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Discontinuous Galerkin methods for first-order hyperbolic problems

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In this paper we consider discontinuous Galerkin (DG) finite element approximations of a model scalar linear hyperbolic equation. We show that in order to ensure continuous stabilization of the method it suffices to add a jump-penalty-term to the discretized equation. In particular, the method does not require upwinding in the usual sense. For a specific value of the penalty parameter we recover the classical discontinuous Galerkin method with upwind numerical flux function. More generally, using discontinuous piecewise polynomials of degree k, the familiar optimal $\mathcal{O}(h^{k+1/2})$ error estimate is proved for any value of the penalty parameter. As precisely the same jump -term is used for the purposes of stabilizing DG approximations of advection-diffusion operators, the discretization proposed here can simplify the construction of discontinuous Galerkin finite element approximations of advection-diffusion problems. Moreover, the use of the jump-stabilization makes the analysis simpler and more elegant.

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1 Introduction

Let Ω be a bounded polygonal domain in $\mathbf{R}^2$, and let the advective velocity field $\boldsymbol{\beta} = (\beta_1,\, \beta_2)^{\mathrm{T}}$ be a vector-valued function defined on $\bar{\Omega}$ with $\beta_i \in \mathrm{C}^1(\bar{\Omega}),\ i = 1, 2$. We define the *inflow* and *outflow* parts of $\Gamma = \partial\Omega$ in the usual fashion:

$$\begin{aligned} \Gamma_- &= \{x \in \Gamma :\ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) < 0\} = \text{inflow}, \\ \Gamma_+ &= \{x \in \Gamma :\ \boldsymbol{\beta}(x) \cdot \mathbf{n}(x) > 0\} = \text{outflow}, \end{aligned}$$

where $\mathbf{n}(x)$ denotes the unit outward normal vector to Γ at $x \in \Gamma$.

Let $\gamma \in \mathrm{C}(\bar{\Omega}),\ f \in \mathrm{L}^2(\Omega),\ g \in \mathrm{L}^2(\Gamma_-)$. Consider the hyperbolic boundary value problem

$$\begin{aligned} \mathcal{L}u \equiv \operatorname{div}(\boldsymbol{\beta} u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma_-. \end{aligned} \tag{1.1}$$

We shall assume the existence of a positive constant c_0 such that

$$\gamma(x) + \frac{1}{2}\operatorname{div}\beta(x) \geq c_0 \qquad \text{for all } x \in \bar{\Omega}. \tag{1.2}$$

The discontinuous Galerkin approximation of (1.1) consists of choosing the space V_h^k of discontinuous piecewise polynomials of degree $k \geq 0$ and seeking $u_h \in V_h^k$ such that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_T (-u_h\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u_h v_h)\,\mathrm{d}x + \sum_{e \not\subset \Gamma_-} \int_e \{\boldsymbol{\beta} u_h\}_u \cdot [\![\, v_h \,]\!]\,\mathrm{d}s \\ = \int_\Omega f v_h\,\mathrm{d}x - \sum_{e \subset \Gamma_-} \int_e (\boldsymbol{\beta} \cdot \mathbf{n})\, g\, v_h\,\mathrm{d}s, \qquad v_h \in V_h^k, \end{aligned} \tag{1.3}$$

where $\{\boldsymbol{\beta} u_h\}_u$ represents the *upwind value* of $\boldsymbol{\beta} u_h$ and, as usual, $[\![\, v_h \,]\!]$ denotes the *jump* of v_h across the edge of an element over which it is evaluated; the precise definition is given in the next section in (2.1) and (2.3).

Here we propose a slightly different stabilization of the problem. Instead of (1.3) we consider

$$\begin{aligned} &\sum_{T \in \mathcal{T}_h} \int_T (-u_h\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u_h v_h)\,\mathrm{d}x + \sum_{e \not\subset \Gamma_-} \int_e \{\boldsymbol{\beta} u_h\} \cdot [\![\, v_h \,]\!]\,\mathrm{d}s \\ &+ \sum_{e \not\subset \Gamma} \int_e c_e(s)\, [\![\, u_h \,]\!] \cdot [\![\, v_h \,]\!]\,\mathrm{d}s = \int_\Omega f v_h\,\mathrm{d}x - \sum_{e \subset \Gamma_-} \int_e (\boldsymbol{\beta} \cdot \mathbf{n})\, g\, v_h\,\mathrm{d}s, \quad v_h \in V_h^k, \end{aligned} \tag{1.4}$$

where $\{\boldsymbol{\beta} u_h\}$ is now the usual average (see (2.2)) and, for every internal edge e, we denoted by c_e a nonnegative function to be chosen (which, in practical implementations, could be defined as constant on e). For related ideas concerning least-squares-type stabilization in the context of discontinuous Galerkin methods, we refer to Section 5 of the paper [5].

