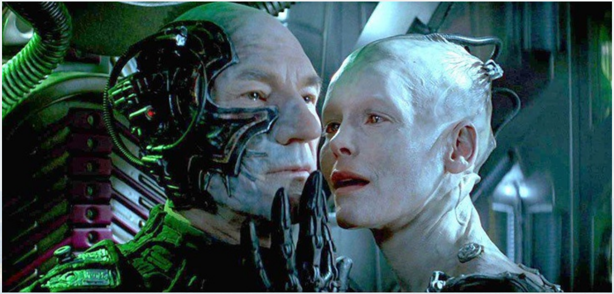

What is Data Assimilation?

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ITT7 (SAMBa Integrative Think Tank)

What is data assimilation?

Definition

Data assimilation is a way of combining observations with a numerical model, to create a better guess of the true state.

Used in

- Weather forecasting
- GPS navigation
- Medical imaging
- Seismology
- and many more places.

Data assimilation setting

We have a state $x_k \in \mathbb{R}^n$ at time t_k.

A numerical model $\mathcal{M}_k$: $\mathbb{R}^n \to \mathbb{R}^n$ such that

$$x_{k+1} = \mathcal{M}_k(x_k) + \eta_k.$$

A background estimate x^b of the truth x_0^*,

$$x_0^* = x^b + e_0.$$

And observations $y_k \in \mathbb{R}^{p_k}$ of the state:

$$y_k = \mathcal{H}_k(x_k^*) + \epsilon_k,$$

where $\mathcal{H}_k : \mathbb{R}^n \to \mathbb{R}^{p_k}$ is an observation operator.

The errors η_k, e_0, ϵ_k are Gaussian with zero mean and covariances $Q_k \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times n}$, $R_k \in \mathbb{R}^{p_k\times p_k}$ respectively.

The aim of data assimilation

The goal is to find an estimate x_0 to the true initial condition x_0^* which minimises

1. the distance to our background state x_0^b,
2. the distances between the **state trajectory** x of this initial state, and our observations y.

Different approaches

Different approaches:

- Sequential Data Assimilation
- Variational Data Assimilation

Kalman Filter

State and error covariance analysis (corrector step)

Given previous forecast x_i^F and error covariance matrix B_i^F

$$
\begin{aligned}
\text{State estimate} \quad x_i^A &= x_i^F + K_i(y_i - H_i x_i^F) \\
\text{where} \quad K_i &= B_i^F H_i^T (H_i B_i^F H_i^T + R_i)^{-1} \\
&\quad \text{(Kalman gain)} \\
\text{Error covariance estimate} \quad B_i^A &= (I - K_i H_i) B_i^F
\end{aligned}
$$

State and error covariance forecast (predictor step)

Given state x_i^A and error covariance matrix B_i^A

$$
\begin{aligned}
\text{State forecast} \quad x_{i+1}^F &= M_{i+1,i} x_i^A \\
\text{Error covariance forecast} \quad B_{i+1}^F &= M_{i+1,i} B_i^A M_{i+1,i}^T + Q_i
\end{aligned}
$$

Different approaches

Different approaches:

- Sequential Data Assimilation
- Variational Data Assimilation

Variational data assimilation

Variational data assimilation

We wish to find an estimate x_0 to x_0^* which minimises

1. the distance to our background state x_0^b,
2. the distances between the state trajectory x of this initial state, and our observations y.

In variational data assimilation, we introduce a cost function J and attempt to minimise that.

$$J(x_0) = \underbrace{\|x_0 - x_0^b\|}_{J_b} + \underbrace{\|y - \mathcal{H}(x)\|}_{J_o}.$$

Variational data assimilation

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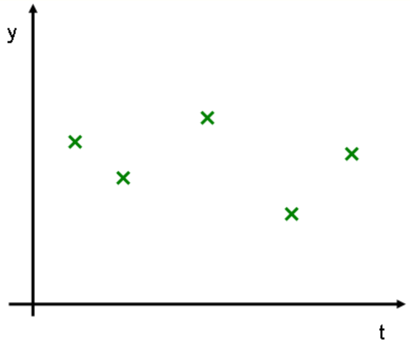
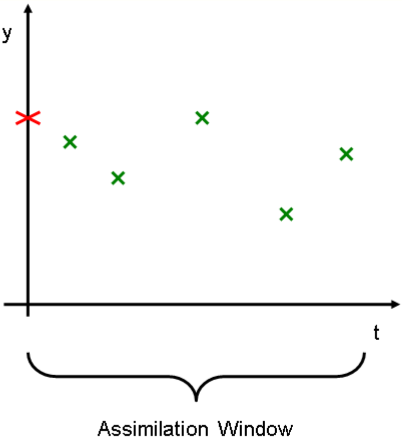


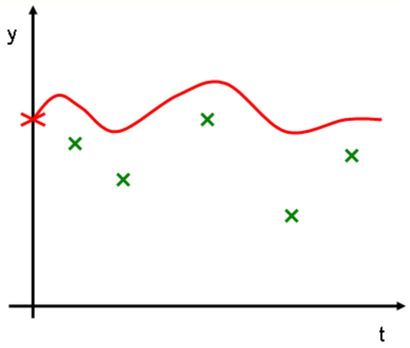
- Take observations *y* of the true dynamical system.

