Sobolev Spaces – Autumn 2018 – Background

Lebesgue Measure.

An *interval (rectangle)* $\prod_{n=1}^{N}(a_n, b_n)$ in $\mathbb{R}^N$ has *volume* $\prod_{n=1}^{N}(b_n - a_n)$.

$S \subset \mathbb{R}^N$ has *zero Lebesgue measure* if, for every $\varepsilon > 0$, S can be covered by a countable collection of rectangles of total volume less than ε.

The *Lebesgue measurable sets* in $\mathbb{R}^N$:

- include all open or closed sets (so $\emptyset$, $\mathbb{R}^N$);
- include the complements of all measurable sets;
- include $\bigcup_{n\in\mathbb{N}} S_N$ and $\bigcap_{n\in\mathbb{N}} S_n$ whenever $\{S_n\}_{n\in\mathbb{N}}$ are measurable;
- include all sets of zero measure.

Lebesgue measure on $\mathbb{R}^N$ has the properties:

- the measure of a rectangle is its volume;
- $|\emptyset| = 0$ and $|\mathbb{R}^N| = \infty$;
- hyperplanes have zero measure;
- if $\{S_n\}_{n=1}^{\infty}$ are measurable and disjoint then

$$\left|\bigcup_{n=1}^{\infty} S_n\right| = \sum_{n=1}^{\infty} |S_n|$$

(without disjointness we have $\leq$);

- if $\{S_n\}_{n=1}^{\infty}$ is an increasing (by inclusion) sequence of sets then

$$\left|\bigcup_{n=1}^{\infty} S_n\right| = \lim_{n\to\infty} |S_n|;$$

- if $\{S_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets, and $|S_1| < \infty$ then

$$\left|\bigcap_{n=1}^{\infty} S_n\right| = \lim_{n\to\infty} |S_n|.$$

A property hold *almost everywhere* if the set of points at which it fails has zero Lebesgue measure. Things that happen almost nowhere make no difference, from the point of view of measure and integration(!)

Lebesgue measurable functions

are $f : \mathbb{R}^N \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ such that

$\{x \mid f(x) \geq \alpha\}$ is measurable for every real α.

A function that is continuous almost everywhere, is measurable.

A pointwise limit of measurable functions is measurable.

If $G : \mathbb{R}^m \to \mathbb{R}$ is continuous and $f_1, \ldots, f_m$ are real-valued measurable functions on Ω then $F(x) = g(f_1(x), \ldots, f_m(x))$ defines a measurable function on Ω, hence sums and products of measurable functions are measurable.

$1_{\mathbb{Q}}$ is discontinuous everywhere, but is nevertheless Lebesgue measurable.

The Lebesgue integral.

• If $\phi = \sum_{n=1}^m c_n 1_{S_n}$ is a *simple function*, where the c_n are positive reals and the S_n are measurable sets, define

$$\int_{\mathbb{R}^N} \phi = \sum_{n=1}^m c_n |S_n|.$$

• If $f \geq 0$ is a measurable function define

$$\int_{\mathbb{R}^N} f = \sup \left\{ \int \phi \,\middle|\, \text{simple } 0 \leq \phi \leq f \right\}$$

• If $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ is measurable define

$$\int_{\mathbb{R}^N} f(x)dx = \int_{\mathbb{R}^N} f = \int f_+ - \int f_-$$

provided this is not $\infty - \infty$, where

$f_+(x) = \max\{f(x), 0\}$ and $f_- = (-f)_+$.

• f is *integrable* if f is measurable and

$\int |f| < \infty$.

If $f : \mathbb{R}^N \to \overline{\mathbb{R}}$ is measurable and $X \subset \mathbb{R}^N$ is measurable define

$$\int_X f = \int_{\mathbb{R}^N} 1_X f.$$

f is *locally integrable* if $\int_K |f| < \infty$ for all compact $K \subset \mathbb{R}^N$.

In what follows, $\Omega \subset \mathbb{R}^N$ is a Lebesgue measurable set (usually open), and integrals over Ω are with respect to Lebesgue measure dx, or Lebesgue measure with a locally integrable non-negative density function, over Ω.

