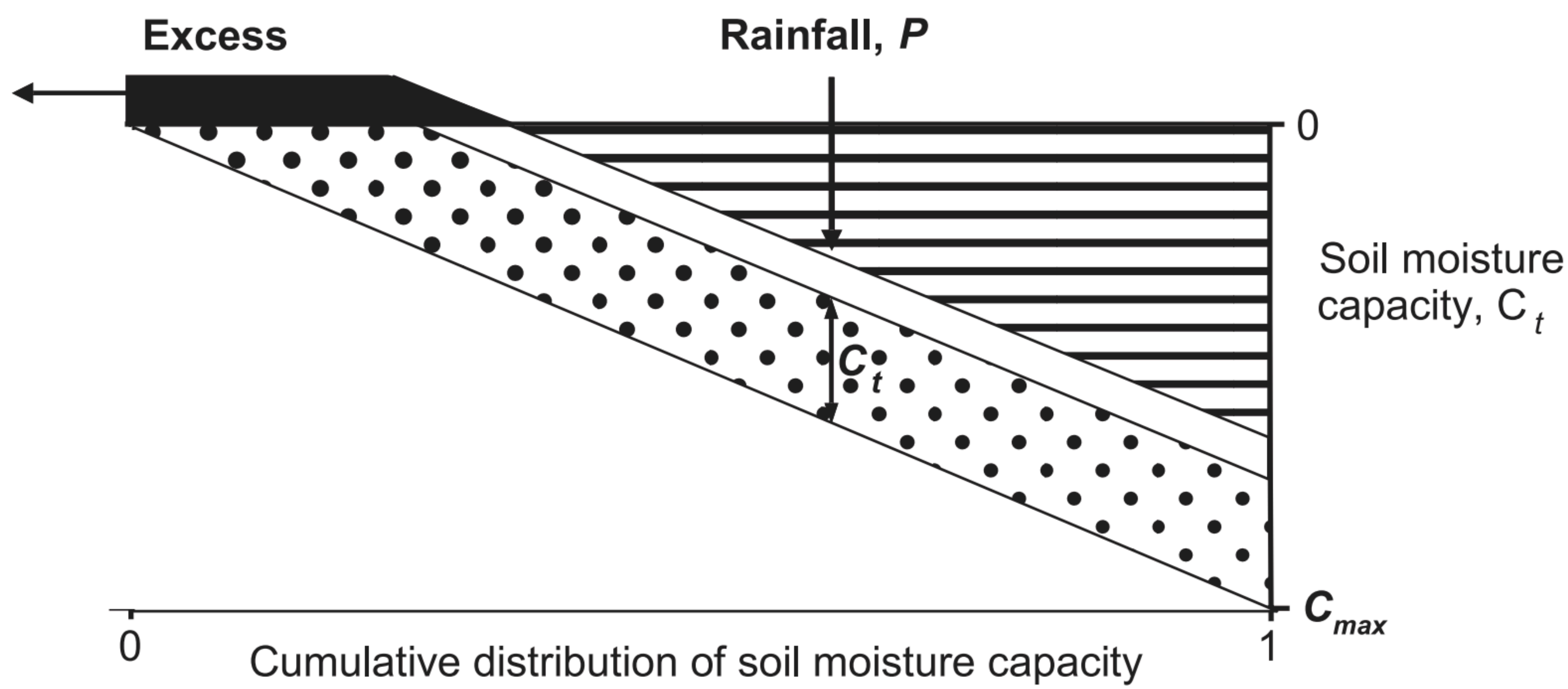


Figure 2.2 *Equal water content C_t across stores of different capacity*

$$\frac{q_t}{P_t} = \frac{C_t}{C_{\max}} + \frac{P_t}{2C_{\max}} \qquad \text{for } t = 1, 2, 3, \ldots \qquad (2.1)$$

Next, they posit a law for how quick-flow accumulates in time, given a unit forcing of rain (i.e. given a unit pulse of rain, the below graph determines the response of the system)

