Geometric solitary waves in a 2D mass-spring lattice

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1 Introduction

In this note we analyse some aspects of the Hamiltonian dynamics of an elastic 2D lattice of particles interacting via interatomic potentials. As in the recent elastostatic investigation [FT01], we consider a cubic lattice where the particles have equal mass and only nearest and next nearest neighbours interact. The associated Hamiltonian dynamical system, obtained by including the kinetic energy of the particles, is a natural 2D analogon of the 1D Fermi-Pasta-Ulam lattice introduced in [FPU55]. (It cannot be simplified into a nearest-neighbour model, which would have no shear resistance and would hence be incapable of capturing 2D elasticity.)

Our interest here is the existence of solitary waves moving through the 2D lattice. We prove the existence of small-amplitude supersonic longitudinal solitary waves moving along the $(1, 0)$ direction, and determine explicitly their asymptotic profile in the near-sonic regime. These results hold for generic potentials, but most interestingly, the solitary waves even exist when the potentials are harmonic, i.e. when the interparticle forces are linear with respect to particle distance.

Note that in this case the Hamiltonian equations of motion are still nonlinear, due to the frame-indifference of the interatomic forces. (In continuum elasticity theory the analogous nonlinearity has been termed 'geometric', to distinguish it from non-universal nonlinearities due to specific modelling assumptions.) This geometric nonlinearity and the ensuing solitary waves are a genuinely 2D phenomenon: in 1D chains with harmonic springs no solitary waves exist [FW94, Sec.7]. Physically this means that the 2D waves are created purely by the coupling between the neighbouring harmonic chains.

These waves are always found to be extension waves, universally with respect to the choice of equilibrium lengths and spring constants for the nearest-neighbour

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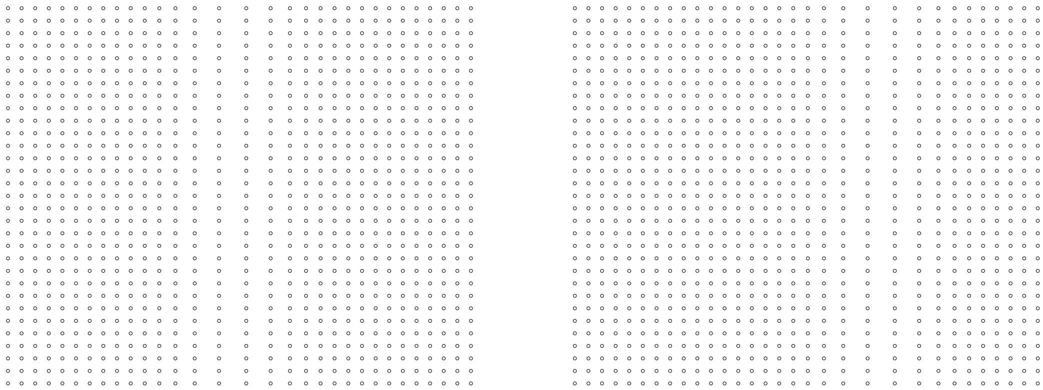

Figure 1: Particle positions for the solitary wave in the lattice with linear springs. The parameter values used here are $K_1 = 5$, $K_2 = a_1 = a_2/\sqrt{2} = 1$, $(c - c_s)/c_s = 1/24$. The solitary wave is always an extension wave and the site of the most extended cell moves along the $(1, 0)$ direction.

and diagonal springs. This is related to the somewhat counterintuitive fact that the effective coupling between horizontal neighbours which ensues from the vertical interactions is found to be always hardening with respect to extension, and softening with respect to compression. (Intuitively, one might have expected that hardening or softening would depend on whether in the cubic equilibrium state of the lattice, the diagonal springs are under compression or under tension.)

Furthermore we show that no transversal solitary waves – and indeed no non-longitudinal solitary waves – exist which propagate in the $(1, 0)$ direction.

The $(1, 0)$ direction has been chosen here because it leads to the smallest number of delays in the ensuing system of differential-difference equations (namely one forward-delay and one backward-delay). The interesting issue of wave propagation in general directions lies beyond the scope of this note. In the simpler case of diffusive evolution of passive scalars on 2D lattices, there has been much recent progress on this issue, see e.g. [CM-PS98, CM-PV99, M-P99]. In these studies, one is dealing with discrete analoga of scalar semi-linear reaction-diffusion equations, where the coupling between the discrete sites is linear and maximum principles hold. By contrast, the elastic lattice studied here is a discrete analogon of a system of quasi-linear conservation laws, where the coupling between sites is necessarily nonlinear due to geometric reasons. Whether some of the techniques of [CM-PS98, CM-PV99, M-P99] can be carried over to such systems remains to be seen.

