1] Discrete Galerkin (D-G.) Method

Theory: Brezzi, Marini, Süli; M3AS, 2004

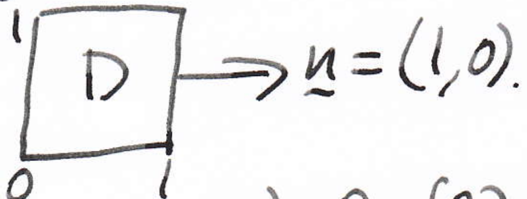
Practice: Jack Blake.

Model 1: Steady state neutron transport

(i) No time variable

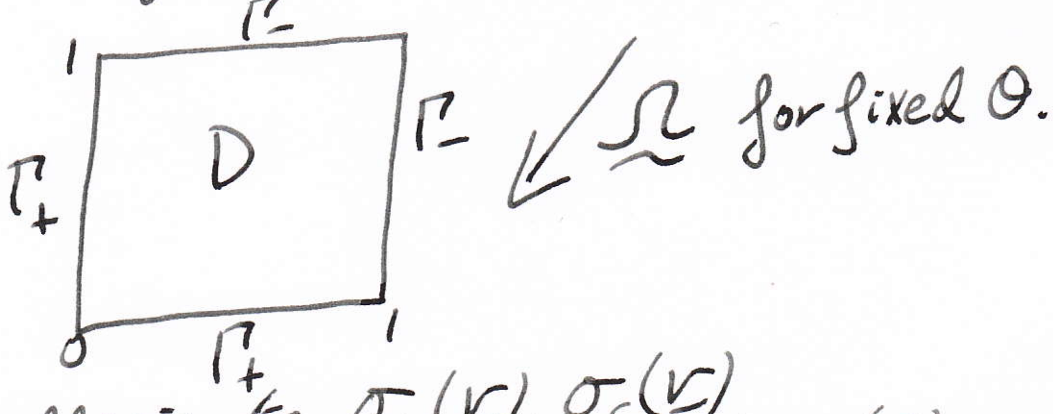
(ii) Position: $\underline{r}=(x,y)\in[0,1]\times[0,1]=:D$, $\quad \underline{\nabla}:=\begin{pmatrix}\partial/\partial x\\ \partial/\partial y\end{pmatrix}$

(iii) $\underline{n}$ = outward unit normal



(iv) Angle: $\underline{\Omega}=(\cos\theta,\sin\theta)$, $\theta\in[0,2\pi)$, indep^t of $\underline{r}$.

(v) Inflow boundary $\Gamma_-=\{\underline{r}\in\partial D \mid \underline{\Omega}\cdot\underline{n}(\underline{r})<0\}$



(vi) Given $\begin{cases}\text{coefficients } \sigma_T(\underline{r}),\ \sigma_S(\underline{r})\\ \text{source } Q(\underline{r}),\ \text{boundary data } g(\underline{r})\end{cases}$

Boundary Value Problem For $\theta\in[0,2\pi)$, $\underline{\Omega}=\underline{\Omega}(\theta)$,

$$\left.\begin{aligned}
&\underline{\Omega}\cdot\underline{\nabla}\psi+\sigma_T(\underline{r})\psi=\sigma_S(\underline{r})\phi(\underline{r})+Q(\underline{r}), && \underline{r}\in D.\\
&\phi(\underline{r}):=\frac{1}{2\pi}\int_0^{2\pi}\psi(\underline{r},\underline{\Omega}(\omega))\,d\omega, && \underline{r}\in D\\
&\psi(\underline{r},\underline{\Omega})=g(\underline{r}), && \underline{r}\in\Gamma_-
\end{aligned}\right\}$$

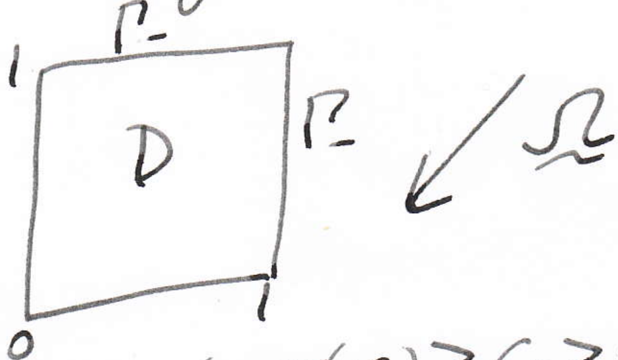
Model 2: Steady hyperbolic

(i) No time

(ii) $\underline{r} \in D = [0,1] \times [0,1]$

(iii) $\underline{\Omega}$ fixed, so $\psi(\underline{r}, \underline{\Omega}) = \psi(\underline{r})$.

