Low-Frequency Vibrations of Strongly Inhomogeneous Elastic Structures

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"Threshold phenomena in spectral analysis and their applications to waves and "tectonics

Summer School, Mexico, 12-15 July 2022

Where is Keele?

Outline

- Introduction
- 1D strongly imhomogeneous structures
  - Multi-component rods
  - Multi-span beams
- Anti-plane problem of elasticity
  - Analysis of dispersion
  - Derivation of the model equations
  - Boundary conditions
- Layered plates
- Surface waves in a coated half-space

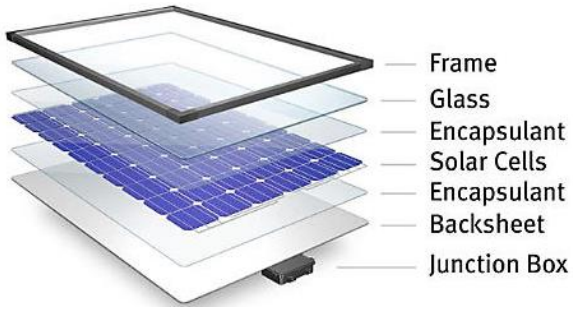
1. Introduction

Preliminary remarks:

- Many modern applications dealing with a diverse variety of configurations
- A lot of enthusiasm based on phenomenological observations
- No consistent multi-parametric approach!
- A number of rigorous, technical contributions on high-contrast homogenisation, but less on multi-component structures.

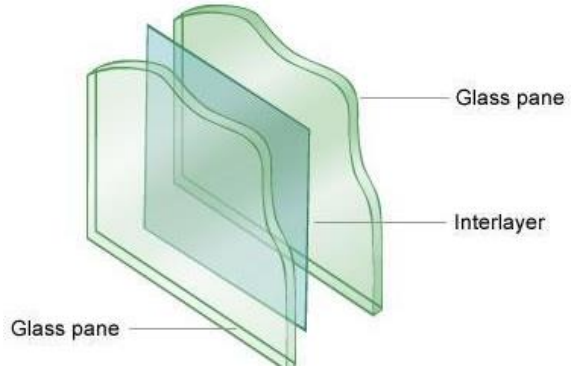
Applications of high-contrast layered structures

- *photovoltaic panels*



www.dupont.com

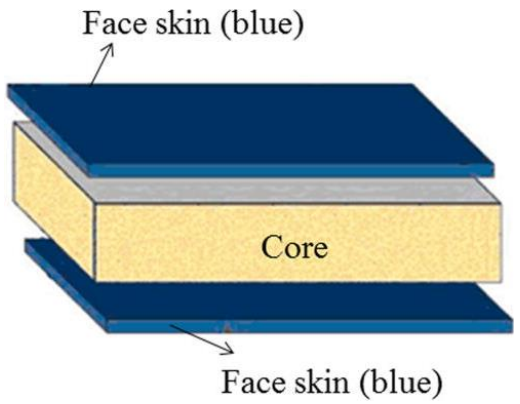
- *laminated glass*



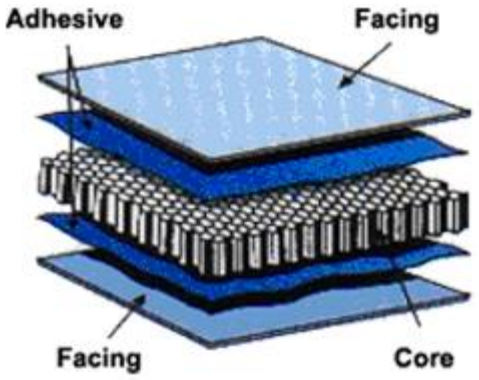
www.gscglass.com

Sandwich structures

- ***Classical sandwich plate***



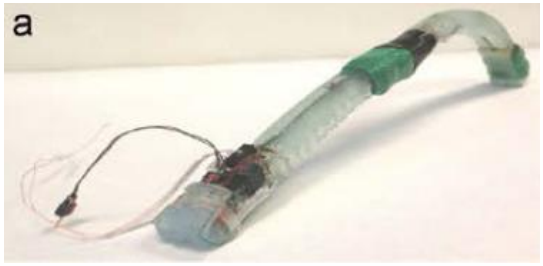
- ***Foam insulation panels***

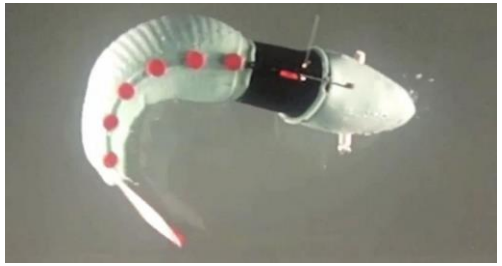


Soft robots

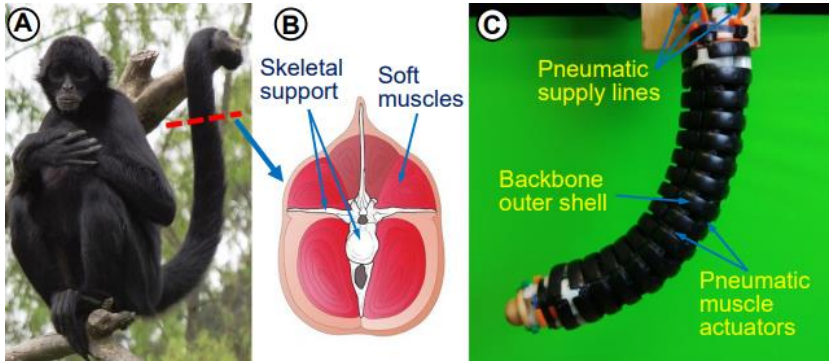
Rus & Tolley, 2015. Design, fabrication and control of soft robots. *Nature*, *521*(7553), 467.

