Branching Particle Linear Semi-group
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Neutron Transport Equation

Andreas E. Kyprianou

Randomly scattered particles

- **Position and velocity:** Think of a particle which is described by the pair $(r_t, \Omega_t) \in \mathbb{R}^3 \times \mathbb{S}_2$ - position and velocity.
- **Randomised scattering:** At times of a Poisson process with rate σ_S, a particle independently picks a new velocity, Ω', on $\mathbb{S}_2$ according to some probability density which depends on its existing velocity, Ω:

$$\mathbb{P}(\text{new velocity } = \Omega' | \text{old velocity } = \Omega) = \Theta_S(\Omega, \Omega')\mathrm{d}\Omega'$$

- **Straight lines:** Between switching velocities, the particle moves in a straight line with velocity Ω, its current velocity:

$$\frac{\mathrm{d}}{\mathrm{d}t} r_t = \Omega$$

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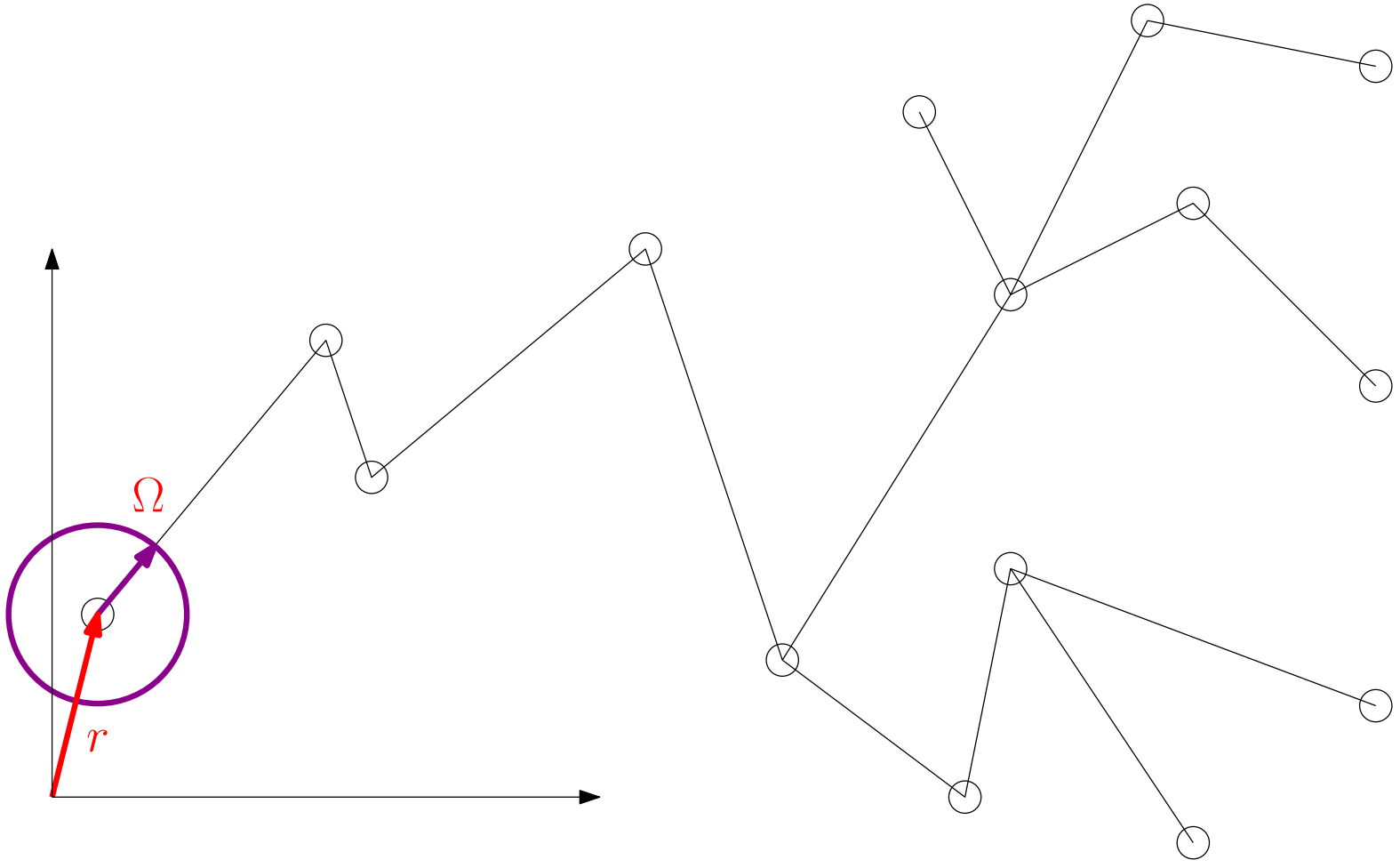
Ω
r