The existence proof for the longitudinal waves proceeds by reduction to an effective 1D Hamiltonian of Fermi-Pasta-Ulam type (which accounts exactly for the vertical coupling), and use of recent work on 1D Fermi-Pasta-Ulam lattices. Specifically we will use the results of [FP99], because they deliver not just existence but also the asymptotic profile shape; other methods to prove existence of travelling waves are given in [FW94, IK99, I00]. By contrast we have to use the 2D structure of the problem for our non-existence proof. In particular we show that a reduction to an effective 1D Hamiltonian is impossible for non-longitudinal waves.

2 Model and Equations of Motion

The particles in our infinite lattice are indexed by $(i,j) \in \mathbb{Z}^2$. The position of the $(i,j)^{th}$ particle at time t is denoted by

$$\begin{pmatrix} r_* i \\ r_* j \end{pmatrix} + q_{i,j}(t) \ \in \mathbb{R}^2.$$

Here $r_* > 0$ is a reference lattice parameter, which will from Section 3 onwards be chosen so that the state $q_{i,j} = 0$ $(i,j \in \mathbb{Z})$ is an equilibrium, in which case the $q_{i,j}$ are the out-of-equilibrium displacements of the particles.

With the usual notation $p_{i,j}(t)$ for the momentum of the $(i,j)^{th}$ particle, the dynamics is described by the infinite-dimensional Hamiltonian

$$\begin{aligned} H &= \sum_{i,j\in\mathbb{Z}} \Big(\frac{1}{2}|p_{i,j}|^2 + V_1(|r_*e_1 + q_{i+1,j} - q_{i,j}|) + V_1(|r_*e_2 + q_{i,j+1} - q_{i,j}|) \\ &\quad + V_2(|r_*(e_1+e_2) + q_{i+1,j+1} - q_{i,j}|) + V_2(|r_*(e_1-e_2) + q_{i+1,j} - q_{i,j+1}|) \Big) \end{aligned} \tag{2.1}$$

Here $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$, e_1 and e_2 are the lattice basis vectors $(1,0)$ and $(0,1)$, V_1 is the potential for the horizontal and vertical interactions and V_2 corresponds to the diagonal interactions. In this section V_1, V_2 can be arbitrary differentiable functions from $(0,\infty) \to \mathbb{R}$; prototypical are the harmonic potentials

$$V_1(r) = \frac{K_1}{2}(r - a_1)^2 \tag{2.2}$$

$$V_2(r) = \frac{K_2}{2}(r - a_2)^2, \tag{2.3}$$

in which case there is a unique equilibrium lattice parameter, given by

$$r_* = \frac{K_1 a_1 + \sqrt{2} K_2 a_2}{K_1 + 2K_2}, \tag{2.4}$$

cf. [FT01].

The equation of motion for each particle contains eight forcing terms, due (in order of appearance) to its nearest neighbours on the right, left, top and bottom and its next nearest neighbours on the top right, bottom left, bottom right and top left: Abbreviating $f(z) := V_1'(|z|)\frac{z}{|z|}$, $g(z) := V_2'(|z|)\frac{z}{|z|}$,

$$\begin{aligned} \dot{q}_{i,j} &= p_{i,j} \\ \ddot{q}_{i,j} &= \dot{p}_{i,j} = -\frac{\partial}{\partial q_{i,j}} H \\ &= -\{ -f(r_*e_1 + q_{i+1,j} - q_{i,j}) + f(r_*e_1 + q_{i,j} - q_{i-1,j}) \\ &\qquad - f(r_*e_2 + q_{i,j+1} - q_{i,j}) + f(r_*e_2 + q_{i,j} - q_{i,j-1}) \\ &\qquad - g(r_*(e_1+e_2) + q_{i+1,j+1} - q_{i,j}) + g(r_*(e_1+e_2) + q_{i,j} - q_{i-1,j-1}) \\ &\qquad - g(r_*(e_1-e_2) + q_{i+1,j-1} - q_{i,j}) + g(r_*(e_1-e_2) + q_{i,j} - q_{i-1,j+1}) \} . \end{aligned} \tag{2.5}$$