Stokes et al. A hybrid combining hard and soft robots. *Soft Robotics* 1.1 (2014): 70-74





Arachchige & Godage, 2022. Hybrid soft robots incorporating soft and stiff elements. *IEEE Robosoft*: 267-272

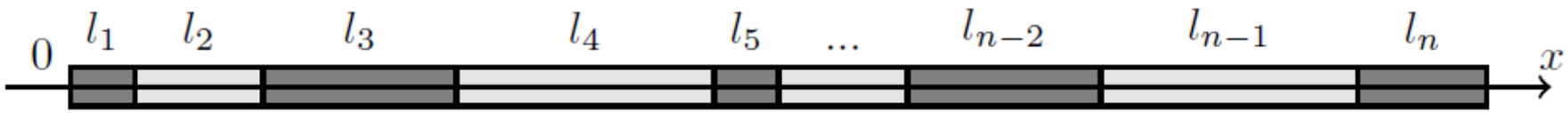


Bio-composites

- *(Teeth, bones, etc., contain both **soft** protein/collagen matrix and **hard** mineral inclusions)*

Slesarenko et al., 2017. Understanding the strength of bioinspired soft composites. *Int. J. Mech. Sci*, *131*, 171-178.

2.1. Low-frequency vibrations of multi-component high-contrast elastic rods



Contrast in

- *Stiffness*
- *Density*
- *Length*

$$\frac{E_i}{E_j}, \qquad \frac{\rho_i}{\rho_j}, \qquad \frac{l_i}{l_j}$$

Small/large parameters → perturbation methods

Physical intuition:

Strong components (free ends) - almost rigid body motions

Weak components (fixed b.c.) - almost homogeneous deformations

- ***Two-component rod***

A.Kudaibergenov, A.Nobili & L.Prikazchikova. On low-frequency vibrations of a composite string with contrast properties for energy scavenging fabric devices. *JOMMS*, 11(3) (2016): 231-243

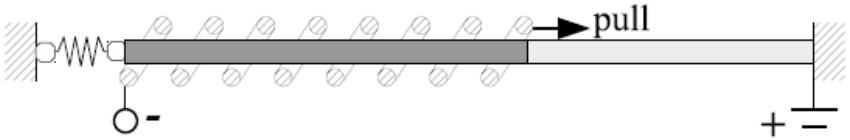


Figure 1. Schematics of a textile energy harvester; compare [Qin et al. 2008, Figure 2a].

Domain: soft component: $0 < x < l_1$; hard component: $l_1 < x < l_1 + l_2$.

Assume $$E = \frac{E_1}{E_2} \ll 1, \qquad l = \frac{l_1}{l_2} \sim 1, \qquad \rho = \frac{\rho_1}{\rho_2} \ll 1.$$

Equations of motion

$$E_i \frac{\partial^2 u_i}{\partial x^2} = \rho_i \frac{\partial^2 u_i}{\partial t^2}, \quad i = 1, 2.$$

Introduce

$$\chi = \frac{x}{l_1}, \qquad \lambda_i = \frac{\omega l_i}{c_i}, \qquad c_i^2 = \frac{E_i}{\rho_i}\,.$$

Assuming time-harmonic dependence

$$u_i = u_i(\chi)e^{-i\omega t}$$

Equations of motion

$$\frac{d^2u_1}{d\chi^2} + \lambda_1^2 u_1 = 0, \qquad \frac{d^2u_2}{d\chi^2} + \lambda_2^2 l^2 u_2 = 0.$$

Continuity conditions

$$u_1(1) = u_2(1), \qquad E_1\frac{du_1(1)}{dx} = E_2\frac{du_2(1)}{dx}\,.$$

Boundary conditions

(a) fixed-free $u_1(0) = \dfrac{du_2(l+1)}{dx} = 0;$

(b) free-fixed $\dfrac{du_1(0)}{dx} = u_2(l+1) = 0;$

(c) fixed-fixed $u_1(0) = u_2(l+1) = 0;$

(d) free-free $\dfrac{du_1(0)}{dx} = \dfrac{du_2(l+1)}{dx} = 0.$

Solving the equations of motion

$$u_1(\chi) = A_1 \cos(\lambda_1 \chi) + B_1 \sin(\lambda_1 \chi),$$

$$u_2(\chi) = A_2 \cos(\lambda_2 l \chi) + B_2 \sin(\lambda_2 l \chi).$$

(a) Fixed-free case

From boundary and continuity conditions

$$A_1 = 0;$$

$$-A_2 \sin(\lambda_2(l+1)) + B_2 \cos(\lambda_2(l+1)) = 0;$$

$$B_1 \sin(\lambda_1) - A_2 \cos(\lambda_2 l) - B_2 \sin(\lambda_2 l) = 0;$$

$$E_1 \lambda_1 B_1 \cos(\lambda_1) - E_2 \lambda_2 l \left[-A_2 \sin(\lambda_2 l) + B_2 \cos(\lambda_2 l)\right] = 0.$$

Hence, from solvability, the frequency equation is

$$E_1\lambda_1\cos(\lambda_1)\cos(\lambda_2) - E_2\lambda_2 l\sin(\lambda_1)\sin(\lambda_2) = 0,$$

or

$$\tan(\lambda_1)\tan(\lambda_2) = \frac{E}{c}\,.$$

Global low-frequency (GLF) regime: $\lambda_1 \ll 1, \quad \text{and} \quad \lambda_2 \ll 1.$

Note $E \ll 1, \quad c \sim 1,$ ⟶ GLF regime is possible!

(b) free-fixed case: $$\tan(\lambda_1)\tan(\lambda_2) = \frac{c}{E};$$

(c) fixed-fixed case $$\tan(\lambda_1 l_1) + \frac{E}{c}\tan(\lambda_2 l_2) = 0;$$

(d) free-free case $$\frac{E}{c}\tan(\lambda_1 l_1) + \tan(\lambda_2 l_2) = 0.$$

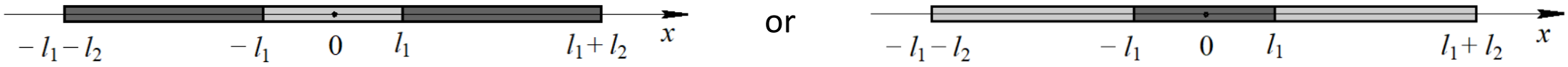
No chance for GLF regime!