Lebesgue's Monotone Convergence theorem. Let $\{f_n\}_{n=1}^\infty$ be an increasing sequence of non-negative measurable functions on Ω, with $f_n \uparrow f$ as $n \to \infty$. Then

$$\int_\Omega f = \lim_{n\to\infty} \int_\Omega f_n.$$

Fatou's Lemma. Let $\{f_n\}_{n=1}^\infty$ be a sequence of non-negative measurable functions on Ω. Then

$$\int_\Omega \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_\Omega f_n.$$

Lebesgue's Dominated Convergence Theorem. Let $g \geq 0$ be integrable on Ω and $\{f_n\}_{n=1}^{\infty}$ a sequence of measurable functions on Ω satisfying $|f_n| \leq g$ a.e. and $f_n \to f$ a.e. as $n \to \infty$. Then

$$\int_\Omega f_n \to \int_\Omega f \quad \text{as } n \to \infty.$$

The *essential supremum* and *essential infimum* of measurable function $f : X \to \overline{\mathbb{R}}$ are

$$\begin{aligned} \text{ess sup}_X f &= \inf\{\xi \in \overline{\mathbb{R}} \mid f \leq \xi \text{ a.e.}\} \\ \text{ess inf}_X f &= \sup\{\xi \in \overline{\mathbb{R}} \mid f \geq \xi \text{ a.e.}\}; \end{aligned}$$

If $1 \leq p < \infty$ then $\mathcal{L}^p(\Omega)$ comprises all finite-valued measurable functions f on Ω for which $\int_\Omega |f|^p < \infty$, and

$$||f||_p = \left(\int_X |f|^p d\mu\right)^{1/p}.$$

$\mathcal{L}^\infty(\Omega)$ comprises all essentially bounded measurable functions on Ω (i.e. finite ess inf and ess sup), and

$$||f||_\infty = \text{ess}\sup_\Omega |f|.$$

Define $L^p(\Omega)$ to be the set of equivalence classes of $\mathcal{L}^p(\Omega)$ under the equivalence relation

$$F \sim g \Leftrightarrow f = g \text{ a.e.}$$

Definition. If $p, q \in (1, \infty)$ with $p^{-1} + q^{-1} = 1$, or $\{p, q\} = \{1, \infty\}$, then we say p and q are *conjugate exponents.*

Young's inequality. Let $1 < p, q < \infty$ be conjugate exponents, x, y non-negative reals. Then

$$xy \leq \frac{x^p}{p} + \frac{y^q}{q}.$$

Hölder's inequality. If $1 \leq p, q \leq \infty$ are conjugate exponents, $f \in L^p(\Omega)$, and $g \in L^q(\Omega)$, then

$$\int_\Omega |fg| \leq \|f\|_p \|g\|_q.$$

Generalised Hölder's inequality. If $p_1, \ldots, p_n \in (1, \infty)$ with $p_1^{-1} + \cdots + p_n^{-1} = 1$, and $u_i \in L^{p_i}(\Omega)$ for $1 \leq i \leq n$, then

$$\int_\Omega |u_1 \cdots u_n| \leq \|u_1\|_{p_1} \cdots \|u_n\|_{p_n}.$$

Interpolation inequality. Let $1 \leq p < q < r < \infty$; choose $0 < \theta < 1$ with $1/q = \theta/p + (1-\theta)/r$. If $u \in L^p(\Omega) \cap L^r(\Omega)$ then

$$\|u\|_q \leq \|u\|_p^\theta \|u\|_r^{1-\theta}.$$

Jensen's inequality. Let $\Psi : \mathbb{R} \to \mathbb{R}$ be a convex function and let ρ be a non-negative measurable function on Ω with $\|\rho\|_1 = 1$. If u is a measurable function on Ω and $\int_\Omega |u(x)|\rho(x)dx < \infty$ then

$$\Psi\left(\int_\Omega u(x)\rho(x)dx\right) \leq \int_\Omega \Psi(u(x))\rho(x)dx.$$

Riesz-Fischer-Young Theorem.

If $1 \leq p \leq \infty$ then $L^p(\Omega)$ with $|| \, ||_p$ is a complete normed vector space (i.e. a Banach space).

Notation for partial derivatives on $\mathbb{R}^N$.

For $1 \leq i \leq N$ write $D_i = \partial/\partial x_i$.

Write $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Any $\alpha = (\alpha_1, \ldots, \alpha_N) \in \mathbb{N}_0^N$ is a *multi-index* of *degree* $|\alpha| = \alpha_1 + \cdots + \alpha_N$. Write $\alpha! = \alpha_1! \cdots \alpha_N!$ and $D^\alpha = D_1^{\alpha_1} \cdots D_N^{\alpha_N}$.

Leibniz's Theorem. If u, v are m-times continuously differentiable functions of n real variables then, for $0 \leq |\alpha| \leq m$,

$$D^\alpha uv = \sum_{0 \leq \beta \leq \alpha} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} D^\beta u D^{\alpha - \beta} v$$

where

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \frac{\alpha!}{\beta!(\alpha - \beta)!}$$

and $\beta \leq \alpha$ signifies $\beta_i \leq \alpha_i$ for $i = 1, \ldots, n$.