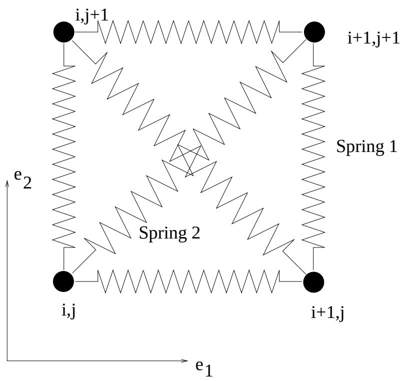

Figure 2: Unit cell of the lattice

3 Solitary waves

We are looking for travelling waves, i.e. we seek solutions of the type

$$q_{i,j}(t) = \tilde{q}(\underline{k} \cdot \begin{pmatrix} i \\ j \end{pmatrix} - ct), \tag{3.1}$$

where $\underline{k} \in \mathbb{R}^2$ ($|\underline{k}| = 1$) is the direction of propagation of the wave and c its speed. We consider here the special case where $\underline{k}$ is parallel to the lattice vectors e_1 or e_2. For $\underline{k} = e_1$ the travelling wave ansatz simplifies to $q_{i,j}(t) = \tilde{q}(i - ct)$ (independently of j) and the equations of motion (2.5) reduce to the following system of differential-difference equations for the profile $\tilde{q}(x)$, $x = i = ct$ (note that two of the eight forcing terms cancel, namely those due to the top and the bottom neighbour):

$$\begin{aligned} c^2\tilde{q}''(x) \;=\; - \;& \{-f(r_*e_1 + \tilde{q}(x+1) - \tilde{q}(x)) + f(r_*e_1 + \tilde{q}(x) - \tilde{q}(x-1)) \\ & -g(r_*(e_1+e_2) + \tilde{q}(x+1) - \tilde{q}(x)) + g(r_*(e_1+e_2) + \tilde{q}(x) - \tilde{q}(x-1)) \\ & -g(r_*(e_1-e_2) + \tilde{q}(x+1) - \tilde{q}(x)) + g(r_*(e_1-e_2) + \tilde{q}(x) - \tilde{q}(x-1))\}\,. \end{aligned} \tag{3.2}$$

So we have here one forward and one backward delay. Travelling waves with arbitrary $\underline{k}$ will give differential-difference equations with up to 4 forward and 4 backward delays, as there are 8 different particles with which a given particle interacts and all can be at different positions of the wave.

Special cases of $\tilde{q}$ are unidirectional waves where the particles move in a single direction, i.e. $\tilde{q}(x) = \underline{d}q(x)$, where q is scalar and $\underline{d} \in \mathbb{R}^2$ is the amplitude director. Pure longitudinal waves ($\underline{d}\,||\,\underline{k}$) and transversal waves ($\underline{d} \cdot \underline{k} = 0$) are examples of these.

By a solitary wave (of speed c) we mean a nonconstant solution $\tilde{q}_c \in C^2(\mathbb{R};\mathbb{R}^2)$ to the travelling wave equation (3.2) for which the relative displacement (or elastic strain) $\tilde{r}_c(x) := \tilde{q}_c(x+1) - \tilde{q}_c(x)$ tends to zero as $x \to \pm\infty$, in the sense that $\tilde{r}_c$ belongs to the Sobolev space $H^1(\mathbb{R})$ of square-integrable functions with square-integrable derivative. (In fact the waves constructed below satisfy $\tilde{r}_c(x) \to 0$ exponentially fast, while our nonexistence results even rule out solitary waves satisfying the above weak localization condition.)

The following theorem holds for generic nonlinear springs, but for simplicity we will now put our emphasis on harmonic springs, where the overall potential is anharmonic due to the geometry.

In the theorem, a certain role is played by the longitudinal speed of sound of the lattice in $(1, 0)$ direction, which can be calculated to be

$$c_s = \sqrt{\frac{2K_1 a_1 + \sqrt{2} K_2 a_2}{2r_*}}. \tag{3.3}$$

(See Section 4 for the dispersion relation of the 2D lattice, its interesting direction-dependence, and possible implications for stability issues.)