(except for pure rigid body motion for fixed-fixed case)

- *Thus, the GLF regime is only possible for **Neumann-type** conditions for **hard** components and **Dirichlet-type** conditions for **soft** ones.*

- ***Antisymmetric motion of a three-component rod***

J.Kaplunov, D.Prikazchikov, O.Sergushova. Multi-parametric analysis of the lowest natural frequencies of strongly inhomogeneous elastic rods. *J. Sound Vib.* 366 (2016): 264-276



Equations of time-harmonic motion

$$E_i \frac{d^2 u_i}{dx^2} + \rho_i \omega^2 u_i = 0, \quad i = 1, 2.$$

Boundary conditions

$$\frac{du_i\left(\pm(l_1+l_2)\right)}{dx} = 0.$$

Continuity conditions

$$u_1(\pm l_1) = u_2(\pm l_1), \quad \text{and} \quad E_1 \frac{du_1(\pm l_1)}{dx} = E_2 \frac{du_2(\pm l_1)}{dx}.$$

Frequency equation

Scaling

$$E = \frac{E_1}{E_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad c = \frac{c_1}{c_2}, \quad l = \frac{l_1}{l_2},$$

and

$$\chi = \frac{x}{l_1}, \quad \lambda_i = \frac{\omega\, l_i}{c_i}, \qquad i = 1, 2.$$

Frequency equation

$$\tan\lambda_1 \tan\lambda_2 = \frac{E}{c}.$$

Low-frequency analysis in view of contrast:

- Global low-frequency behaviour ($\lambda_i \ll 1, \quad i = 1, 2$)
- Local low-frequency behaviour ($\lambda_i \ll 1, \lambda_k \gtrsim 1, \quad i \neq k$)

Displacement

$$u = \sin \lambda_1 \chi, \quad |\chi| \leq 1,$$

$$u = \sin \lambda_1 \cos l\lambda_2(\chi - 1) + \frac{E}{c} \cos \lambda_1 \sin l\lambda_2(\chi - 1), \qquad 1 < |\chi| \leq 1 + l^{-1}.$$

- Now look into Global and Local low-frequency modes

GLF behaviour

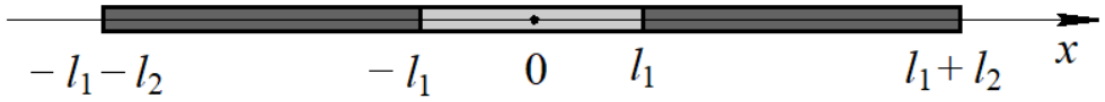
Approximate frequency equation

$$\lambda_1\lambda_2 = \frac{E}{c} \quad \Rightarrow \quad \lambda_1 = \sqrt{\rho l}, \qquad \lambda_2 = \sqrt{\frac{E}{l}}$$

Conditions on material parameters

$$\lambda_1 \ll 1, \quad \lambda_2 \ll 1 \quad \Rightarrow \quad \boxed{E \ll l \ll \rho^{-1}}$$

GLF behaviour

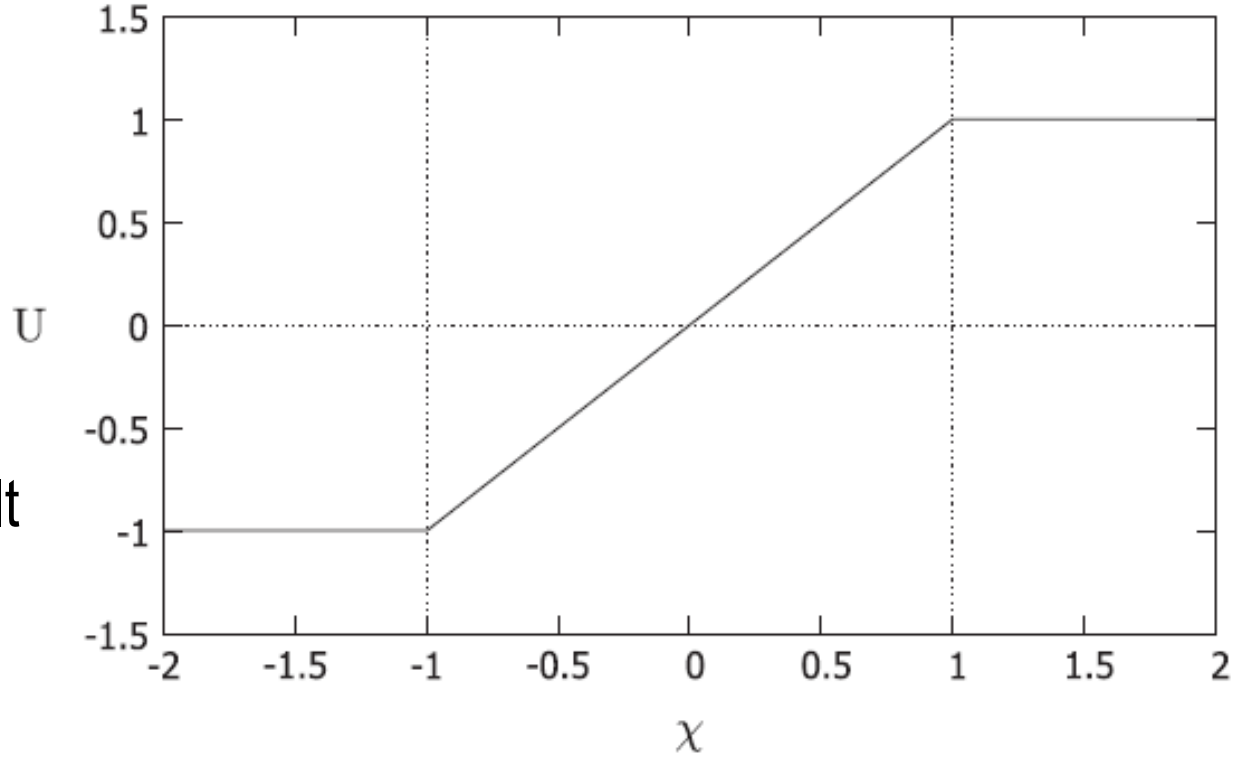


Approximate polynomial eigenform

$$U=\begin{cases}1, & |\chi|>1;\\ \chi, & |\chi|\le 1.\end{cases}$$

- Note: a perturbation approach is also possible, giving the same result

$$\lambda_i^2=\varepsilon(\lambda_{i0}^2+\varepsilon\lambda_{i1}^2+\varepsilon^2\lambda_{i2}^2+\cdots),$$
$$u=u_0+\varepsilon u_1+\varepsilon^2 u_2+\cdots.$$