Variational data assimilation



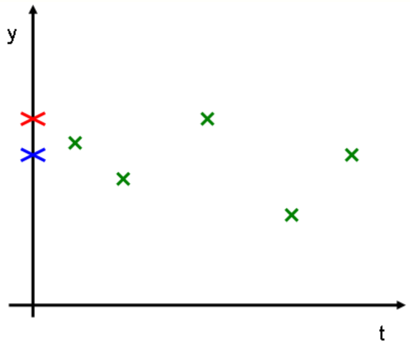
- Estimate the initial condition x_0^b, for the numerical model $x_{i+1} = M_{i+1,i}(x_i)$, simulating the true dynamical system.

Variational data assimilation

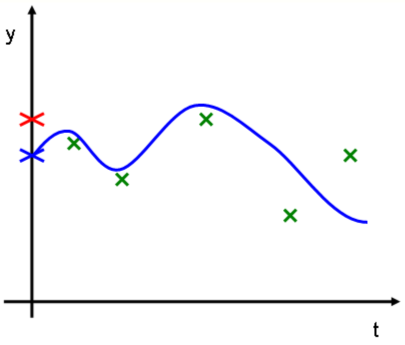


- Run the numerical model using the estimated initial condition.

Variational data assimilation

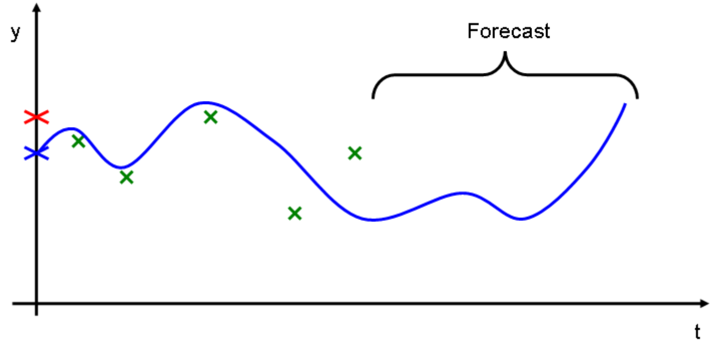


- Minimise cost function $J(x_0)$ to find an improved initial condition x^A, i.e.: $\nabla J(x^A) = 0$.



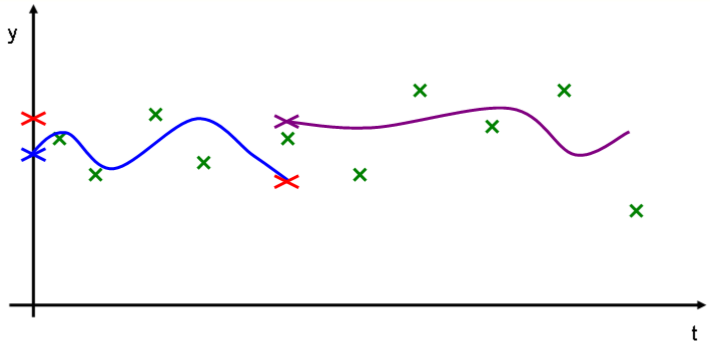
- The numerical model is run using x^A as an initial condition.

Variational data assimilation



- The simulation is continued to create a forecast for the true dynamical system.

Variational data assimilation



- The process is repeated for new observations.
- This is called cyclic 4DVar.

(Plots from Melina Freitag)

$$J(x_0) = \underbrace{\|x_0 - x_0^b\|}_{J_b} + \underbrace{\|y - \mathcal{H}(x)\|}_{J_o}.$$

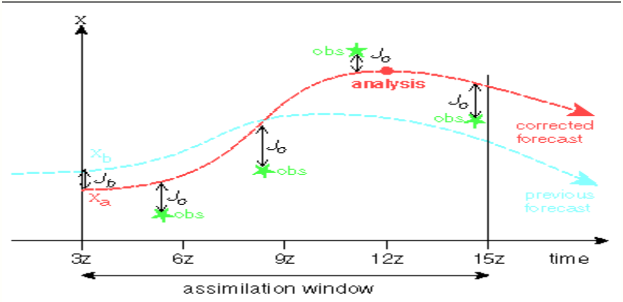


Figure 1: Copyright:ECMWF

Variational data assimilation

$$J(x_0) = \underbrace{\|x_0 - x_0^b\|}_{J_b} + \underbrace{\|y - \mathcal{H}(x)\|}_{J_o}.$$

Earlier we saw that

$$x_0^* = x_0^b + e_0,$$

e_0 Gaussian with zero mean and covariance $B \in \mathbb{R}^{n\times n}$.