Theorem 1 *Assume that the interaction potentials and the lattice parameter are given by (2.2), (2.3), (2.4). Let c_s be as defined in (3.3), and recall that e_1 denotes the lattice basis vector $(1, 0)$. Then the following results hold.*

i. *(Existence and asymptotic profile of longitudinal waves)*

a) For all supersonic wavespeeds $c > c_s$ which are sufficiently close to c_s, there exists a unique longitudinal single-pulse solitary wave $\tilde{q}_c(x) = e_1 q_c(x)$, q_c scalar, propagating in e_1 direction. Here unique means unique up to the trivial invariances of the travelling wave equation under spatial translation and additive constants, and single-pulse means that the derivative of the associated relative displacement profile $r_c(x) = q_c(x+1) - q_c(x)$ vanishes only at one point.

b) The above wave has the following properties: it is an extension wave (i.e. $\tilde{r}_c(x) > 0$ for all $x \in \mathbb{R}$), for any choice of the spring constants $K_1 > 0$, $K_2 > 0$ and spring equilibrium lengths $a_1 > 0$, $a_2 > 0$; r_c is even with respect to reflection at the point x_0 where r_c is maximal; r_c tends to zero exponentially as $x \to \pm\infty$ in the sense that $e^{b_0(c)|x-x_0|} r_c(x)$ converges to a finite nonzero limit as $x \to \pm\infty$, where $b_0(c)$ is the unique positive root of the equation $c/c_s = (sinh\frac{b_0}{2})/\frac{b_0}{2}$.

c) In the regime of close-to-sonic wavespeed, the profile r_c has the following asymptotic shape: Abbreviating $\varepsilon := \sqrt{24(c - c_s)/c_s}$, r_c has characteristic width of order $1/\varepsilon$ and height of order ε^2 (i.e. it is a small-amplitude long wave): more precisely there exist constants $C > 0$ and $\varepsilon_0 > 0$ such that for all $\varepsilon \le \varepsilon_0$

$$\left\| \frac{1}{\varepsilon^2} r_c\left(\frac{\cdot}{\varepsilon}\right) - \Phi \right\|_{H^1} \le C\varepsilon^2, \tag{3.4}$$

where Φ is the KdV soliton profile

$$\Phi(x) = \frac{(2\sqrt{2}K_1 a_1 + 2K_2 a_2) r_*}{3K_2 a_2} \Big(\frac{1}{2} sech(\frac{1}{2}x)\Big)^2. \tag{3.5}$$

ii. *(Nonexistence of other unidirectional waves) There exists a universal constant $\delta > 0$ such that for no wavespeed $c \in \mathbb{R}$ and no amplitude director $\underline{d} \neq e_1$ do there exist any unidirectional solitary waves with amplitude* $\max_{x\in\mathbb{R}} |\tilde{r}_c(x)| < \delta$ *propagating in e_1 direction. In particular there are no transversal solitary waves.*

iii. *(Nonexistence of non-unidirectional waves) There exists a universal constant $\delta > 0$ such that for no supersonic wavespeed $c > c_s$ does there exist any (not necessarily unidirectional) solitary wave propagating in e_1 direction which is not longitudinal (i.e. $\tilde{r}_c(x) \cdot e_2$ not identically zero) and whose amplitude satisfies* $\max_{x\in\mathbb{R}} |\tilde{r}_c(x)| < \delta$.

The proofs of i., ii. and iii. are each based on different methods. To show i. we will reduce the problem to a scalar 1D lattice problem and apply results for those by Friesecke and Pego [FP99]. The proof of ii. is based on a monotonicity argument which relies on the vectorial nature of the equation, while for iii. we analyse the Fourier transform of (3.2).

3.1 Existence and asymptotic profile of longitudinal solitary waves

For longitudinal waves $q_{i,j}(t) = e_1 q(i - ct)$, or more general longitudinal motions

$$q_{i,j}(t) \cdot e_2 \equiv 0, \quad q_{i,j}(t) \cdot e_1 \equiv q_i(t) \text{ independently of } j, \tag{3.6}$$

the vectorial equation of motion (2.5) can be reduced to a scalar one. First we claim that the second component (2.5)$\cdot e_2$ holds automatically. To see this note first that the left hand side of (2.5) is orthogonal to e_2, as are the first two terms on the right hand side. Next, the third and fourth term on the right hand side cancel, due to the j-independence of the $q_{i,j}$. Finally the terms $g(r_*(e_1 \pm e_2) + q_{i+1,j\pm1} - q_{i,j}) \cdot e_2$ have opposite and equal sign and equal magnitude and hence cancel, as do the terms $g(r_*(e_1 \pm e_2) + q_{i,j} - q_{i-1,j\mp1}) \cdot e_2$.