Local low-frequency (LLF) behaviour

Consider LLF behaviour of the inner component

$$\lambda_1 \ll 1, \quad \lambda_2 \gtrsim 1$$

Approximate frequency equation

$$\tan(\lambda_2) = \frac{E}{l\,\lambda_2}$$

Displacement profile

$$U = \chi, \quad |\chi| \le 1,$$

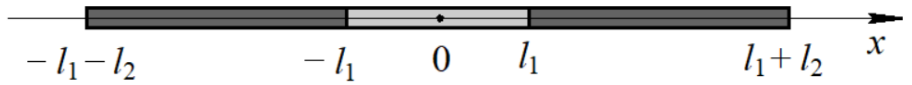
$$U = \cos\, l\,\lambda_2(\chi - 1) + \frac{E}{h\lambda_2} \sin\, l\,\lambda_2(\chi - 1), \quad 1 \le |\chi| \le 1 + l^{-1}.$$

LLF behaviour for inner component (particular cases)

Case (a) – near-resonant vibrations of the strong outer components with free ends

$$\frac{E}{l\,\lambda_2} \ll 1 \quad \Rightarrow \quad \tan(\lambda_2) \ll 1 \quad \Rightarrow \quad \lambda_2 \approx \pi n, \quad n = 1, 2, 3, \ldots$$

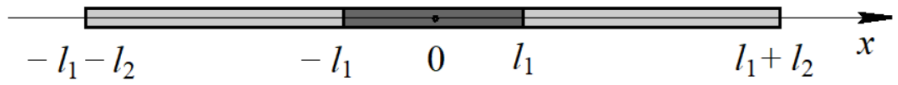
occurs for $\dfrac{E}{l} \ll n \ll \dfrac{c}{l}$.



Case (b): near-resonant vibrations of the weak outer components with fixed-free ends

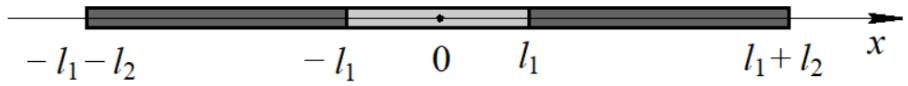
$$\frac{E}{l\,\lambda_2} \gg 1 \quad \Rightarrow \quad \lambda_2 \approx \frac{\pi}{2} + \pi n, \quad n = 1, 2, 3, \ldots$$

occurs for $n \ll \min\left[\dfrac{c}{l}, \dfrac{E}{l}\right]$.

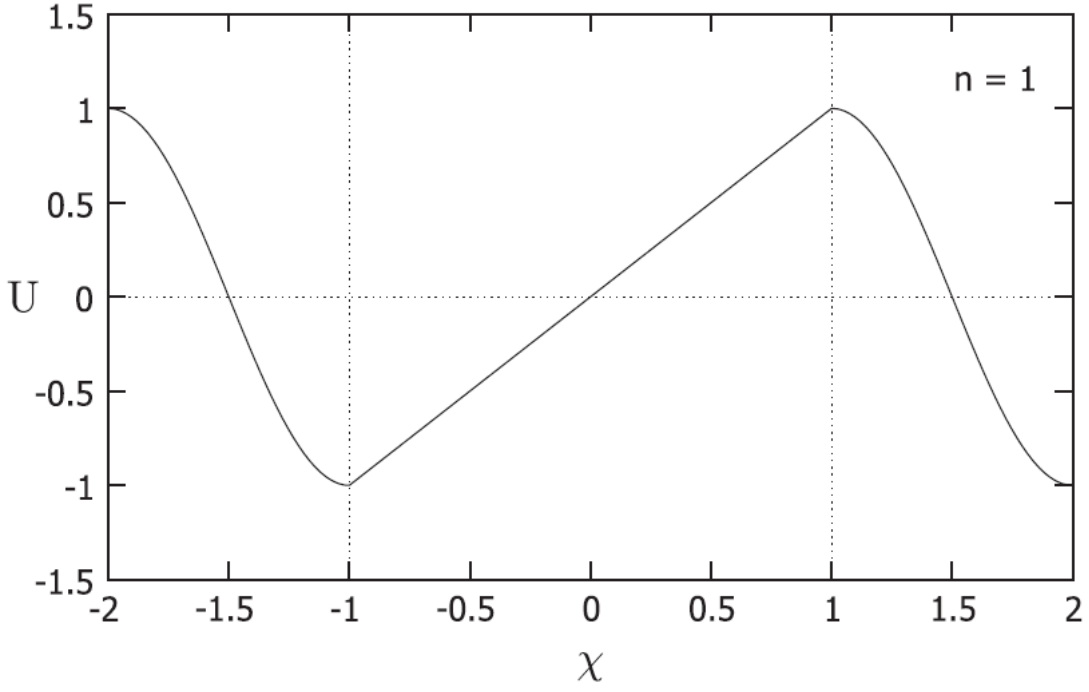


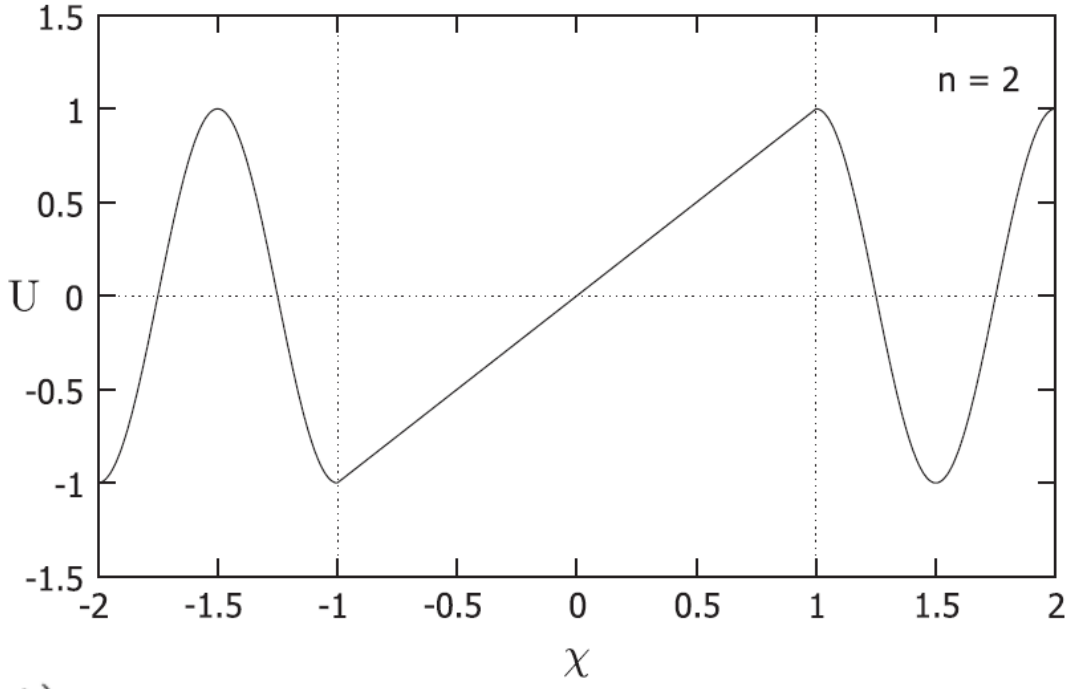
Approximate displacement profiles $\lambda_1 \ll 1, \quad \lambda_2 \gtrsim 1$