(iv) Inflow boundary $\Gamma_- := \{\underline{r} \in \partial D \mid \underline{\Omega} \cdot \underline{n}(\underline{r}) < 0\}$



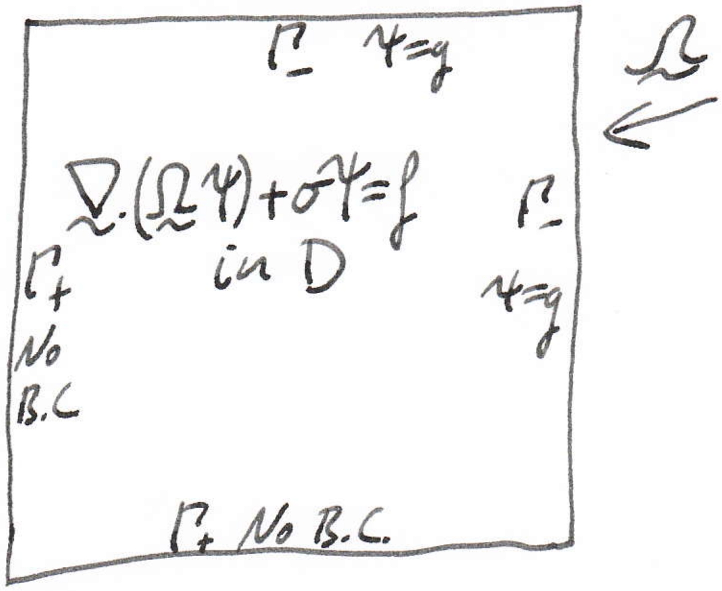
(v) Given $\begin{cases} \text{coefficient } \sigma(\underline{r}) \geq c_0 > 0. \\ \text{source } f(\underline{r}) \\ \text{boundary data } g(\underline{r}) \end{cases}$

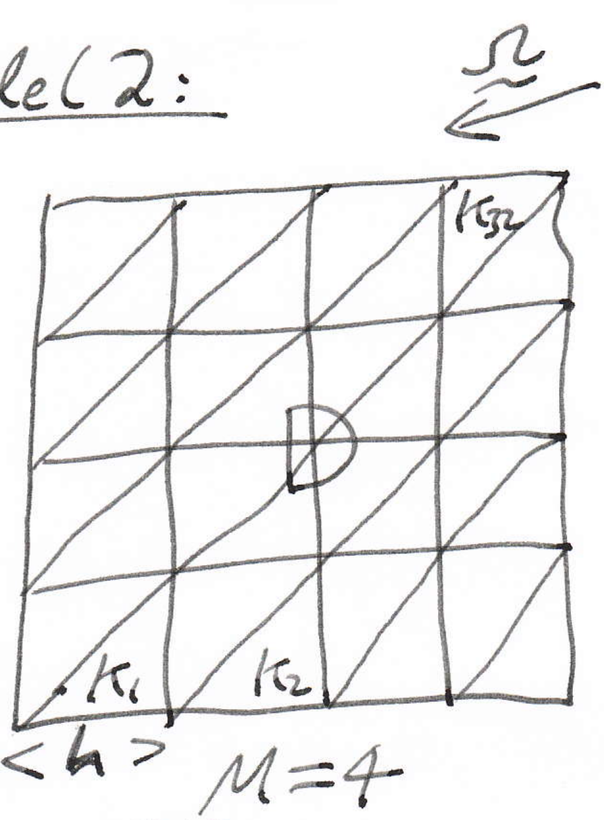
<u>B.V.P.</u> $\quad \underline{\nabla} \cdot (\underline{\Omega}\psi) + \sigma(\underline{r})\psi = f(\underline{r})$ in D

$$\psi(\underline{r}) = g(\underline{r}) \text{ on } \Gamma_-$$

[<u>Note:</u> $\underline{\Omega}$ indep^t of $\underline{r} \Rightarrow \underline{\nabla} \cdot \underline{\Omega} = 0 \Rightarrow \underline{\nabla} \cdot (\underline{\Omega}\psi) = \underline{\Omega} \cdot \underline{\nabla}\psi$.