We shall prove that, when the stabilization function is taken to be $c_e = |\boldsymbol{\beta} \cdot \mathbf{n}|/2$ then (1.4) collapses to the original discontinuous Galerkin method (1.3). This fact is

essentially known, and has already been used, for instance, by Cockburn and Shu [4] in selecting the numerical flux functions for LDG methods in order to reduce the stencil. However, the discontinuous Galerkin method (1.4) with jump-stabilization is stable more generally, whenever there exists a $\theta_0 > 0$ such that

$$c_e \geq \theta_0 \, |\boldsymbol{\beta} \cdot \mathbf{n}_e| \qquad \text{for each internal edge } e. \tag{1.5}$$

This approach, in our opinion, has several potential advantages. In the first place we have a way to tune-up the *amount of upwinding* that we are willing to use. Admittedly, this is a rather insignificant advantage for a problem as simple as our model problem (1.1); however, the technique can be relevant in more complicated situations: for instance, in the case of advection-diffusion equations where a certain amount of viscosity is present, possibly only in subsets of the computational domain, particularly when such subsets are unknown a priori, or change with time, or when it is necessary to alter the amount of local numerical dissipation in the course of an iterative procedure. In fact, if a diffusive term is present and is also approximated by means of a discontinuous Galerkin method, it is quite likely that a jump-penalty, identical or very similar to the present one, is already included into the diffusive part of the discretization, and we can therefore treat the two jump-penalty stabilizations together, both from the theoretical viewpoint as well as in the actual implementation of the method. Finally, we believe that the present way of dealing with upwinding provides a simpler and more elegant analysis even in the case when we take exactly $c_e \equiv |\boldsymbol{\beta} \cdot \mathbf{n}|/2$.

The paper is organized as follows. In the next section we formulate our hypotheses, and we derive the *unstabilized* discontinuous Galerkin method for our model problem. Then, in Section 3, we introduce the jump-stabilization and arrive at the ultimate form of our method. The consistency and the stability of the method are explored in Section 4, and the final a priori error estimates are proved in Section 5. In particular, in the case of discontinuous piecewise polynomial finite element approximations of degree $k \geq 0$, we prove an optimal error estimate of the form

$$\|u - u_h\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_h} \|c_e^{1/2} [\![\, u - u_h \,]\!]\|_{0,e}^2 \leq C h^{2k+1} \|u\|_{k+1,\Omega}^2, \tag{1.6}$$

familiar from the theory of stabilized finite element methods for first-order hyperbolic problems.

2 The discontinuous finite element approximation

Let $\mathcal{T}_h$ be a regular family of decompositions of Ω into triangles T; let h_T denote the diameter of T, and let $h = \max_{T \in \mathcal{T}_h} h_T$. In order to define a discontinuous finite element approximation of problem (1.1) we first need to introduce typical tools such as *jumps* and *averages* of scalar- and vector-valued functions across the edges of $\mathcal{T}_h$. Following the notation of [3], let e be an interior edge shared by elements T_1 and T_2. Define the unit normal vectors $\mathbf{n}^1$ and $\mathbf{n}^2$ on e pointing exterior to T_1 and T_2, respectively. For a

function φ, piecewise smooth on $\mathcal{T}_h$, with $\varphi^i := \varphi|_{T_i}$ we define

$$\{\varphi\} = \frac{1}{2}(\varphi^1 + \varphi^2), \quad [\![\, \varphi \,]\!] = \varphi^1 \mathbf{n}^1 + \varphi^2 \mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ, \tag{2.1}$$

where $\mathcal{E}_h^\circ$ is the set of interior edges e. For a vector-valued function $\boldsymbol{\tau}$, piecewise smooth on $\mathcal{T}_h$, with analogous meaning for $\boldsymbol{\tau}^1$ and $\boldsymbol{\tau}^2$, we define

$$\{\boldsymbol{\tau}\} = \frac{1}{2}(\boldsymbol{\tau}^1 + \boldsymbol{\tau}^2), \quad [\![\, \boldsymbol{\tau} \,]\!] = \boldsymbol{\tau}^1 \cdot \mathbf{n}^1 + \boldsymbol{\tau}^2 \cdot \mathbf{n}^2 \quad \text{on } e \in \mathcal{E}_h^\circ. \tag{2.2}$$