Case (a) $U = \chi, \quad |\chi| \le 1;$



$$U = \cos\left(\pi n l(\chi - 1)\right), \quad 1 < |\chi| \le 1 + l^{-1}.$$

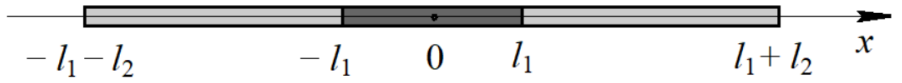




$(l = 1)$

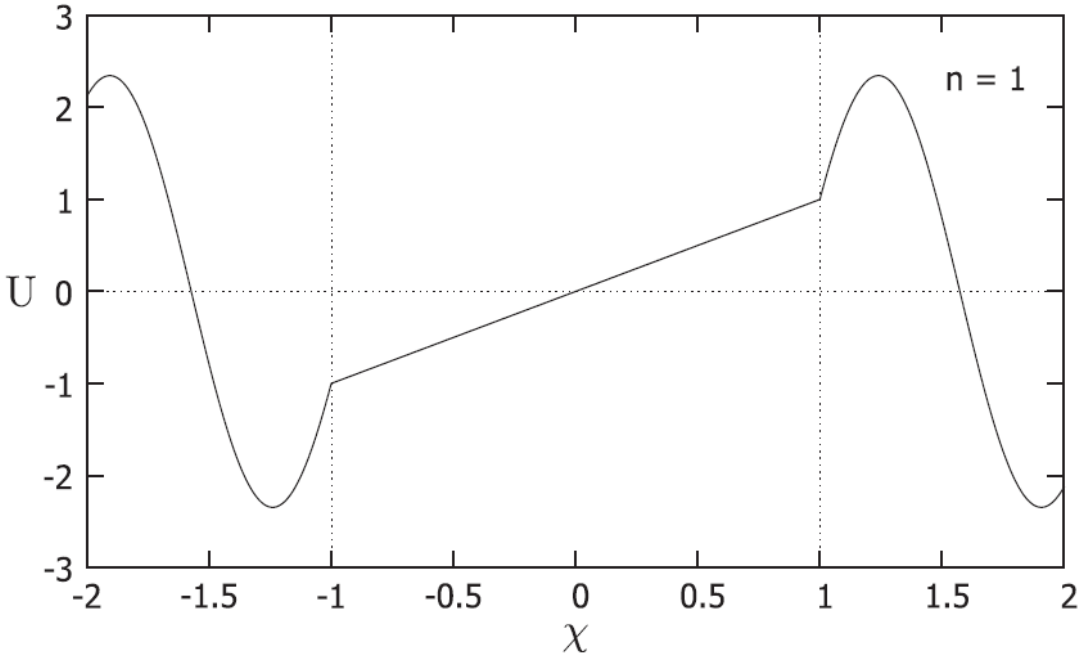
Approximate displacement profiles $\lambda_1 \ll 1, \quad \lambda_2 \gtrsim 1$

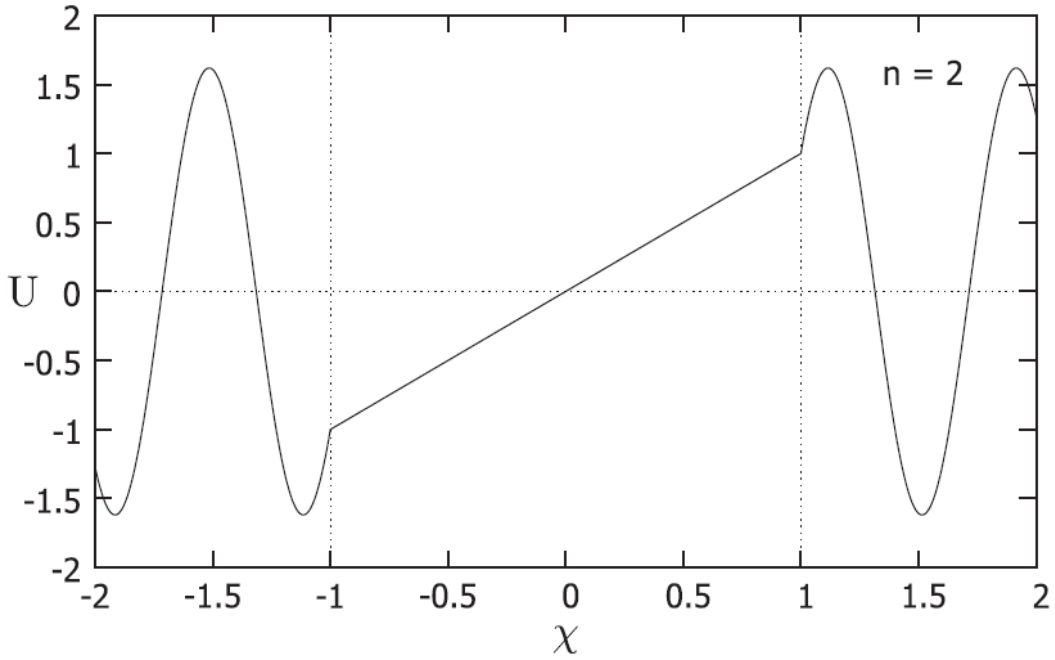
Case (b)



$$U = \chi, \quad |\chi| \le 1;$$

$$U = \cos\frac{\pi(2n+1)}{2}\, l\,(\chi-1) + \frac{2E}{l\pi(2n+1)}\,\sin\frac{\pi(2n+1)}{2}\, l\,(\chi-1), \quad 1 < |\chi| \le 1 + l^{-1}.$$