3| Discretization of Model 2:





- Divide D into triangles $\{K_i\}_{i=1}^{2M^2} = \mathcal{T}_h$
- Grid width $= h = \frac{1}{M}$, $M \in \mathbb{N}$.
- Code needs node $\Longleftrightarrow$ triangle maps (MA3O170)

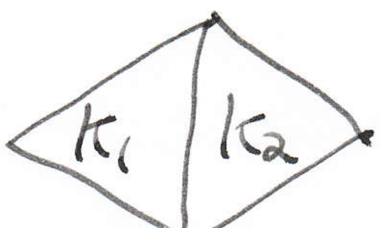
D.G. Function Space V_h^2 –

- Given $K \in \mathcal{T}_h$, consider $\mathbb{P}_2(K)$ space of quadratic polynomials with support on K; i.e.

$$v \in \mathbb{P}_2(K) \Rightarrow v(x,y) = \begin{cases} c_0 + c_{11}x + c_{12}y + c_{21}x^2 + c_{22}xy + c_{23}y^2 & \text{for } (x,y)\in K \\ 0, & (x,y) \notin K. \end{cases}$$

- $V_h^2 = \{ v_h : D \to \mathbb{R} \mid v_h|_K \in \mathbb{P}_2(K) \text{ for all } K \in \mathcal{T}_h \} \subset L_2$
- Discontinuous, since $c_0, c_{11}, \dots, c_{23}$ depend on K

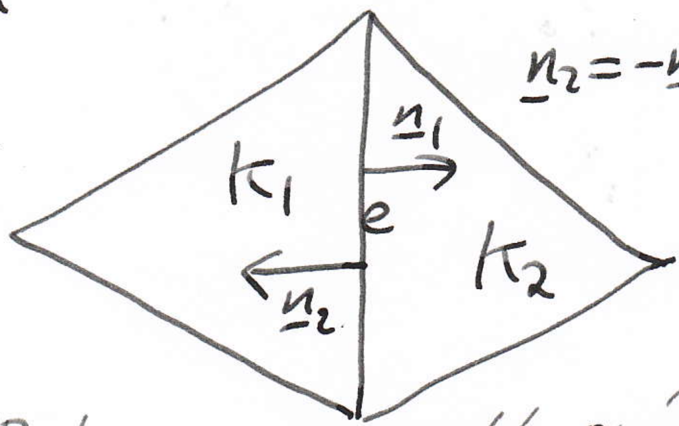
v_h is 2-valued on ∂K



4| $V_h^2 := \{ v_h : D \to \mathbb{R} \mid v_h|_K \in \mathbb{P}_2(K) \ \forall K \in \mathcal{T}_h \} \cap L_2(D)$.

<u>Interior Jumps</u>

$e \in \mathcal{E}_h^o$ set of interior edges.



$\underline{n}_2 = -\underline{n}_1$

Let scalar $\phi : D \to \mathbb{R}$ be p-w smooth on $\mathcal{T}_h$

Let vector $\underset{\sim}{\tau} : D \to \mathbb{R}^2$ be p-w " " ".

Let $\phi^i := \phi|_{K_i}$, $\underset{\sim}{\tau}^i := \underset{\sim}{\tau}|_{K_i}$, $i = 1, 2$.

$\{\phi\} := \frac{1}{2}(\phi^1 + \phi^2)$ averaging operator

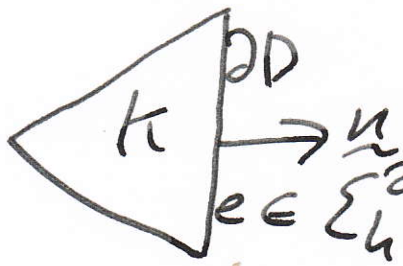
$\{\underset{\sim}{\tau}\} := \frac{1}{2}(\underset{\sim}{\tau}^1 + \underset{\sim}{\tau}^2)$ " "

$[\![\phi]\!] := \phi^1 \underset{\sim}{n}_1 + \phi^2 \underset{\sim}{n}_2$ scalar jump operator

$[\![\underset{\sim}{\tau}]\!] := \underset{\sim}{\tau}^1 \cdot \underset{\sim}{n}_1 + \underset{\sim}{\tau}^2 \cdot \underset{\sim}{n}_2$ vector jump operator

scalars map to vectors & conversely

<u>Boundary Jumps.</u>



$e \in \mathcal{E}_h^\partial$ set of boundary edges

$[\![\phi]\!] = \phi \underset{\sim}{n}$

$\{\tau\} = \underset{\sim}{\tau}$

- Scalar averages/vector jumps not used on ∂D.