Notice that the jump $[\![\, \varphi \,]\!]$ of the scalar function φ across $e \in \mathcal{E}_h^\circ$ is a vector parallel with the normal to e, and the jump $[\![\, \boldsymbol{\tau} \,]\!]$ of the vector function $\boldsymbol{\tau}$ is a scalar quantity. The advantage of these definitions is that they do not depend on the ordering that is assigned to the elements T_i. For $e \in \mathcal{E}_h^\partial$, the set of boundary edges, we let

$$[\![\, \varphi \,]\!] = \varphi \mathbf{n}, \quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau} \quad \text{on } e \in \mathcal{E}_h^\partial. \tag{2.3}$$

We do not require either of the quantities $\{\varphi\}$ or $[\![\, \boldsymbol{\tau} \,]\!]$ on boundary edges, and leave them undefined there.

Next, with any integer $k \geq 0$ we associate the finite element space of discontinuous piecewise polynomial functions

$$V_h^k = \{v \in \mathrm{L}^2(\Omega) : \ v_{|T} \in \mathrm{P}_k(T) \quad T \in \mathcal{T}_h\}, \tag{2.4}$$

where, as usual, P_k is the space of polynomials of degree k or less. On multiplying equation (1.1) by a function $v_h \in V_h^k$ and integrating by parts, we get

$$\sum_{T \in \mathcal{T}_h} \Big(\int_T (-u\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h)\, \mathrm{d}x + \int_{\partial T} (\boldsymbol{\beta} \cdot \mathbf{n})\, u\, v_h\, \mathrm{d}s \Big) = \int_\Omega f v_h\, \mathrm{d}x. \tag{2.5}$$

Recall the following identity (see [3]) which holds for vectors $\boldsymbol{\tau}$ and scalars φ, piecewise smooth on $\mathcal{T}_h$:

$$\sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\tau} \cdot \mathbf{n})\, \varphi\, \mathrm{d}s = \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\tau}\} \cdot [\![\, \varphi \,]\!]\, \mathrm{d}s + \sum_{e \in \mathcal{E}_h^\circ} \int_e [\![\, \boldsymbol{\tau} \,]\!]\, \{\varphi\}\, \mathrm{d}s. \tag{2.6}$$

Now, from (2.6) with $\boldsymbol{\tau} = \boldsymbol{\beta} u$ and $\varphi = v_h$, since $[\![\, \boldsymbol{\beta} u \,]\!] = 0$ on internal edges, we deduce that

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\beta} \cdot \mathbf{n})\, u\, v_h\, \mathrm{d}s &= \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\beta} u\} \cdot [\![\, v_h \,]\!]\, \mathrm{d}s \\ &= \sum_{e \not\subset \Gamma_-} \int_e \{\boldsymbol{\beta} u\} \cdot [\![\, v_h \,]\!]\, \mathrm{d}s + \sum_{e \subset \Gamma_-} \int_e \boldsymbol{\beta} \cdot \mathbf{n}\, g\, v_h\, \mathrm{d}s. \end{aligned} \tag{2.7}$$

Setting

$$a_h(u, v_h) = \sum_{T \in \mathcal{T}_h} \int_T (-u\,(\boldsymbol{\beta} \cdot \nabla v_h) + \gamma u v_h)\, \mathrm{d}x, \tag{2.8}$$

$$b_h\,(u, v_h) = \sum_{e \not\subset \Gamma_-} \int_e \{\boldsymbol{\beta} u\} \cdot [\![\, v_h \,]\!]\, \mathrm{d}s, \tag{2.9}$$

$$(f, v_h) = \int_\Omega f v_h \,\mathrm{d}x, \quad \langle g, v_h \rangle = -\sum_{e \subset \Gamma_-} \int_e (\boldsymbol{\beta} \cdot \mathbf{n})\, g\, v_h \,\mathrm{d}s, \tag{2.10}$$

we then have that

$$a_h\,(u, v_h) + b_h\,(u, v_h) = (f, v_h) + \langle g, v_h \rangle, \qquad v_h \in V_h^k. \tag{2.11}$$