$(E = 10,\ l = 1)$

Local low-frequency (LLF) behaviour of outer components

$$\lambda_1 \gtrsim 1, \quad \lambda_2 \ll 1$$

Approximate frequency equation

$$\tan(\lambda_1) = \frac{\rho l}{\lambda_1}\,.$$

Displacement profile

$$u = \sin\, \chi\lambda_1, \quad |\chi| \leq 1,$$

$$u = \sin\, \lambda_1 + E\lambda_1(\chi - 1)\, \cos\, \lambda_1, \qquad 1 < |\chi| \leq 1 + l^{-1}.$$

LLF behaviour for outer components (particular cases)

$$\tan(\lambda_1) = \frac{\rho l}{\lambda_1}\,.$$

Case (a) $$\frac{\rho l}{\lambda_1} \ll 1 \quad \Rightarrow \quad \lambda_1 = \pi n,\ n = 1, 2, 3, \ldots$$

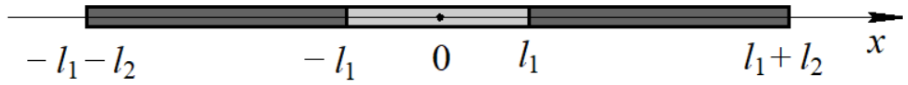
occurs for $$\rho l \ll n \ll \frac{l}{c}\,.$$

Case (b) $$\frac{\rho l}{\lambda_1} \gg 1 \quad \Rightarrow \quad \lambda_1 = \frac{\pi}{2} + \pi n,\ n = 1, 2, 3, \ldots$$

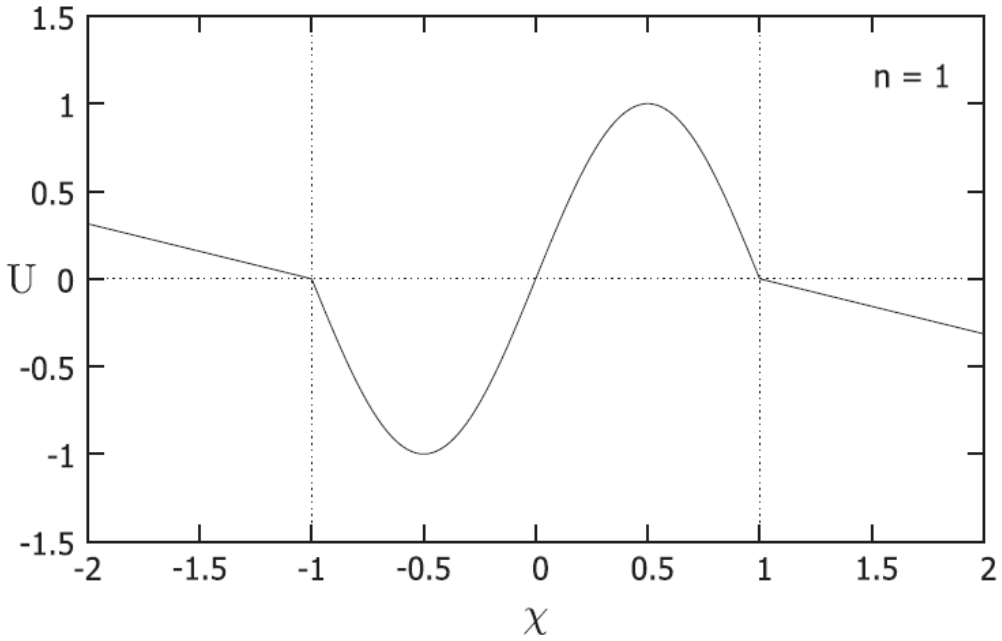
occurs for $$n \ll \min\left[\rho l, \frac{l}{c}\right].$$

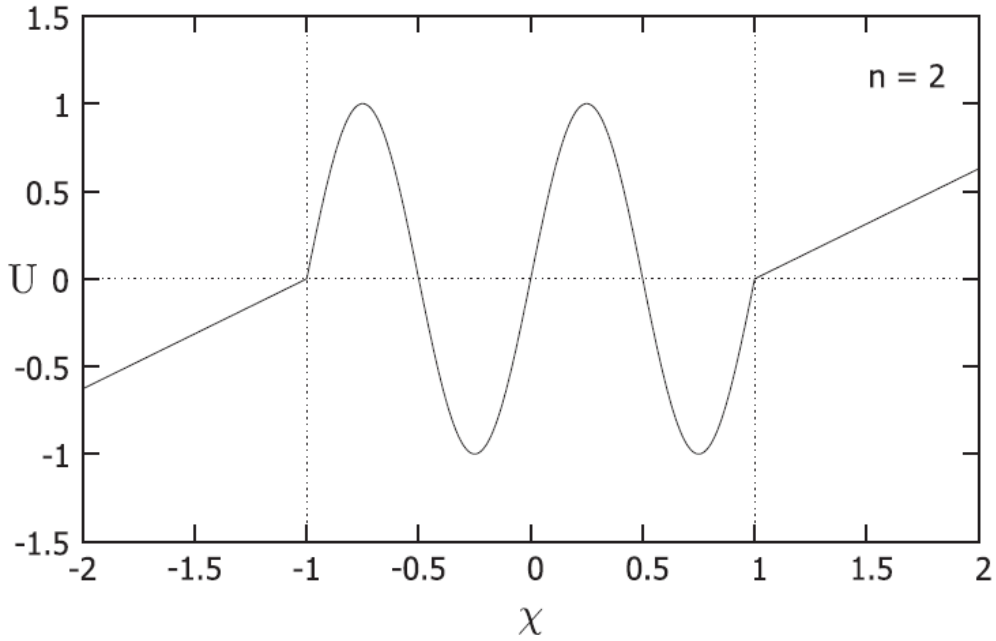
Approximate displacement profiles $\lambda_1 \ll 1, \quad \lambda_2 \gtrsim 1$