5] Integration by parts (Divergence Theorem)

Let $v, \psi : K \to \mathbb{R}$ be smooth. Then,

$$\int_K v \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} \psi) + \psi \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} v) = \int_K \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} v \psi) = \int_{\partial K} (\underset{\sim}{n} \cdot \underset{\sim}{\Omega}) v \psi$$

$$\Rightarrow \int_K v \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} \psi) = -\int_K \psi \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} v) + \int_{\partial K} v \psi (\underset{\sim}{n} \cdot \underset{\sim}{\Omega}).$$

RHS well-defined even when $\underset{\sim}{\nabla} \psi$ does not exist. Helps to defined a weak form.

Now suppose $\underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} \psi) + \sigma \psi = f$ in $\underline{D}$

Multiply by $v_h \in V_h^2$ & integrate by parts over each K:

$$\sum_{K \in \mathcal{T}_h} \left(\int_K -\psi \underset{\sim}{\nabla} \cdot (\underset{\sim}{\Omega} v_h) + \sigma \psi v_h + \int_{\partial K} (\underset{\sim}{\Omega} \cdot \underset{\sim}{n}) \psi v_h \right) = \int_D f v_h$$

Need edge identity, using ψ <u>not</u> discontinuous:

$$\sum_{K \in \mathcal{T}_h} \int_{\partial K} (\underset{\sim}{\Omega} \cdot \underset{\sim}{n}) \psi v_h = \sum_{e \in \mathcal{E}_h} \int_e \{\underset{\sim}{\Omega} \psi\} \cdot [[v_h]]$$

$$= \sum_{e \notin \Gamma_-} \int_e \{\underset{\sim}{\Omega} \psi\} [[v_h]] + \sum_{e \in \Gamma_-} \int_e \underset{\sim}{\Omega} \cdot \underset{\sim}{n} \, g v_h$$

$$\left(\sum_{K\in\mathcal{T}_h}\int_K -\Psi\,\underset{\sim}{\Omega}\cdot\underset{\sim}{\nabla} v_h + \sigma\Psi v_h\right) + \sum_{e\notin\Gamma_-}\int_e \{\underset{\sim}{\Omega}\Psi\}\cdot[\![v_h]\!]$$

$$= \int_D f v_h - \sum_{e\subseteq\Gamma_-}\int_e (\underset{\sim}{\Omega}\cdot\underset{\sim}{n})\, g\, v_h$$

$$\Longleftrightarrow a_h(\Psi, v_h) + b_h(\Psi, v_h) = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in V_h^2.$$

<u>Numerical Method I (Weak Form)</u>

Find $\Psi_h \in V_h^2$ s.t. $\Psi_h \approx \Psi$ satisfies

$$a_h(\Psi_h, v_h) + b_h(\Psi_h, v_h) = (f, v_h) + \langle g, v_h\rangle \quad \forall v_h \in V_h^2.$$

* This method is stable in L_2, but may have local oscillations (H^1-instability)
* <u>Upwind stabilization</u> of $\{\underset{\sim}{\Omega}\Psi_h\}$ in $b_h(\Psi_h, v_h)$

$$\{\underset{\sim}{\Omega}\Psi_h\}_u := \begin{cases} \underset{\sim}{\Omega}\Psi_h^1 & \text{if } \underset{\sim}{\Omega}\cdot\underset{\sim}{n}_1 > 0 \\ \underset{\sim}{\Omega}\Psi_h^2 & \text{if } \underset{\sim}{\Omega}\cdot\underset{\sim}{n}_2 > 0 \\ \underset{\sim}{\Omega}\left(\dfrac{\Psi_h^1+\Psi_h^2}{2}\right) & \text{if } \underset{\sim}{\Omega}\cdot\underset{\sim}{n}_1 = 0 \end{cases} \qquad \left[\underset{\sim}{\Omega}\cdot\underset{\sim}{n}_2 = -\underset{\sim}{\Omega}\cdot\underset{\sim}{n}_1\right]$$

Now, $\{\underset{\sim}{\Omega}\psi_h\}_u \cdot \underset{\sim}{n} = (\{\underset{\sim}{\Omega}\psi_h\} + c^*[[\psi_h]]) \cdot \underset{\sim}{n}$ for $c^* := \frac{|\underset{\sim}{\Omega} \cdot \underset{\sim}{n}|}{2}$. This motivates

$$b_h^u(\psi_h, v_h) := b_h(\psi_h, v_h) + \sum_{e \in \mathcal{E}_h^o} \int_e c_e^* [[\psi_h]][[v_h]]$$

[Note: $[[\psi]] = 0$ so still consistent.]

