Eigenvalues of covariance matrices

Mark Opmeer

An autoregressive-moving-average (ARMA) model

$$x_{k+1} = Ax_k + Bw_k, \quad x_0 = 0, \quad y_k = Cx_k + w_k.$$

- $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times 1}$, $C \in \mathbb{R}^{1 \times n}$.
- w_k i.i.d. random variables, zero mean, variance σ^2.

Alternative description:

$$y_k = \sum_{j=0}^{k} h_j w_{k-j}, \qquad h_j = \begin{cases} I & j = 0 \\ CA^{j-1}B & j > 0. \end{cases}$$

Of interest: (eigenvalues of) covariance matrix of

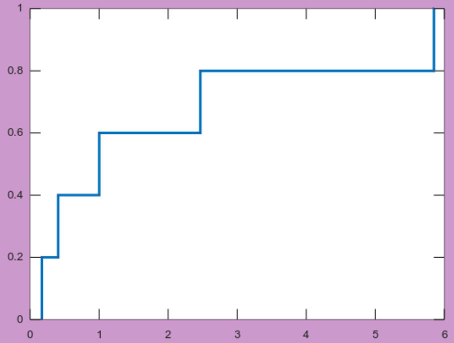
$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix},$$

for $N \to \infty$.

Eigenvalue distribution

Example: $n = 1,\ A = \frac{1}{2},\ B = C = 1,\ \sigma = 1,\ N = 5$

0.17
0.41
1.00
2.46
5.85



Eigenvalue distribution function of a symmetric $N \times N$ matrix

$$D_N : \mathbb{R} \to [0,1], \quad D_N(x) := \frac{\#\text{eigenvalues} \leq x}{N}.$$

- Compare: Empirical Distribution Function.

Eigenvalue distribution

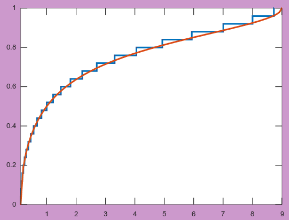
Eigenvalue distribution function of a symmetric $N \times N$ matrix

$$D_N : \mathbb{R} \to [0,1], \quad D_N(x) := \frac{\#\text{eigenvalues} \le x}{N}.$$

EDF of a sequence (T_N) where T_N is a symmetric $N \times N$ matrix

$$D : \mathbb{R} \to [0,1], \qquad D(x) = \lim_{N\to\infty} D_N(x).$$

Example: $n = 1,\ A = \frac{1}{2},\ B = C = 1,\ \sigma = 1,\ N = 25$



- Compare: Cumulative Distribution Function.

Recap and goal

ARMA Model

$$x_{k+1} = Ax_k + Bw_k, \quad x_0 = 0, \quad y_k = Cx_k + w_k.$$

- $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times 1}$, $C \in \mathbb{R}^{1\times n}$.
- w_k i.i.d. random variables, zero mean, variance σ^2.

Of interest: the *eigenvalue distribution function* of the sequence of covariance matrices of

$$\begin{bmatrix} y_0 \\ y_1 \\ \vdots \\ y_{N-1} \end{bmatrix}.$$

Goal

Understand the eigenvalue distribution function of the sequence of covariance matrices in terms of A, B, C and σ.

How to find the EDF for an ARMA Covariance matrix?

Recall

State space description:

$$x_{k+1} = Ax_k + Bw_k, \quad x_0 = 0, \quad y_k = Cx_k + w_k.$$

Discrete convolution description:

$$y_k = \sum_{j=0}^{k} h_j w_{k-j}, \qquad h_j = \begin{cases} I & j = 0 \\ CA^{j-1}B & j > 0. \end{cases}$$

Transfer function and frequency response

$$G(z) = \sum_{j=0}^{\infty} h_j z^j = I + Cz(I - zA)^{-1}B, \qquad \phi(t) = |G(e^{it})|^2.$$

For our example ($A = \frac{1}{2},\ B = C = 1$):

$$G(z) = \frac{2+z}{2-z}, \qquad \phi(t) = \frac{5 + 4\cos(t)}{5 - 4\cos(t)}.$$

How to find the EDF for an ARMA Covariance matrix?