Case (a) $u = \sin \pi n \chi, \quad |\chi| \le 1,$



$$u = (-1)^n E \pi n (\chi - 1), \quad 1 < |\chi| \le 1 + l^{-1}.$$





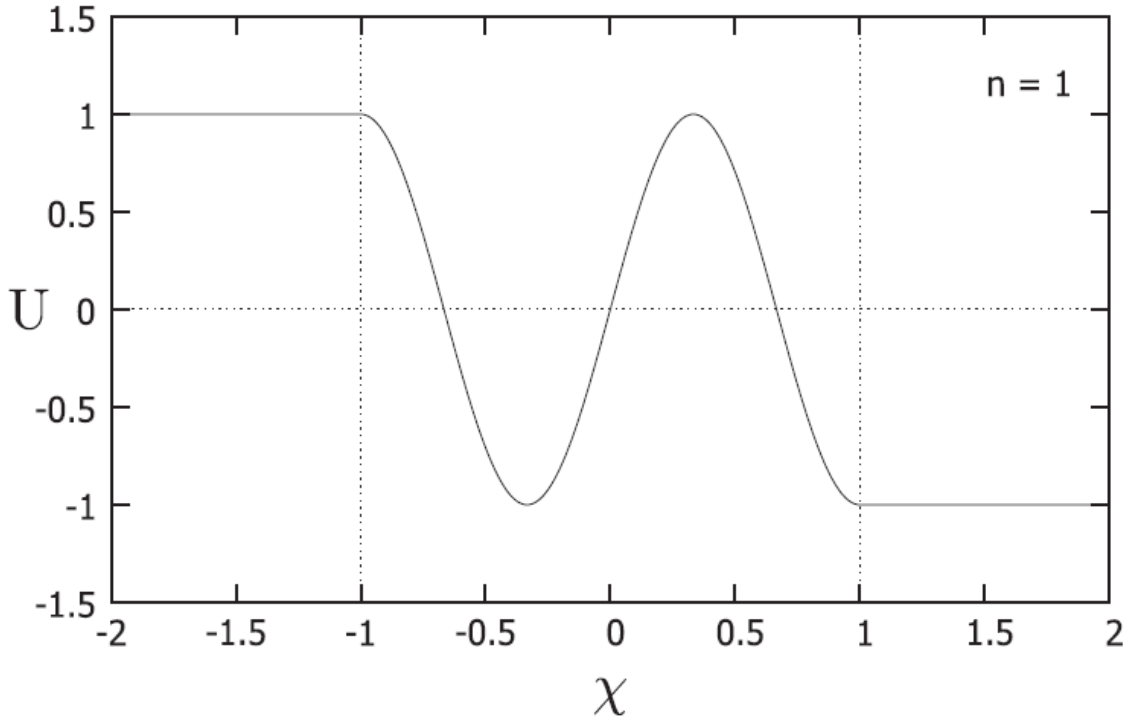
$(E = 0.1,\ l = 1)$

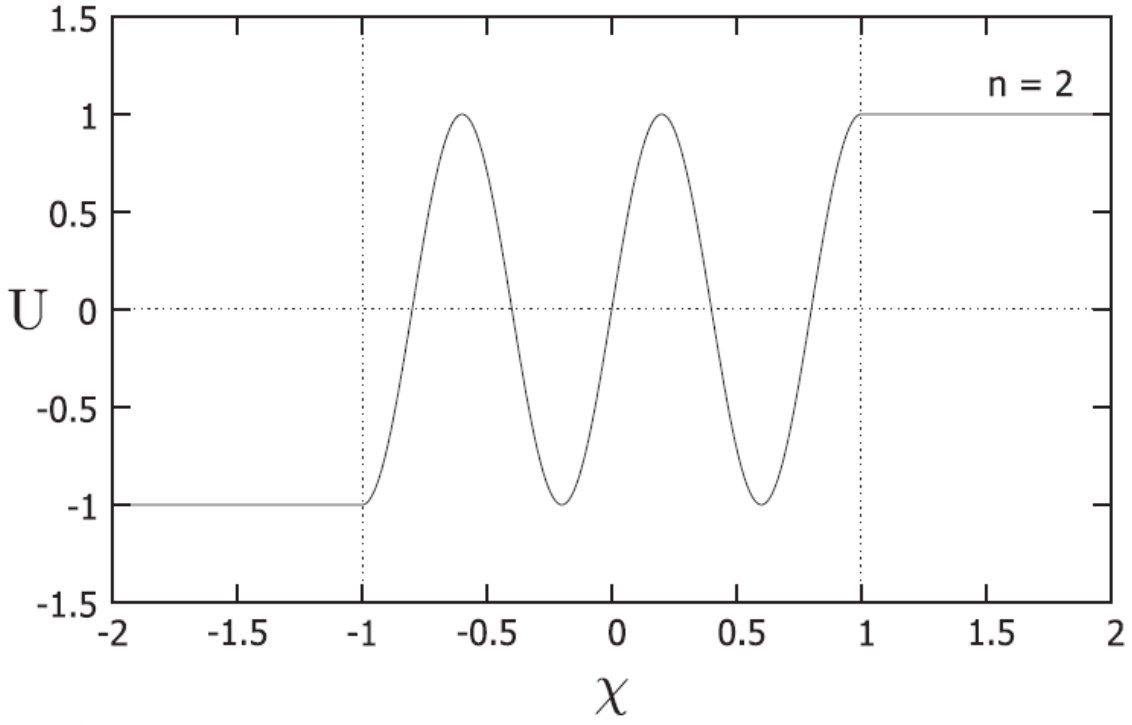
Approximate displacement profiles $\lambda_1 \ll 1, \quad \lambda_2 \gtrsim 1$

Case (b)

$$u = \sin\frac{\pi(2n+1)}{2}\chi, \quad |\chi| \le 1,$$

$$u = (-1)^n, \qquad 1 < |\chi| \le 1 + l^{-1}.$$





$(l = 1)$

Symmetric motion of a three-component rod with fixed ends

Frequency equation

$$\cot \lambda_1 \cot \lambda_2 = \frac{E}{c}.$$

Displacement

$$u = \cos \lambda_1 \chi, \quad |\chi| \le 1,$$

$$u = \cos \lambda_1 \cos l\lambda_2(\chi - 1) - \frac{E}{c} \sin \lambda_1 \sin l\lambda_2(\chi - 1), \quad 1 < |\chi| \le 1 + l^{-1}.$$