So we take

$$\begin{aligned} J_b &= \frac{1}{2}\|x_0 - x_0^b\|_B^2 \\ &= \frac{1}{2}(x_0 - x_0^b)^T B^{-1}(x_0 - x_0^b). \end{aligned}$$

Variational data assimilation

$$J(x_0) = \underbrace{\|x_0 - x_0^b\|}_{J_b} + \underbrace{\|y - \mathcal{H}(x)\|}_{J_o}.$$

With one observation at x_0,

$$\begin{aligned} J_o &= \frac{1}{2}\|y_0 - \mathcal{H}_0(x_0)\|_{R_0}^2 \\ &= \frac{1}{2}(y_0 - \mathcal{H}_0(x_0))^T R_0^{-1}(y_0 - \mathcal{H}_0(x_0)). \end{aligned}$$

Variational data assimilation

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With 4D-Var we consider all observations in our cost function to obtain instead:

$$\begin{aligned} J_o &= \frac{1}{2}\sum_{k=0}^{N}\|y_k - \mathcal{H}_k(x_k)\|_{R_k}^2 \\ &= \frac{1}{2}\sum_{k=0}^{N}(y_k - \mathcal{H}_k(x_k))^T R_k^{-1}(y_k - \mathcal{H}_k(x_k)) \end{aligned}$$

4D-Var cost function

4D-Var cost function

$$J(x_0) = \underbrace{\frac{1}{2}(x_0 - x_0^b)^T B^{-1} (x_0 - x_0^b)}_{J_b}$$

$$+ \underbrace{\frac{1}{2}\sum_{k=0}^{N}(y_k - \mathcal{H}_k(x_k))^T R_k^{-1} (y_k - \mathcal{H}_k(x_k))}_{J_o}\,.$$

4D-Var cost function

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$$+ \underbrace{\frac{1}{2}\sum_{k=0}^{N}(y_k - \mathcal{H}_k(\mathcal{M}_{k,0}(x_0)))^T R_k^{-1}(y_k - \mathcal{H}_k(\mathcal{M}_{k,0}(x_0)))}_{J_o}.$$

Where $\mathcal{M}_{k,0}(x_0) = \mathcal{M}_k(\mathcal{M}_{k-1}(\cdots \mathcal{M}_1(x_0))) = x_k$

4D-Var cost function

4D-Var cost function

$$J(x_0) = \frac{1}{2}(x_0 - x_0^b)^T B^{-1}(x_0 - x_0^b) + \frac{1}{2}\sum_{k=0}^{N}(y_k - \mathcal{H}_k(\mathcal{M}_{k,0}(x_0)))^T R_k^{-1}(y_k - \mathcal{H}_k(\mathcal{M}_{k,0}(x_0))).$$

4D-Var Cost function gradient

$$\nabla J(x_0) = B^{-1}(x_0 - x_0^b) - \sum_{k=0}^{N}(H_k M_{k,0})^T R_k^{-1}(y_k - \mathcal{H}_k((\mathcal{M}_{k,0}(x_0)))$$

where H_k and $M_{k,0}$ are the Jacobians of $\mathcal{H}_k$ and $\mathcal{M}_{k,0}$.

Weak 4D-Var

Weak 4D-Var cost function

$$J(x) = \underbrace{\frac{1}{2}(x_0 - x_0^b)^T B^{-1}(x_0 - x_0^b)}_{J_b}$$

$$+ \underbrace{\frac{1}{2}\sum_{k=0}^{N}(y_k - \mathcal{H}_k(x_k))^T R_k^{-1}(y_k - \mathcal{H}_k(x_k))}_{J_o}$$

$$+ \underbrace{\frac{1}{2}\sum_{k=1}^{N}(x_k - \mathcal{M}_k(x_{k-1}))^T Q_k^{-1}(x_k - \mathcal{M}_k(x_{k-1}))}_{J_q}.$$

$$x = \left[x_0^T, x_1^T, \ldots, x_N^T\right]^T.$$

Incremental 4D-Var

Incremental 4D-Var

Operationally, Incremental 4D-Var is used.

- This is a form of Gauss-Newton iteration, with
- linearised quadratic cost function $\tilde{J}(\delta x)$.

$$\tilde{J}(\delta x^{(\ell)}) = \frac{1}{2}(\delta x_0^{(\ell)} - b_0^{(\ell)})^T B^{-1}(\delta x_0^{(\ell)} - b_0^{(\ell)}) + \frac{1}{2}\sum_{i=0}^{N}(d_i^{(\ell)} - H_i \delta x_i^{(\ell)})^T R_i^{-1}(d_i^{(\ell)} - H_i \delta x_i^{(\ell)}).$$

- Increment at iterate ℓ,

$$\delta x_0^{(\ell)} = x_0^{(\ell+1)} - x_0^{(\ell)}.$$

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- Increment at iterate ℓ,

$$\delta x_0^{(\ell)} = x_0^{(\ell+1)} - x_0^{(\ell)}.$$

You can form it into

$$\mathcal{A}x = b$$

More information

For more information and references:

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Thank you for listening.
Any Questions?