The first component (2.5)$\cdot e_1$ becomes (using that $g(r_*(e_1+e_2)+(q_j-q_k)e_1)\cdot e_1 = g(r_*(e_1 - e_2) + (q_j - q_k)e_1) \cdot e_1$)

$$\begin{aligned}\ddot{q}_i = \ - \ & \{-f((r_* + q_{i+1} - q_i)e_1) \cdot e_1 + f((r_* + q_i - q_{i-1})e_1) \cdot e_1 \\ & -2g(r_*(e_1 + e_2)(q_{i+1} - q_i)e_1) \cdot e_1 + 2g(r_*(e_1 + e_2) + (q_i - q_{i-1})e_1) \cdot e_1\}.\end{aligned}$$

This is (denoting $\dot{q}_i =: p_i$) the equation of motion $\ddot{q}_i = \dot{p}_i = -\frac{\partial H_{eff}}{\partial q_i}$ for a new effective Hamiltonian

$$H_{eff} = \sum_{i\in\mathbb{Z}} \Big(\tfrac{1}{2}|p_i|^2 + V_{eff}(q_{i+1}-q_i)\Big), \quad V_{eff}(\rho) = V_1(|r_*+\rho|) + 2V_2(\sqrt{{r_*}^2 + (r_* + \rho)^2}).$$

(In particular, any dynamics for this 1D anharmonic lattice can be embedded into the 2D harmonic lattice.) Specifying to travelling waves $q_{i,j}(t) = e_1 q_c(i - ct)$, eqn. (3.2) reduces to the travelling wave equation for the 1D lattice,

$$c^2 q_c''(x) = V_{eff}'(q_c(x + 1) - q_c(x)) - V_{eff}'(q_c(x) - q_c(x - 1)).$$

Applying [FP99, Theorem 1.1] gives existence and all properties of the waves described in part i. of Theorem 1, noting that the required hypotheses on the potential ($V'(0) = 0$, $V''(0) > 0$, $V'''(0) \neq 0$) are satisfied, for all choices of spring constants K_1, $K_2 > 0$ and spring equilibrium lengths a_1, $a_2 > 0$: Using (2.2), (2.3), (2.4) one calculates

$$\begin{aligned} V'_{eff}(0) &= 0 \\ V''_{eff}(0) &= \frac{K_1 a_1 + (\sqrt{2}/2) K_2 a_2}{r_*} \\ V'''_{eff}(0) &= \frac{3K_2 a_2}{2\sqrt{2} {r_*}^2}. \end{aligned}$$

Interestingly, the anharmonic coefficient $V'''(0)$ (which governs the sign of the strain profile, appearing in particular as a prefactor in the asymptotic shape formula $\Phi(x) = \frac{V''(0)}{V'''(0)}(\frac{1}{2}\mathrm{sech}(\frac{x}{2}))^2$ of [FP99]) is always positive, yielding extension waves.

3.2 Non-existence of unidirectional solitary waves of arbitrary speed

We now rule out non-longitudinal unidirectional solitary waves, i.e. waves of form

$$\tilde{q}(x) = \underline{d} q(x) \text{ for } \underline{d} \notin \mathrm{Span}\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\} \text{ and } q : \mathbb{R} \to \mathbb{R}.$$

A necessary condition for the existence of such waves is

$$\underline{d}^{\perp} \cdot \tilde{q}'' \equiv 0, \tag{3.7}$$

where $\underline{d} = \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$ and $\underline{d}^{\perp} = \begin{pmatrix} -d_2 \\ d_1 \end{pmatrix}$. Using eq. (3.2) together with the identities $(r_* e_1 + \tilde{q}(x_1) - \tilde{q}(x_2)) \cdot \underline{d}^{\perp} = -r_* d_2$, $(r_*(e_1 + e_2) + \tilde{q}(x_1) - \tilde{q}(x_2)) \cdot \underline{d}^{\perp} = r_*(d_1 - d_2)$, $(r_*(e_1 - e_2) + \tilde{q}(x_1) - \tilde{q}(x_2)) \cdot \underline{d}^{\perp} = r_*(-d_1 - d_2)$ (for arbitrary x_1, $x_2 \in \mathbb{R}$), we calculate

$$\underline{d}^{\perp} \cdot \tilde{q}''(x) = h(q(x+1) - q(x)) - h(q(x) - q(x-1)),$$

with

$$\begin{aligned} h(s) &= \frac{V_1'(|r_* e_1 + s\underline{d}|)}{|r_* e_1 + s\underline{d}|}(-r_* d_2) \\ &+ \frac{V_2'(|r_*(e_1 + e_2) + s\underline{d}|)}{|r_*(e_1 + e_2) + s\underline{d}|} r_*(d_1 - d_2) + \frac{V_2'(|r_*(e_1 - e_2) + s\underline{d}|)}{|r_*(e_1 - e_2) + s\underline{d}|} r_*(-d_1 - d_2). \end{aligned}$$

