Some mathematical background for seismic imaging

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Seismic inversion - brief overview

- Full Waveform Inversion (FWI): reconstructs the earth's subsurface properties from local measurements of a seismic wavefield

- FWI aims to interpret data from all seismic events (including complicated events like wave diffraction, missed by some approximate methods)

- Minimises the "misfit" between numerically predicted and physically recorded data (see Evren's talk for more)

- The optimisation cycle:
  - Make a guess of the subsurface properties ("the model")
  - Solve a "forward problem" (usually a linear hyperbolic PDE) to predict the wavefield
  - Apply one step of a minimisation algorithm to obtain a better model
  - Iterate the process

Mathematical set-up

Find the model m which minimses the misfit function :

$$f(m) = \frac{1}{2} \sum_{s=1}^{N_s} \sum_{\ell=1}^{N_\omega} \| R_s u_s(m, \omega_\ell) - d_s(\omega_\ell) \|^2$$

- ω_ℓ are the relevant frequencies
- The wavefield $u_s(m, \omega_\ell)$ is the solution of the forward problem with source ϕ_s (the“shot”) frequency ω_ℓ and current model m.
- $d_s(\omega_\ell)$ are the observations (data) made at the sensors
- R_s represents the evaluation of the wavefield at the point where the observation was made
- Forward problem is of the form

$$A(m(x), \omega) u(x, \omega) = \phi(x, \omega)$$

Linear in u, nonlinear in m.

Marine Seismic

In marine seismic data collected by sensors on streamers towed by a boat. This is noisy - see Evren's talk

Waves produced by "shots" from the same boat.

A nice review of the minimisation process:
Métivier, Brossier, Operto, Virieux Full Waveform Inversion and the Truncated Newton Method - SIAM Review vol 59 pp153-195 (2017)

Seismic exploration important in earthquake prediction, detecting hydrocarbon deposits , understanding water tables etc etc.

Survey configuration

Marine seismic

The forward model - elastodynamics

$$\rho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \mathrm{div} \mathbf{C} \boldsymbol{\varepsilon}(\mathbf{u}) \;=\; \mathbf{f}$$

$\mathbf{u}$ = displacement $\quad$ $\boldsymbol{\varepsilon}$ = strain trensor

ρ = density $\quad$ $\mathbf{C}$ = stiffness tensor

$\mathbf{u}$ is the output

Model m (to be reconstructed): elasticity parameters λ, μ and density ρ.

Ignoring shear waves gives acoustic approximation

$$\frac{1}{\rho c^2} \frac{\partial^2 u}{\partial t^2} - \mathrm{div} \left(\frac{1}{\rho} \nabla u \right) \;=\; \mathrm{div} \left(\frac{1}{\rho} \mathbf{f} \right)$$

c = wave speed, ρ = density

The forward model - frequency domain

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{(setting density} \quad \rho = 1)$$

Frequency domain (Fourier Transform in time):

$$-\left(\frac{\omega}{c}\right)^2 u - \Delta u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.

The forward model - frequency domain

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{or its elastic variant}$$

Frequency domain:

$$-\left(\frac{\omega L}{c}\right)^2 u - \Delta u = f, \qquad \omega = \quad \text{frequency}$$

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Large domain of characteristic length L.

Helmholtz equation - effectively high frequency

The forward model - frequency domain

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} - \Delta u = f \quad \text{or its elastic variant}$$

Frequency domain:

$$-\Delta u - \left(\frac{\omega L}{c}\right)^2 u = f, \qquad \omega = \quad \text{frequency}$$

solve for u with approximate c.

Large domain of characteristic length L.

Helmholtz equation - effectively high frequency

Another problem: signal in seismic exploration is not narrow band - so many ω potentially needed

Time domain solvers

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

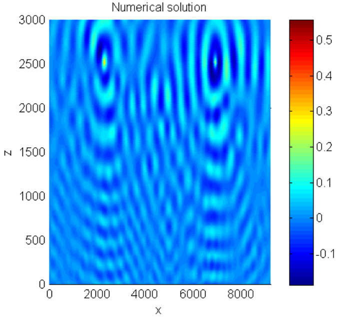
initial conditions on u and $\partial u/\partial t$ at $t = 0$

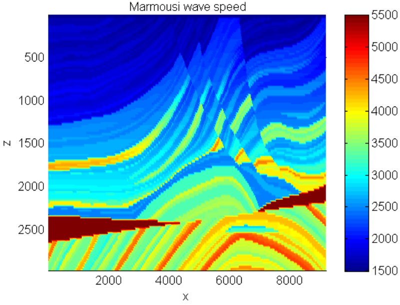
FDTD ("leapfrog scheme" - second order):

$$\frac{1}{(c_j^n)^2}\frac{U_j^{n+1} - 2U_j^n + U_j^{n-1}}{(\Delta t)^2} = \frac{U_{j-1}^n - 2U_j^n + U_{j-1}^n}{(\Delta x)^2}$$

Stability constraint: $\dfrac{c\Delta t}{\Delta x} \leq 1$

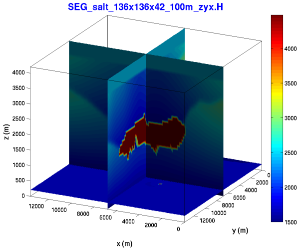
Marmousi Model Problem

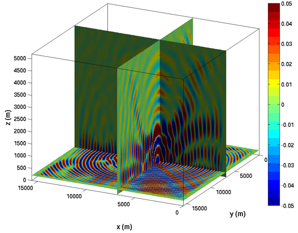




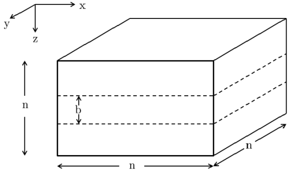
A 3D model problem

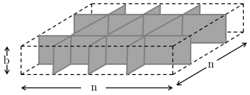
3D SEG Salt model





“Domain decomposition” method for the frequency domain problem





P. Childs, IGG, J.D. Shanks, 2015

Many interesting mathematical and statistical problems, e.g.

- stable data acquisition (Evren's talk)
- Inversion process: good starting guesses
- Inversion process: avoid cycle skipping
- Inversion process: spurious minima of misfit fuction
- Inversion process: Fast forward solvers
- Dealing with heterogeneity (trapped waves)
- Dealing with uncertainty: small fluctuations in parameters of the wave equation can have a large effect:
- $-\Delta u - (\omega/c)^2 u$ very different from $-\nabla.(c\nabla u)$