Inspired by the identity (2.11), we now define the unstabilized discrete problem as

$$\begin{cases} \text{find } u_h \in V_h^k : \\ a_h\,(u_h, v_h) + b_h\,(u_h, v_h) = (f, v_h) \; + \langle g, v_h \rangle, \qquad v_h \in V_h^k. \end{cases} \tag{2.12}$$

3 Stabilization with a jump-penalty

The formulation (2.12) is stable, but only in the $\mathrm{L}^2(\Omega)$-norm. The practical consequences of this can be detrimental: discontinuities in the boundary data may trigger large, nonphysical oscillations in the numerical solution. In order to design a formulation that is stable in a stronger norm, on every internal edge e, common to the triangles T^1 and T^2, one usually substitutes the average $\{\boldsymbol{\beta} u_h\}$ that appears in $b_h\,(u_h, v_h)$ (see (2.9)) by the *upwind value* of $\boldsymbol{\beta} u_h$, defined as

$$\{\boldsymbol{\beta} u_h\}_u = \begin{cases} \boldsymbol{\beta} u_h^1 & \text{if} \quad \boldsymbol{\beta} \cdot \mathbf{n}^1 > 0 \\ \boldsymbol{\beta} u_h^2 & \text{if} \quad \boldsymbol{\beta} \cdot \mathbf{n}^1 < 0 \\ \boldsymbol{\beta} \{u_h\} & \text{if} \quad \boldsymbol{\beta} \cdot \mathbf{n}^1 = 0. \end{cases} \tag{3.1}$$

As $\{\boldsymbol{\beta} u_h\}_u$, in $b_h\,(u_h, v_h)$, is multiplied by $[\![\, v_h \,]\!]$, which is directed as the normal $\mathbf{n}$ to e, it is clear that only the *normal component* of $\{\boldsymbol{\beta} u_h\}_u$ will feature in the scheme.

On the other hand, it is a simple matter to check that, if $\mathbf{n}$ is normal to e, then $\{\boldsymbol{\beta} u_h\}_u \cdot \mathbf{n}$ can also be written as

$$\{\boldsymbol{\beta} u_h\}_u \cdot \mathbf{n} = (\{\boldsymbol{\beta} u_h\} + c^* [\![\, u_h \,]\!]) \cdot \mathbf{n} \tag{3.2}$$

where $\{\boldsymbol{\beta} u_h\}$ is again the usual average and c^* is given by

$$c^* = \, |\boldsymbol{\beta} \cdot \mathbf{n}|/2. \tag{3.3}$$

Motivated by (3.2) and (3.3), we now hypothesize (and will prove later on) that we could still achieve stability in a norm that is stronger than $\|\cdot\|_{0,\Omega}$ if we replace the upwind average $\{\boldsymbol{\beta} u_h\}_u$ by $\{\boldsymbol{\beta} u_h\} + c_e [\![\, u_h \,]\!]$, provided that c_e is a nonnegative function chosen on each e in such a way that

$$c_e \geq \theta_0\, |\boldsymbol{\beta} \cdot \mathbf{n}|, \tag{3.4}$$

with θ_0 a positive constant independent of e and h. In order to make our proofs more elegant, it will be convenient to define c_e on the boundary $\partial\Omega$ as well by setting

$$c_e = \begin{cases} \boldsymbol{\beta} \cdot \mathbf{n}_\Omega/2 & \text{on } \Gamma_+ \\ -\boldsymbol{\beta} \cdot \mathbf{n}_\Omega/2 & \text{on } \Gamma_-, \end{cases} \tag{3.5}$$

where $\mathbf{n}_\Omega$ is the unit normal vector to $\partial\Omega$.