Recall

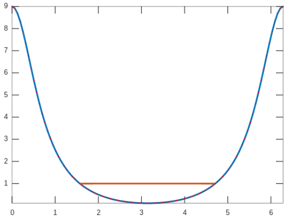
$$G(z) = \sum_{j=0}^{\infty} h_j z^j = I + Cz(I - zA)^{-1}B, \qquad \phi(t) = |G(\mathrm{e}^{it})|^2.$$

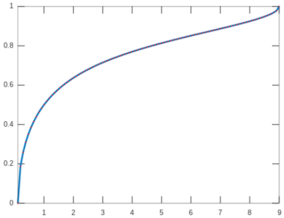
For our example ($A = \frac{1}{2},\ B = C = 1$):

$$G(z) = \frac{2+z}{2-z}, \qquad \phi(t) = \frac{5 + 4\cos(t)}{5 - 4\cos(t)}.$$

Then

$$D(x) = \frac{1}{2\pi}\text{measure}\{t \in [0, 2\pi] : \phi(t) \le x\},$$





$$D : [1/9, 9] \to [0, 1], \qquad D(x) = 1 - \frac{1}{\pi}\arccos\left(\frac{5(x-1)}{4(x+1)}\right).$$

Moral of the story

If we understand the frequency response function,
then we understand the eigenvalue distribution function.

How to prove this?

Gábor Szegö (1895-1985)

- The covariance matrix sequence ≈ symmetric Toeplitz matrix sequence.
- This symmetric Toeplitz matrix sequence ≈ symmetric circulant matrix sequence.
- For this symmetric circulant matrix sequence the eigenvalues are easily calculated.

- $(S_N)_N \approx (T_N)_N$ if $\lim_{N\to\infty} \frac{1}{N}\|S_N - T_N\|_F = 0$;
- $(S_N)_N \approx (T_N)_N$ implies $D_{(S_N)} = D_{(T_N)}$.

What do frequency response functions look like?

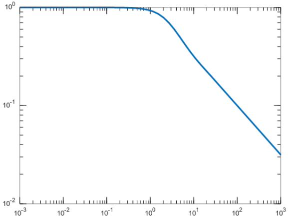
Heat equation:

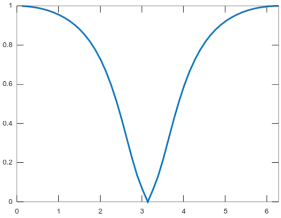
$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial \xi^2}, \quad u(t,0) = 0, \quad \frac{\partial u}{\partial \xi}(t,1) = w(t), \quad y(t) = u(t,1).$$

Continuous-time and discrete-time transfer functions:

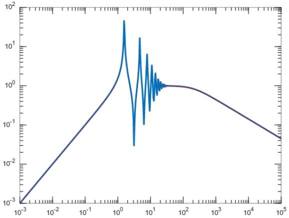
$$G_c(s) = \frac{\tanh\sqrt{s}}{\sqrt{s}}, \qquad G(z) = G_c\left(\frac{z-1}{z+1}\right).$$

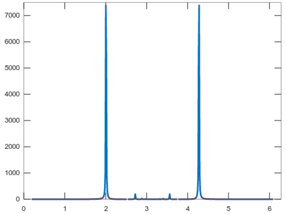
Continuous Bode plot of G_c and discrete time plot of ϕ:



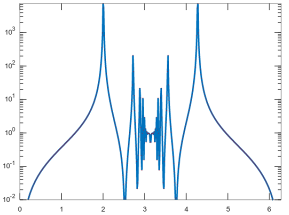


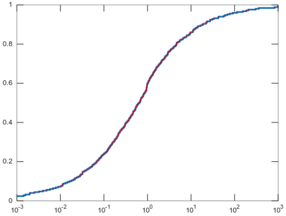
Continuous Bode plot of G_c and discrete time plots of ϕ:





Log scale for ϕ and (data based) EDF





Moral of the story

If we understand the frequency response function,
then we understand the eigenvalue distribution function.
In some cases we understand the frequency response function.