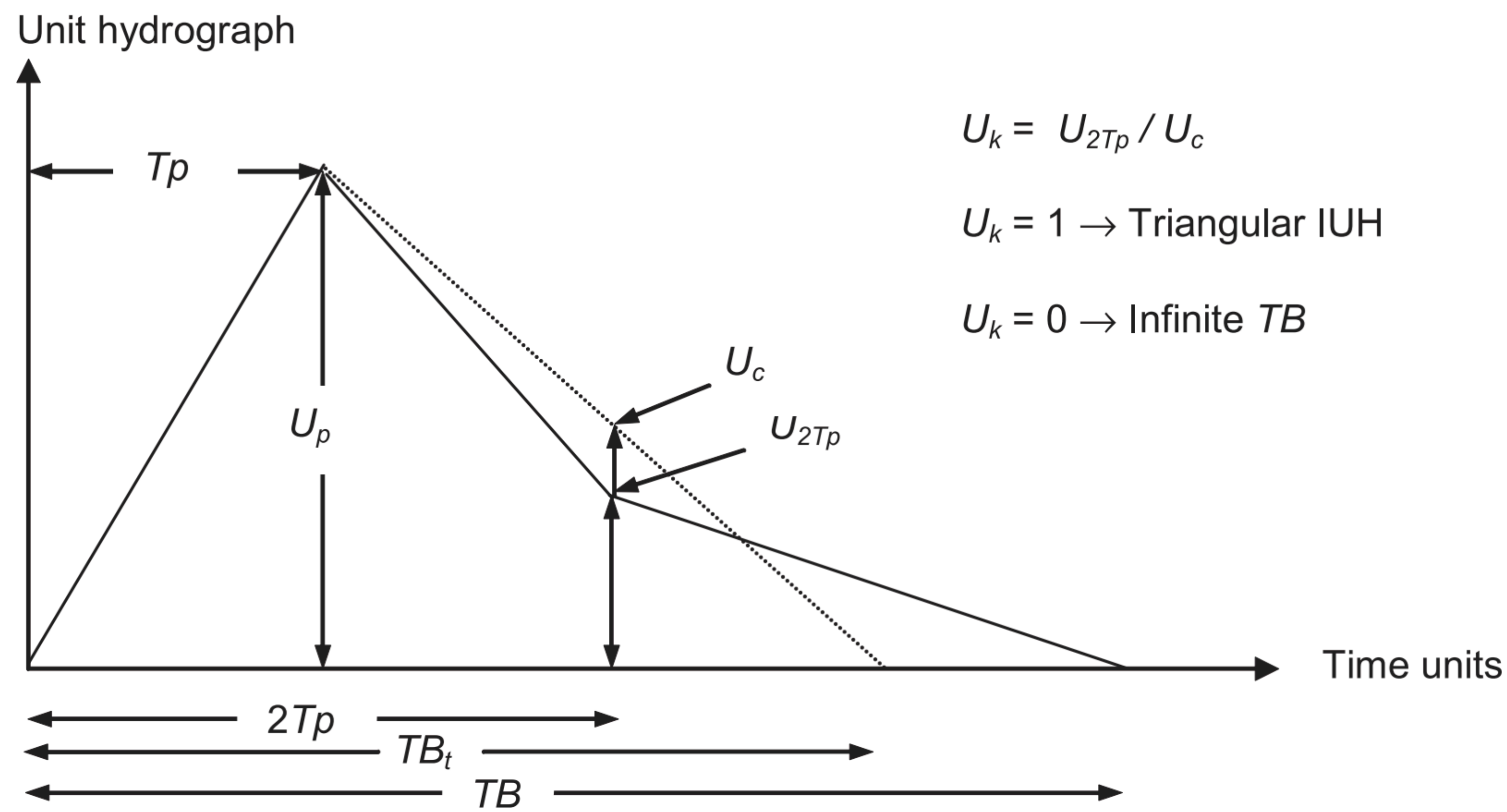


Figure 2.3 *Shape of standard instantaneous unit hydrograph adopted in ReFH*

The quick-flow then results from a convolution with the rain profile in time:

$$q_t = \sum_{i=1}^{t} P_i u_{t-i+1} \quad \text{for} \quad t = 1, 2, 3, \ldots$$

Finally, the evolution of the base flow is determined by a rate-exchange equation.

$$z_{t+\Delta t} = k_1^* q_t + k_2^* q_{t+\Delta t} + k_3^* z_t$$

$$k_1^* = BR\left(\frac{BL}{\Delta t}(1-k_3) - k_3\right)$$

$$k_2^* = BR\left(1 - (1-k_3)\frac{BL}{\Delta t}\right)$$

$$k_3^* = \exp\left(-\frac{\Delta t}{BL}(1+BR)\right)$$

(Somehow this takes in account natural decay of the base flow, and growth/recharge due to the quick-flow).

Criticisms of the model?

Need inputs
C_min, C_max, uniform soil distribution
Shape of the unit hydrograph response, T_p, U_p,
BL, BR [rate constants for base flow]
Initial values

Need assumptions
Soil saturation occurs independently(?) from dynamics
Uncoupled base/quick flows
No geometrical parameters
Later various ad-hoc fixes (to account for seasonality)
Various dubious steps to determine ICs/parameters