We emphasize here that, trivially, the conditions (3.4) and (3.5) imply that

$$c_e \geq 0 \qquad \text{for all } e \in \mathcal{E}_h. \tag{3.6}$$

We therefore assume that c_e satisfies (3.4) and we replace $b_h(u_h, v_h)$ in (2.12) by its *stabilized version*

$$b_h^{\mathrm{s}}(u_h, v_h) := b_h(u_h, v_h) + \sum_{e \in \mathcal{E}_h^\circ} \int_e c_e [\![u_h]\!] \cdot [\![v_h]\!] \, \mathrm{d}s, \tag{3.7}$$

thus obtaining the stabilized discrete problem

$$\begin{cases} \text{find } u_h \in V_h^k : \\ a_h(u_h, v_h) + b_h^{\mathrm{s}}(u_h, v_h) = (f, v_h) + \langle g, v_h \rangle, \qquad v_h \in V_h^k. \end{cases} \tag{3.8}$$

We note, in particular, that (3.8) includes, as a special case, the classical discontinuous Galerkin finite element method (see, [7] and [8]) with the numerical flux function taken as the upwind flux. Indeed, we can always choose $c_e = c^*$, given by (3.3).

We also note that in certain cases taking $\{\boldsymbol{\beta} u_h\} + c_e [\![u_h]\!]$ instead of the usual average corresponds to taking a *different type of average*. To see this, consider an internal edge e (common to the triangles T_1 and T_2) and assume in particular that the function c_e vanishes whenever $\boldsymbol{\beta} \cdot \mathbf{n}_e$ does. In this case we can define, for $i = 1, 2$,

$$\alpha^i = \begin{cases} \dfrac{1}{2} + \dfrac{c_e}{\boldsymbol{\beta} \cdot \mathbf{n}^i} & \text{if } \boldsymbol{\beta} \cdot \mathbf{n}^i \neq 0 \\ \dfrac{1}{2} & \text{if } \boldsymbol{\beta} \cdot \mathbf{n}^i = 0. \end{cases} \tag{3.9}$$

Clearly, $\alpha^1 + \alpha^2 = 1$. We can, therefore, define the *tilted average*

$$\{\boldsymbol{\beta} u_h\}_\alpha := \boldsymbol{\beta}(u_h^1 \alpha^1 + u_h^2 \alpha^2). \tag{3.10}$$

It then follows that, whenever $\mathbf{n}_e$ is orthogonal to e, we have that

$$(\{\boldsymbol{\beta} u_h\} + c_e [\![u_h]\!]) \cdot \mathbf{n}_e = \{\boldsymbol{\beta} u_h\}_\alpha \cdot \mathbf{n}_e. \tag{3.11}$$

Therefore our jump-stabilization could also be seen as using the tilted average (3.10) instead of the usual average.

4 Consistency and stability of the method

Consistency. Consistency follows immediately from (2.11) and (3.7) upon observing that, since $\boldsymbol{\beta} \cdot [\![u]\!] = 0$ on internal edges, $b_h^{\mathrm{s}}(u, v_h) \equiv b_h(u, v_h)$. In particular, Galerkin orthogonality holds:

$$a_h(u - u_h, v_h) + b_h^{\mathrm{s}}(u - u_h, v_h) = 0, \qquad v_h \in V_h^k. \tag{4.1}$$

Stability. We shall prove stability and error estimates in the norm

$$|||\cdot||| = \left(\|\cdot\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_h} \|c_e^{1/2} [\![\cdot]\!]\|_{0,e}^2 \right)^{1/2}. \tag{4.2}$$

The norm (4.2) is well defined on $H^1(\Omega) + V_h^k$, thanks to (3.6). After integration by parts, the definition (2.8) of $a_h(\cdot,\cdot)$ yields that

$$a_h(v_h, v_h) = \sum_{T \in \mathcal{T}_h} \int_T (\frac{1}{2}\mathrm{div}\boldsymbol{\beta} + \gamma)\, v_h^2 \,\mathrm{d}x - \frac{1}{2} \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\beta} \cdot \mathbf{n}) v_h^2 \,\mathrm{d}s. \tag{4.3}$$

Furthermore, from (2.6) with $\boldsymbol{\tau} = \boldsymbol{\beta}$ and $\varphi = v_h^2$, since $[\![\boldsymbol{\beta}]\!] = 0$, we have

$$\begin{aligned} \sum_{T \in \mathcal{T}_h} \int_{\partial T} (\boldsymbol{\beta} \cdot \mathbf{n}) v_h^2 \,\mathrm{d}s &= \sum_{e \in \mathcal{E}_h} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s \\ &= \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s + \sum_{e \in \mathcal{E}_h^\partial} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s. \end{aligned} \tag{4.4}$$