Generators

- **Stochastic calculus:** Because the number of changes of velocity up to time t is random, the pair (r_t, Ω_t) (current position and current velocity at time t) is random, but Markovian.
- The randomness in the process (r_t, Ω_t), $t \geq 0$, is captured through the semi-group equation.
- For $(r, \Omega) \in \mathbb{R}^3 \times \mathbb{S}_2$ write

$$u_g(r, \Omega, t) = \mathbb{E}_{(r,\Omega)}[g(r_t, \Omega_t)]$$

- Then

$$\begin{aligned} \frac{\partial}{\partial t} u_g(r, \Omega, t) &= \Omega \cdot \nabla u_g(r, \Omega, t) + \sigma_S \int_{\mathbb{S}_2} u_g(r, \Omega', t) \Theta_S(\Omega, \Omega') \mathrm{d}\Omega' - \sigma_S u_g(r, \Omega, t) \\ &=: \Omega \cdot \nabla g + \mathcal{S} g. \end{aligned}$$

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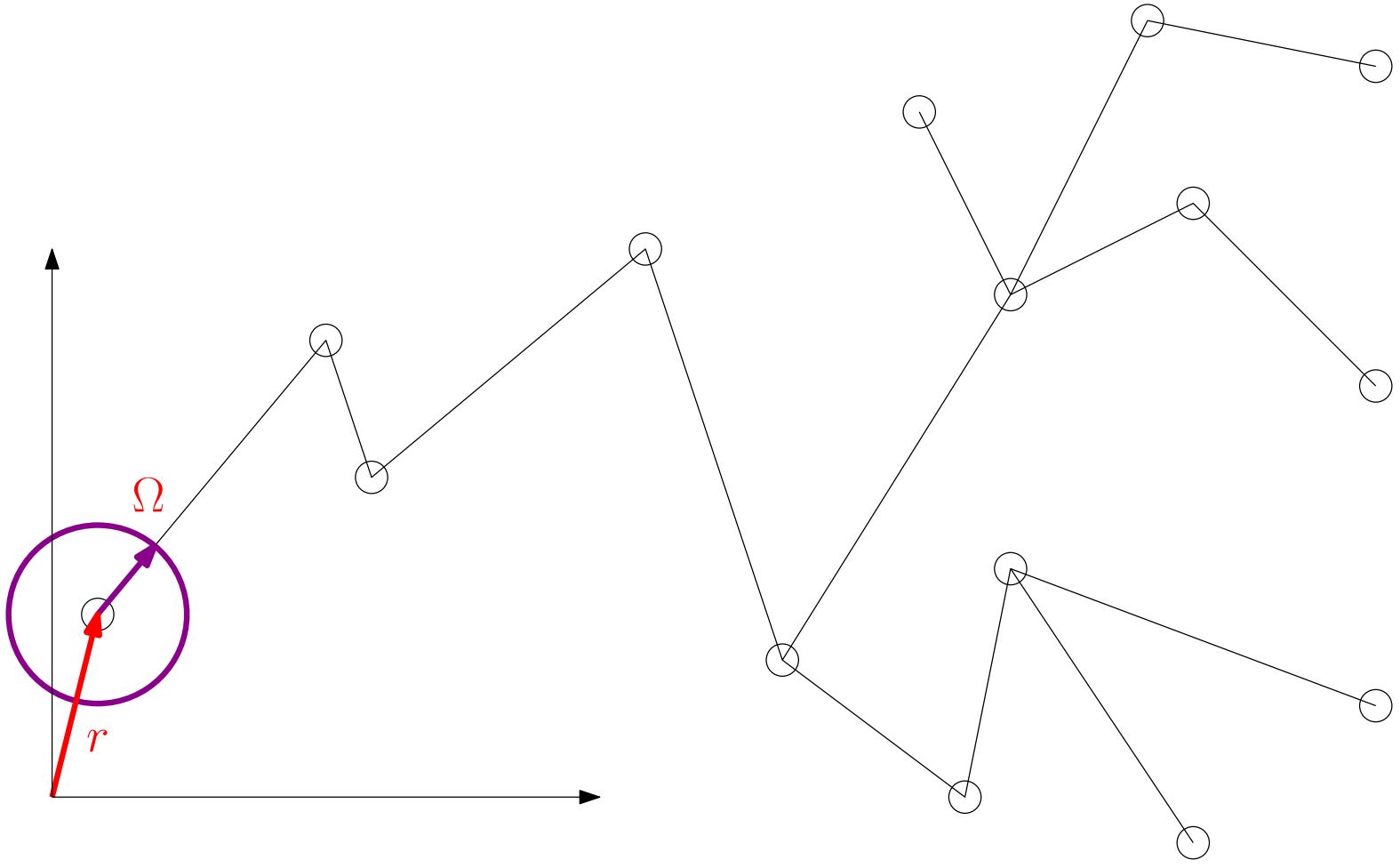
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Branching Markov process