But for $|\underline{d}| = 1$, h is analytic for $|s| < r_*$, hence (since h is nonconstant) its level sets are discrete. Hence provided $\sup_{x \in \mathbb{R}} |q(x+1) - q(x)| < r_*$, the equation $h(q(x+1) - q(x)) - h(q(x) - q(x-1)) = 0$ together with the continuous dependence of q on x implies $q(x+1) - q(x) \equiv const.$ But $q(x+1) - q(x) \to 0$ as $|x| \to \infty$ and hence $q(x) \equiv const.$ This establishes Theorem 1 ii., with explicit constant $\delta = r_*$. (We remark that the above argument has not required analyticity of the potentials but merely the generic property that h is not identically constant in a neighbourhood of zero. The nice feature of the harmonic potentials (2.2), (2.3) is that no restrictions whatsoever were needed on the spring constants K_1, K_2 and spring equilibrium lengths a_1, a_2.)

3.3 Non-existence of general solitary waves of supersonic speed

Finally we rule out solitary waves without a fixed amplitude director, i.e. waves of form $q_{i,j}(t) = \tilde{q}_c(i - ct)$ ($\tilde{q} : \mathbb{R} \to \mathbb{R}^2$).

If the displacement profile $\tilde{q}_c$ solves eq. (3.2) then the relative displacement profile $\tilde{r}_c(x) = \tilde{q}_c(x+1) - \tilde{q}_c(x))$ solves the following centered difference equation (where we use the symbolic notation $e^{\pm\partial}\tilde{r}(x,t) = \tilde{r}(x \pm 1, t)$)

$$c^2\tilde{r}_c'' = (e^{\partial} - 2 + e^{-\partial})\Big(f(r_*e_1 + \tilde{r}_c) + g(r_*(e_1+e_2) + \tilde{r}_c) + g(r_*(e_1-e_2) + \tilde{r}_c)\Big). \quad (3.8)$$

So it suffices to rule out non-longitudinal solutions (i.e. $\tilde{r}_c(x) \cdot e_2$ not identically zero) to (3.8).

To analyse this equation we follow the method introduced in a 1D context in [FP99], albeit our goal (to prove nonexistence) is different from theirs (to prove existence). Separating out the linear contributions to f and g, i.e. writing the forcing terms in the form

$$\begin{aligned} & f(r_*e_1 + \underline{r}) + g(r_*(e_1+e_2) + \underline{r}) + g(r_*(e_1-e_2) + \underline{r}) = A\underline{r} + N(\underline{r}), \text{ with} \\ & A \in M^{2\times 2}, \quad \frac{N(\underline{r})}{|\underline{r}|} \to 0 \ (|\underline{r}| \to 0), \end{aligned} \quad (3.9)$$

and applying the Fourier transform with the normalisation $\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)dx$ transforms (3.8) into the system

$$-c^2\xi^2\hat{\tilde{r}}_c(\xi) = -4\sin^2\frac{\xi}{2}\left[A\hat{\tilde{r}}_c + \widehat{N(\tilde{r}_c)}\right]. \quad (3.10)$$

A tedious calculation shows that

$$A = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_2 \end{pmatrix}, \quad \alpha_1 = V_{eff}''(0) = \frac{K_1a_1 + (\sqrt{2}/2)K_2a_2}{r_*}, \quad \alpha_2 = \frac{(\sqrt{2}/2)K_2a_2}{r_*}.$$

In particular $\alpha_1 - \alpha_2 = K_1a_1/r_* > 0$. This will be essential below. For supersonic c, i.e. $c > \sqrt{\alpha_1}$, eq. (3.10) can be rewritten as a fixed point equation involving a certain matrix-valued Fourier multiplier $p(\xi)$,

$$\begin{aligned} \hat{\tilde{r}}_c(\xi) &= p(\xi)\widehat{N(\tilde{r}_c)}(\xi), \\ p(\xi) &= \begin{pmatrix} \frac{4\sin^2(\frac{\xi}{2})}{c^2\xi^2 - 4\alpha_1\sin^2(\frac{\xi}{2})} & 0 \\ 0 & \frac{4\sin^2(\frac{\xi}{2})}{c^2\xi^2 - 4\alpha_2\sin^2(\frac{\xi}{2})} \end{pmatrix} = \begin{pmatrix} \frac{\mathrm{sinc}^2(\frac{\xi}{2})}{c^2 - \alpha_1\mathrm{sinc}^2(\frac{\xi}{2})} & 0 \\ 0 & \frac{\mathrm{sinc}^2(\frac{\xi}{2})}{c^2 - \alpha_2\mathrm{sinc}^2(\frac{\xi}{2})} \end{pmatrix} \end{aligned}$$