Combining (4.3) and (4.4), and splitting the contributions on $\mathcal{E}_h^\partial$ into their parts on Γ_+ and Γ_- we can then write

$$\begin{aligned} a_h(v_h, v_h) = &\sum_{T \in \mathcal{T}_h} \int_T (\frac{1}{2}\mathrm{div}\boldsymbol{\beta} + \gamma)\, v_h^2 \,\mathrm{d}x - \frac{1}{2} \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s \\ &- \frac{1}{2} \sum_{e \subset \Gamma_+} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s - \frac{1}{2} \sum_{e \subset \Gamma_-} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s. \end{aligned} \tag{4.5}$$

On the other hand, using the continuity of $\boldsymbol{\beta}$ and the definitions of averages and jumps (2.1), (2.2), and (2.3), we have that

$$\{\boldsymbol{\beta} v_h\} \cdot [\![v_h]\!] \equiv \{\boldsymbol{\beta}\} [\![v_h^2]\!]. \tag{4.6}$$

Formula (4.6) is straightforward, but crucial. Its validity allows a simpler treatment of the jump-stabilization (where the usual average still appears explicitly), compared with the classical upwind stabilization. Indeed from (4.6) we immediately have

$$\begin{aligned} b_h(v_h, v_h) &= \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\beta} v_h\} \cdot [\![v_h]\!] \,\mathrm{d}s + \sum_{e \subset \Gamma_+} \int_e \{\boldsymbol{\beta} v_h\} \cdot [\![v_h]\!] \,\mathrm{d}s \\ &= \frac{1}{2} \sum_{e \in \mathcal{E}_h^\circ} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s + \sum_{e \subset \Gamma_+} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \,\mathrm{d}s. \end{aligned} \tag{4.7}$$

Consequently, using (4.7) and (3.7) with (3.4) we obtain

$$\begin{aligned} b_h^{\mathrm{s}}\left(v_h, v_h\right) = & \frac{1}{2} \sum_{e \in \mathcal{E}_h^{\circ}} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \, \mathrm{d}s \\ & + \sum_{e \subset \Gamma_+} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \, \mathrm{d}s + \sum_{e \in \mathcal{E}_h^{\circ}} \int_e c_e |[\![v_h]\!]|^2 \, \mathrm{d}s. \end{aligned} \tag{4.8}$$

Finally, we note that the conditions on the boundary (3.5) and (2.3) imply that

$$c_e |[\![v_h]\!]|^2 = \begin{cases} \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!]/2 & \text{on} \quad \Gamma_+ \\ -\{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!]/2 & \text{on} \quad \Gamma_-. \end{cases} \tag{4.9}$$

Collecting (4.5) and (4.8), using (4.9), then (1.2) and (3.6), and finally (4.2), we obtain

$$\begin{aligned} a_h\left(v_h, v_h\right) & + b_h^{\mathrm{s}}\left(v_h, v_h\right) \\ & = \sum_{T \in \mathcal{T}_h} \int_T (\frac{1}{2} \mathrm{div} \boldsymbol{\beta} + \gamma) \, v_h^2 \, \mathrm{d}x - \frac{1}{2} \sum_{e \subset \Gamma_-} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \, \mathrm{d}s \\ & \quad + \frac{1}{2} \sum_{e \subset \Gamma_+} \int_e \{\boldsymbol{\beta}\} \cdot [\![v_h^2]\!] \, \mathrm{d}s + \sum_{e \in \mathcal{E}_h^{\circ}} \int_e c_e |[\![v_h]\!]|^2 \, \mathrm{d}s \\ & = \sum_{T \in \mathcal{T}_h} \int_T (\frac{1}{2} \mathrm{div} \boldsymbol{\beta} + \gamma) \, v_h^2 \, \mathrm{d}x + \sum_{e \in \mathcal{E}_h} \int_e c_e |[\![v_h]\!]|^2 \, \mathrm{d}s \\ & \geq c_0 \|v_h\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}_h} \|c_e^{1/2} [\![v_h]\!]\|_{0,e}^2 \geq C_S |||v_h|||^2, \end{aligned} \tag{4.10}$$

with $C_S := \min\{c_0, 1\}$.