- **Initial ancsetor:** A particle starts at $x \in \mathbb{R}^3$ and velocity $\Omega \in \mathbb{S}_2$ and moves randomly with infinitesimal generator $L = \Omega \cdot \nabla + \mathcal{S}$.
- **Branching:** A particle dies and branches at rate σ_F, but now produces a mean number of offspring, say m, at its spatial point of death. Each offspring is assigned a new, independent, velocity which is chosen from the probability density $\Theta_F(\Omega, \Omega')$, meaning

$$\mathbb{P}(\text{offspring is assigned velocity } \Omega' | \text{ parent has velocity } \Omega) = \Theta_F(\Omega, \Omega')\mathrm{d}\Omega'$$

- **State space:** This is a Markov process in the space of spatial counting measures. A typical realisation of the process takes the form

$$Z_t(\cdot) = \sum_{i=1}^{n_t} \delta_{(r_i(t), \Omega_i(t))}(\cdot)$$

where n_t are the number of particles at time t and $r_i(t)$ are their positions and $\Omega_i(t)$ are their velocities.

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Branching semi-group

- **Function of Z:** Because Z is a counting measure, the natural 'function f of Z' is represented through

$$\langle f, Z_t \rangle := \sum_{i=1}^{n_t} f(r_i(t), \Omega_i(t)).$$

- **Nonetheless:** With $u_f(r, \Omega, t) = \mathbb{E}_{\delta_{(r,\Omega)}}[\langle f, Z_t \rangle]$ it turns out that $\partial u_f / \partial t = L u_f$, becomes

$$\begin{aligned} L u_f &= \Omega \cdot \nabla u_f + \sigma_S \int_{\mathbb{S}_2} u_f(r, \Omega', t) \Theta_S(\Omega, \Omega') \mathrm{d}\Omega' \\ &\quad + m\sigma_F \int_{\mathbb{S}_2} u_f(r, \Omega', t) \Theta_F(\Omega, \Omega') \mathrm{d}\Omega' - (\sigma_F + \sigma_S) u_f \\ &= \Omega \cdot \nabla u_f + \mathcal{S} u_f + \mathcal{F} u_f \end{aligned}$$

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Backward vs Forwards equation

- **Backwards equations:** Semi-group equations for Markov processes are backward equations. That is, they describe the evolution in t through the spatial variables pertaining to the initial configuration of the system.
- **Forwards equations:** The forwards equations are generally much more difficult to write down. One describes the evolution of the process in time in terms of the spatial variables pertaining to the configuration of the system at time t (in which case one needs to condition on the configuration at time t taking certain values).
- **Dualty or adjoint:** The forward equations are the same as the backwards albeit that the operator L is replaced by its dual (for the probabilist) L^* (adjoint for the analyst).
- **Spectral properties:** The spectral properties of the forward and backwards equations are easily related (modulo taking care of left- and right- eigen functions through duality) from the point of view of the lead eigen value and associated positive harmonic functions.

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Monte-Carlo simulation

- The leading eigenvalue associated to the operator L is the object of interest.
- This is the growth rate of the branching process.
- For example, with appropriate assumptions, we would expect a Perron-Frobenius-type result:

$$\lim_{t\to\infty} \mathrm{e}^{-\lambda t}\mathbb{E}_{(r,\Omega)}[\langle f, Z_t\rangle] = \varphi(r,\Omega)\int_{\mathbb{R}^3\times\mathbb{S}_2}[f(r,\Omega)\tilde{\varphi}(r,\Omega)]\mathrm{d}r\mathrm{d}\Omega,$$

  where λ is the lead eigenvalue associated to L with left and right eigenfunctions $\tilde{\varphi}$ and φ respectively.
- Boundary conditions have natural interpretations for the probabilistic setting as much as they do for the mechanics-analytic setting - and they will play out equally in growth results such as the one above.

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