where $\text{sinc}(z) = \sin(z)/z$. By Plancherel's formula and the fact that $|\text{sinc}(z)| \leq 1$ for all $z \in \mathbb{R}$,

$$\begin{aligned} ||\tilde{r}_c \cdot e_2||_{L^2(\mathbb{R})} &= \frac{1}{\sqrt{2\pi}}||\widehat{\tilde{r}_c} \cdot e_2||_{L^2(\mathbb{R})} \\ &\leq \frac{1}{\sqrt{2\pi}}||\frac{\text{sinc}^2}{c^2 - \alpha_2 \text{sinc}^2}||_{L^\infty(\mathbb{R})}||\widehat{N(\tilde{r}_c)} \cdot e_2||_{L^2(\mathbb{R})} \\ &= \frac{1}{\alpha_1 - \alpha_2}||N(\tilde{r}_c) \cdot e_2||_{L^2(\mathbb{R})}. \end{aligned}$$

Lemma 3.1 *There exists a constant $C > 0$ such that the nonlinearity N satisfies*

$$|N(\underline{r}) \cdot e_2| \leq C|\underline{r} \cdot e_2||\underline{r}|$$

for all $\underline{r} \in \mathbb{R}^2$ with $|\underline{r}| \leq r^*/2$.

Proof: According to (3.9),

$$N(\underline{r}) \cdot e_2 = \Big(f(\underline{r} + r^* e_1) + g(\underline{r} + r^*(e_1 + e_2)) + g(\underline{r} + r^*(e_1 - e_2))\Big) \cdot e_2 - \alpha_2(\underline{r} \cdot e_2).$$

First we claim that for $\underline{r}$ with $\underline{r} \cdot e_2 = 0$ we have $N(\underline{r}) \cdot e_2 = 0$. Indeed the first term in N points in e_1 direction, the e_2-components of the next two terms cancel, and the last (linear) term is zero. Thus we can write

$$N(\underline{r}) \cdot e_2 = (\underline{r} \cdot e_2)\tilde{N}(\underline{r}), \tag{3.11}$$

where the function $\tilde{N}$ is continuously differentiable for $|\underline{r}| < r^*$, and $\lim_{\underline{r} \to 0} \tilde{N}(\underline{r}) = 0$ holds by (3.9). By the mean-value theorem we have for, say, $|\underline{r}| \leq r^*/2$ the estimate

$$|\tilde{N}(\underline{r})| \leq \Big(\max_{|\underline{r}| \leq r^*/2} |\nabla \tilde{N}(\underline{r})|\Big)|\underline{r}| = C|\underline{r}|.$$

The assertion of the lemma now follows from (3.11). □

It follows that $||\tilde{r}_c \cdot e_2||_{L^2} \leq \frac{C}{\alpha_1 - \alpha_2}||\tilde{r}_c||_{L^\infty}||\tilde{r}_c \cdot e_2||_{L^2}$. This implies $\tilde{r}_c \cdot e_2 \equiv 0$ provided $||\tilde{r}_c||_{L^\infty} < (\alpha_1 - \alpha_2)/C$. The proof of Theorem 1 is complete.

4 Remarks on stability

The longstanding question of dynamic stability of the solitary wave in the 1D Fermi-Pasta-Ulam chain has recently been resolved affirmatively by Friesecke and Pego in a series of papers [FP01]. This result directly gives the stability of the 2D solitary wave found in this paper with respect to purely longitudinal perturbations, i.e. perturbations which can be realized as perturbed initial conditions in the effective Hamiltonian of section 3.1.

Next we discuss dynamic stability with respect to non-longitudinal perturbations. It is crucial in the 1D results that the wave is supersonic, i.e. its wavespeed exceeds that of any acoustic phonon. We proceed to calculate the dispersion relation for phonons in the 2D lattice. That is we seek solutions of form

$$q_{j,\ell}(t) = \underline{a} e^{i(\underline{k}\cdot\underline{x} - \omega t)}, \quad \underline{x} = \begin{pmatrix} r_* j \\ r_* \ell \end{pmatrix}$$

to the linearized equation of motion

$$\begin{aligned} \ddot{q}_{j,\ell} &= A_1(r_* e_1)(q_{j+1,\ell} - 2q_{j,\ell} + q_{j-1,\ell}) + A_1(r_* e_1)(q_{j,\ell+1} - 2q_{j,\ell} + q_{j,\ell-1}) \\ &+ A_2(r_*(e_1+e_2)(q_{j+1,\ell+1} - 2q_{j,\ell} + q_{j-1,\ell-1}) + A_2(r_*(-e_1+e_2)(q_{j-1,\ell+1} - 2q_{j,\ell} + q_{j+1,\ell-1}) \end{aligned}$$