5 A priori error estimates

In what follows C will denote a generic positive constant which depends only on the degree k of the polynomials, on the minimum angle of the mesh, and on the maximum value of the stabilizing functions c_e.

Let P_h^k be the L^2-projector onto V_h^k, for which the following standard estimate holds

$$\|u - P_h^k u\|_{r,p,T} \leq C h^{k+1-r} \|u\|_{k+1,p,T}, \quad r = 0, 1, \quad 1 \leq p \leq \infty, \quad T \in \mathcal{T}_h. \tag{5.1}$$

We recall the following trace inequality (see [1], [2]):

$$\|u - P_h^k u\|_{0,e}^2 \leq C(|e|^{-1} \|u - P_h^k u\|_{0,T}^2 + |e| |u - P_h^k u|_{1,T}^2), \tag{5.2}$$

with C a positive constant depending only on the minimum angle of T. Thus, from (5.1)-(5.2) we deduce that

$$\|u - P_h^k u\|_{0,e} \leq C h_T^{k+1/2} \|u\|_{k+1,T}, \qquad e \in \mathcal{E}_h. \tag{5.3}$$

Let us define

$$\eta = u - P_h^k u, \qquad \delta = u_h - P_h^k u.$$

Hence from (4.10) and (4.1) we have that

$$C_S \, |||\delta|||^2 \leq a_h(\delta,\delta) + b_h^{\mathrm{s}}(\delta,\delta) = a_h(\eta,\delta) + b_h^{\mathrm{s}}(\eta,\delta). \tag{5.4}$$

Next, observe that $\nabla\delta \in V_h^k$, so that, by the definition of the projector P_h^0,

$$\int_T (P_h^0 \boldsymbol{\beta} \cdot \nabla\delta)\, \eta \, \mathrm{d}x = 0. \tag{5.5}$$

Using this, together with (5.1), the Cauchy-Schwarz inequality and the inverse inequality, we deduce that

$$\begin{aligned} a_h(\eta,\delta) &= \sum_{T\in\mathcal{T}_h} \int_T (-\eta\,(\boldsymbol{\beta}\cdot\nabla\delta) + \gamma\delta\eta)\,\mathrm{d}x \\ &= \sum_{T\in\mathcal{T}_h} \int_T ((P_h^0\boldsymbol{\beta} - \boldsymbol{\beta})\cdot\nabla\delta)\,\eta + \gamma\delta\eta)\,\mathrm{d}x \\ &\leq C\Big(\sum_{T\in\mathcal{T}_h} \|P_h^0\boldsymbol{\beta} - \boldsymbol{\beta}\|_{0,\infty,T}|\delta|_{1,T} + \|\delta\|_{0,T}\Big)\|\eta\|_{0,T} \\ &\leq C\Big(\sum_{T\in\mathcal{T}_h} h_T|\boldsymbol{\beta}|_{1,\infty,T}h_T^{-1}\|\delta\|_{0,T} + \|\delta\|_{0,T}\Big)\|\eta\|_{0,T} \\ &\leq Ch^{k+1}\|\delta\|_{0,\Omega}\|u\|_{k+1,\Omega}. \end{aligned} \tag{5.6}$$

It remains to estimate $b_h^{\mathrm{s}}(\eta,\delta)$. For this purpose, we first make use of (3.4) and the continuity of $\boldsymbol{\beta}$ to obtain, for every edge e and for every unit vector $\mathbf{n}$ normal to e,

$$|\{\boldsymbol{\beta}\eta\}\cdot\mathbf{n}| = |\boldsymbol{\beta}\cdot\mathbf{n}||\{\eta\}| \leq \frac{c_e}{\theta_0}|\{\eta\}|. \tag{5.7}$$

Making use of the fact that $[\![\, \delta \,]\!]$ is also normal to e, and using (3.6) once again, we then have that

$$\int_e \{\boldsymbol{\beta}\eta\}\cdot[\![\, \delta \,]\!]\,\mathrm{d}s \leq \frac{1}{\theta_0}\|c_e^{1/2}\{\eta\}\|_{0,e}\;\|c_e^{1/2}[\![\, \delta \,]\!]\|_{0,e}. \tag{5.8}$$