Upwind-stabilized D.G. Method:

Find $\psi_h \in V_h^2$ such that

$$a_h(\psi_h, v_h) + b_h^u(\psi_h, v_h) = (f, v_h) + \langle g, v_h \rangle \quad \forall v_h \in V_h^2. \quad (*)$$

Implementation $\psi_h|_K \in \mathbb{P}_2(K)$, $\psi_h|_K = c_0 + c_1 x + \dots + c_{23} y^2$

On each K, $v_h \in \text{Span}\{1, x, y, x^2, xy, y^2\}$

Thus, only need consider these basis $f^{\underline{n}}$s in $(*)$. 6 basis $f^{\underline{n}}$s / triangle $\times$ $2M^2$ triangles

$\Longrightarrow \begin{cases} 12M^2 \text{ unknown coefficients for } \psi_h \\ 12M^2 \text{ trial functions } v_h \text{ all linearly independent} \end{cases}$

Linear system of equations $12M^2 \times 12M^2$.

8|

Consistency As $[[\Psi]]=0$ on edges,

$$a_h(\Psi, v_h) + b_h^u(\Psi, v_h) = (f, v_h) + \langle g, v_h \rangle, \quad \forall v_h \in V_h^2.$$

Subtract numerical solution:

$$a_h(\Psi - \Psi_h, v_h) + b_h^u(\Psi - \Psi_h, v_h) = 0 \quad \forall v_h \in V_h^2.$$

Galerkin orthogonality: error $\perp^r$ to V_h^2.

Stability (Coercivity) May prove $\forall v_h \in V_h^2$,

$$a_h(v_h, v_h) + b_h^u(v_h, v_h) \geq C_s |||v_h|||^2$$

$$|||v_h|||^2 := \|v_h\|_{L^2(D)}^2 + \sum_{e \in \mathcal{E}_h} \|c_e^{*1/2} [[v_h]]\|_{L^2(e)}^2$$

Projection Error Bounds (Usually proved via interpolant)

$P_h^2 : L^2(D) \to V_h^2$: Given $\Psi \in L^2(D)$, find $\Psi_h \in V_h^2$ so that $\|\Psi - \Psi_h\|_{L^2(D)}$ is minimized $\Leftrightarrow$ $\|\Psi - \Psi_h\|_{L^2(K)}$ min $\forall K$

$$\|\Psi - \Psi_h\|_{L^2(K)} \leq C h^3 \|\Psi\|_{H^3(K) = W^{3,2}(K)} \quad \forall K \in \mathcal{T}_h$$

$$\|\Psi - \Psi_h\|_{L^2(e)} \leq C h^{5/2} \|\Psi\|_{H^3(K)}$$

9| Error bound

$$C_s |||u_h - P_h^2 u|||^2 \leq a_h(u_h - P_h^2 u, u_h - P_h^2 u) + b_h(u_h - P_h^2 u, u_h - P_h^2 u)$$

$$= a_h(u - P_h^2 u, u_h - P_h^2 u) + b_h(u - P_h^2 u, u_h - P_h^2 u)$$

[Galerkin orthogonality]

[Bit of work] $$\leq C h^3 \|u\|_{H^3} \|u_h - P_h^2 u\| + C h^{5/2} \|u\|_{H^3} \left(\sum_{e \in \mathcal{E}_h} \| C_e^{*1/2} [[u_h - P_h^2 u]] \|^2 \right)^{\frac{1}{2}}$$

$$\leq C h^{5/2} \|u\|_{H^3} |||u_h - P_h^2 u|||$$

$$\Rightarrow \quad |||u_h - P_h^2 u||| \leq \frac{C h^{5/2}}{C_s} \|u\|_{H^3(D)}$$

But projection error estimates give

$$|||u - P_h^2 u||| \leq C h^{5/2} \|u\|_{H^3(D)}.$$

Combining last two inequalities,

$$\boxed{|||u - u_h||| \leq C h^{5/2} \|u\|_{H^3(D)}.}$$

Recall

$$|||u_h|||^2 := \|u_h\|^2_{L^2(D)} + \sum_{e \in \mathcal{E}_h} \| C_e^{*1/2} [[u_h]] \|^2_{L^2(e)}$$

Strictly positive C_e implies a bound on jumps between elements; i.e. local instability.