where A_1, A_2 are the 2×2 matrices $A_1(q_0) = D_q^2 V_1(|q_0+q|)|_{q=0}$, $A_2(q_0) = D_q^2 V_2(|q_0+q|)|_{q=0}$. Under this ansatz the equation reduces to the matrix eigenvalue problem

$$D\underline{a} = \omega^2 \underline{a}$$

with the dispersion matrix

$$\begin{aligned} D &= 4\sin^2 \tfrac{r_* k_1}{2} A_1(r_* e_1) + 4\sin^2 \tfrac{r_* k_2}{2} A_1(r_* e_2) \\ &+ 4\sin^2 \tfrac{r_*(k_1+k_2)}{2} A_2(r_*(e_1+e_2)) + 4\sin^2 \tfrac{r_*(-k_1+k_2)}{2} A_2(r_*(-e_1+e_2)) \end{aligned}$$

where

$$\begin{aligned} A_1(r_* e_1) &= K_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 - \frac{a_1}{r_*} \end{pmatrix}, \quad A_1(r_* e_2) = K_1 \begin{pmatrix} 1 - \frac{a_1}{r_*} & 0 \\ 0 & 1 \end{pmatrix}, \\ A_2(r_*(e_1+e_2)) &= K_2 \Big[\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + (1 - \frac{a_2}{\sqrt{2} r_*}) \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \Big], \\ A_2(r_*(-e_1+e_2)) &= K_2 \Big[(1 - \frac{a_2}{\sqrt{2} r_*}) \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{pmatrix} \Big]. \end{aligned}$$

Hence the dispersion relation $\omega(\underline{k})$ is given by

$$\begin{aligned} \frac{1}{4}(\omega_\pm(k))^2 &= K_1 \left(1 - \frac{a_1}{2r_*}\right) \left(\sin^2 \frac{r_* k_1}{2} + \sin^2 \frac{r_* k_2}{2}\right) \\ &+ K_2 \left(1 - \frac{a_2}{2\sqrt{2} r_*}\right) \left(\sin^2 \frac{r_*(k_1+k_2)}{2} + \sin^2 \frac{r_*(k_1-k_2)}{2}\right) \\ &\pm \Bigg\{ \left[\frac{K_1 a_1}{2r_*} \left(\sin^2 \frac{r_* k_1}{2} - \sin^2 \frac{r_* k_2}{2}\right)\right]^2 \\ &\quad + \left[\frac{K_2 a_2}{2\sqrt{2} r_*} \left(\sin^2 \frac{r_*(k_1+k_2)}{2} - \sin^2 \frac{r_*(k_1-k_2)}{2}\right)\right]^2 \Bigg\}^{\frac{1}{2}} \end{aligned} \tag{4.12}$$

Depending on the spring constants and spring equilibrium lengths K_1, K_2 and a_1, a_2, the maximal group velocity $|\nabla_{\underline{k}} \omega(\underline{k})|$ is attained for different directions $\underline{k}$. For $K_1 = 0$ and $a_1 = a_2/\sqrt{2}$, the dispersion relation reduces to

$$\frac{\omega_\pm^2(k)}{4} = K_2 \sin^2(r_* \frac{k_1 \pm k_2}{2})$$

and the maximal group velocity is attained in $(1,1)$ direction. For $K_2 = 0$ and $a_1 = a_2/\sqrt{2}$ one has

$$\frac{\omega_\pm^2(k)}{4} = \frac{K_1}{2}\Big(\sin^2\frac{r_*k_1}{2} + \sin^2\frac{r_*k_2}{2}\Big) + \frac{K_1}{2}\Big|\sin^2\frac{r_*k_1}{2} - \sin^2\frac{r_*k_2}{2}\Big|$$

and the maximal group velocity is attained in $(1,0)$ direction. Numerical plots of $\omega_\pm$ for different parameters suggest that only these two directions occur, with a sharp crossover. So in the last example, the solitary wave is supersonic with respect to the entire acoustic spectrum. By contrast in the first example the solitary wave is slower than transversal phonons. A possible scenario for the stability of the solitary wave with respect to non-longitudinal perturbations is then as follows: The solitary wave could still be stable when it is supersonic, whereas it looses stability when there are faster transversal phonons.

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