Next,

$$\int_e c_e[\![\, \eta \,]\!]\cdot[\![\, \delta \,]\!]\,\mathrm{d}s \leq \|c_e^{1/2}[\![\, \eta \,]\!]\|_{0,e}\|c_e^{1/2}[\![\, \delta \,]\!]\|_{0,e}. \tag{5.9}$$

Inserting (5.8) and (5.9) into the definition (3.7) of b_h^{s}, and then using the Cauchy-Schwarz inequality and (5.3), it follows that

$$\begin{aligned} b_h^{\mathrm{s}}(\eta,\delta) &\leq \sum_{e\in\mathcal{E}_h}\Big(\frac{1}{\theta_0}\|c_e^{1/2}\{\eta\}\|_{0,e} + \|c_e^{1/2}[\![\, \eta \,]\!]\|_{0,e}\Big)\|c_e^{1/2}[\![\, \delta \,]\!]\|_{0,e} \\ &\leq Ch^{k+1/2}\|u\|_{k+1,\Omega}\Big(\sum_{e\in\mathcal{E}_h}\|c_e^{1/2}[\![\, \delta \,]\!]\|_{0,e}^2\Big)^{1/2}. \end{aligned} \tag{5.10}$$

Substituting (5.6) and (5.10) in (5.4), and using the definition (4.2) we obtain

$$C_S |||\delta|||^2 \leq Ch^{k+1/2} \|u\|_{k+1,\Omega} |||\delta|||, \tag{5.11}$$

which implies immediately that

$$|||\delta||| \leq Ch^{k+1/2} \|u\|_{k+1,\Omega}. \tag{5.12}$$

Hence

$$|||u - u_h||| \leq Ch^{k+1/2}, \tag{5.13}$$

thus completing the error analysis of the method.

6 Conclusions

We considered discontinuous Galerkin finite element approximations of a model scalar linear hyperbolic equation $\mathrm{div}(\boldsymbol{\beta} u) + \gamma u = f$ in $\Omega \subset \mathbb{R}^2$, subject to nonhomogeneous boundary condition $u = g$ at the inflow part of $\partial\Omega$. We showed that in order to ensure continuous stabilization of the method it suffices to add a jump-penalty term to the discretized equation. A particular value of the penalty functions c_e results in the standard upwind scheme, but, using discontinuous piecewise polynomials of degree k, an optimal $\mathcal{O}(h^{k+1/2})$ error estimate was proved for any choice of penalty functions c_e that satisfy $c_e \geq \theta_0 |\boldsymbol{\beta} \cdot \mathbf{n}_e|$ with $\theta_0 > 0$ independent of e and h. The latter property can be easily ensured by simply choosing the penalty function as a suitable constant on each edge. As precisely the same jump-term is used for stabilizing DG approximations of diffusion operators, the discretization proposed here can simplify the analysis and the implementation of discontinuous Galerkin finite element approximations of advection-diffusion problems.

If the jump-penalty terms are omitted from the scheme by formally setting $c_e \equiv 0$ on each $e \in \mathcal{E}_h^\circ$, then the scheme (3.8) collapses to (2.12). Since the latter is only stable in the $\mathrm{L}^2(\Omega)$-norm, in the error analysis of (2.12) one is forced to use the inverse inequality

$$\|[\![\, \delta \,]\!]\|_{0,e} \leq Ch_e^{-1/2} \|\delta\|_{0,T^1 \cup T^2}$$

to revert from the edgewise L^2-norm to the elemental L^2-norm in the course of bounding the left-hand side of (5.8). Hence, instead of (5.8) one has

$$\int_e \{\boldsymbol{\beta}\eta\} \cdot [\![\, \delta \,]\!] \, \mathrm{d}s \leq C\|\beta\|_{0,\infty,e} \|\{\eta\}\|_{0,e} h_e^{-1/2} \|\delta\|_{0,T^1 \cup T^2} \leq Ch^k \|\delta\|_{0,T^1 \cup T^2}, \tag{6.1}$$

where T^1 and T^2 are the two triangles whose common edge is e; the inequality (6.1), in turn, results in the suboptimal error bound $\|u - u_h\|_{0,\Omega} \leq Ch^k$ for (2.12), — in sharp contrast with the optimal-order error bound (1.6) for the stabilised scheme (3.8). This undesirable loss of optimality of the unstabilised scheme (2.12) further highlights the helpful role played in the stabilized scheme (3.8) by the jump-penalty terms.

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