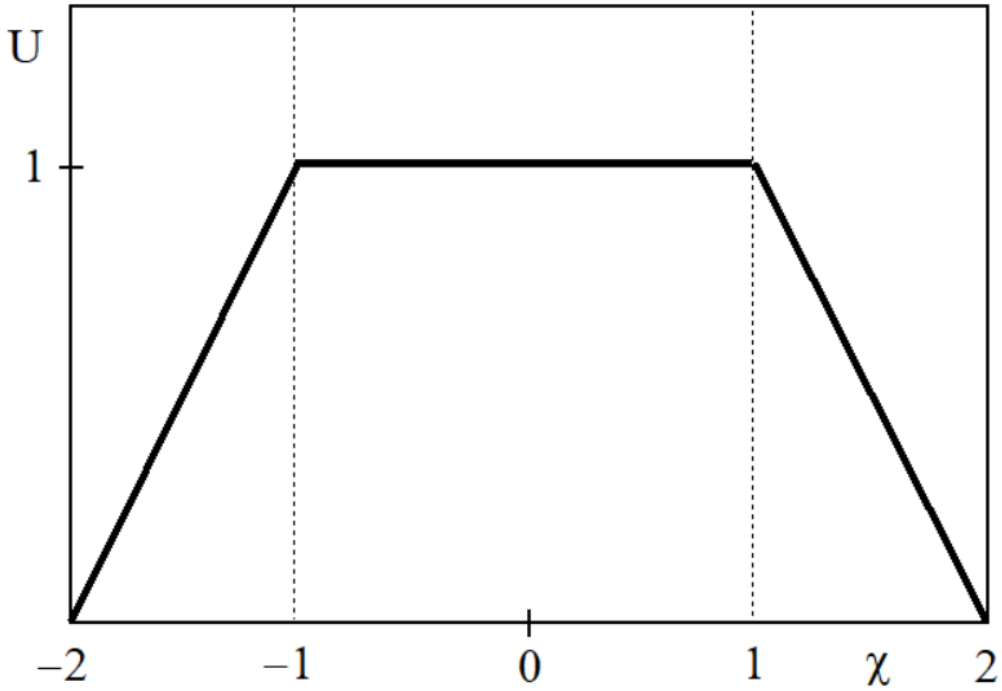
GLF behaviour

$$\lambda_1\lambda_2 = \frac{c}{E}, \quad \Rightarrow \quad \lambda_1 = \sqrt{\frac{l}{E}}, \quad \lambda_2 = \sqrt{\frac{1}{\rho l}} \qquad (\rho^{-1} \ll l \ll E)$$

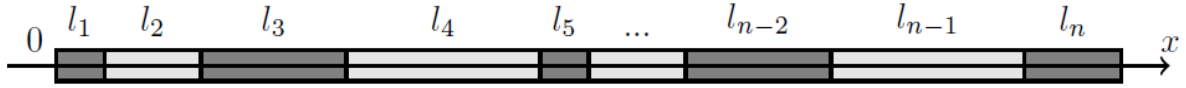
Approximate polynomial eigenform

$$u = 1, \quad |\chi| \le 1,$$

$$u = 1 - E\lambda_1^2(\chi - 1), \quad 1 < |\chi| \le 1 + l^{-1}.$$



- **Multi-component high-contrast elastic rods**



Perturbation approach $\varepsilon = \dfrac{E_2}{E_1} << 1, \qquad c_1 \sim c_2, \qquad L_i^j = \dfrac{l_j}{l_i} \sim 1, \qquad c_m^2 = \dfrac{E_m}{\rho_m}.$

Dimensionalising $X_i = \dfrac{x_i}{l_i}, \quad \Omega_i = \dfrac{\omega l_i}{c_m}, \quad b_i \le X_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1,n}; \quad m = 1,2.$

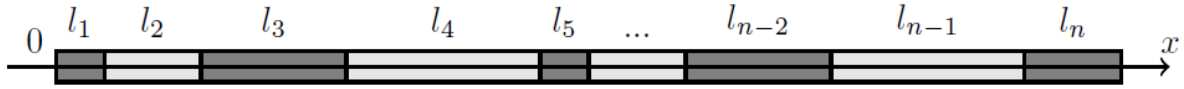
Equations of motion $\dfrac{d^2 u_i}{dX_i^2} + \Omega_i^2 u_i = 0,$

Boundary conditions $\left.\dfrac{du_1}{dX_1}\right|_{X_1=0} = \left.\dfrac{du_n}{dX_n}\right|_{X_n=b_n+1} = 0,$

Continuity $u_i\big|_{X_i=b_i+1} = u_{i+1}\big|_{X_{i+1}=b_{i+1}}, \qquad \left.\dfrac{du_i}{dX_i}\right|_{X_i=b_i+1} = \varepsilon^j L_{i+1}^i \left.\dfrac{du_{i+1}}{dX_{i+1}}\right|_{X_{i+1}=b_{i+1}}.$

($j = 1$ or -1 for i^{th} component being stiff or soft, respectively)

Multi-component high-contrast elastic rods



Asymptotic expansions $u_i = u_{i0} + \varepsilon u_{i1} + \dots,$

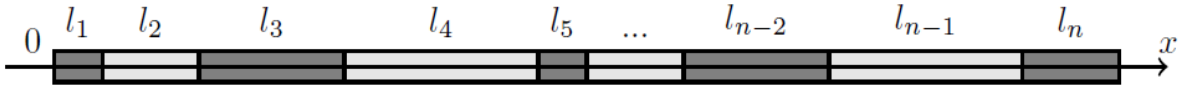
GLF regime $\Omega_i^2 \sim \varepsilon, \quad \Omega_i^2 = \varepsilon\left(\Omega_{i0}^2 + \varepsilon\,\Omega_{i1}^2 + \dots\right).$

- Leading order problem for stiff components

$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad \left.\frac{du_{i0}}{dX_i}\right|_{X_i = b_i} = \left.\frac{du_{i0}}{dX_i}\right|_{X_i = b_i + 1} = 0.$$

(almost rigid body motion) $u_{i0} = C_i = \text{const}$

Multi-component high-contrast elastic rods



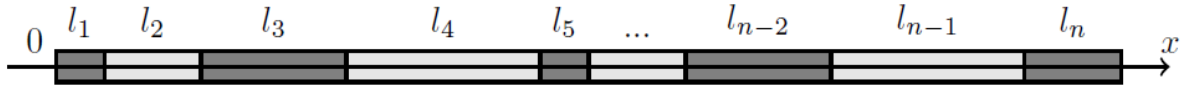
- Leading order problem for soft components

$$\frac{d^2 u_{i0}}{dX_i^2} = 0, \qquad u_i\big|_{X_i=b_i} = C_{i-1}, \quad u_i\big|_{X_i=b_i+1} = C_{i+1}.$$

$$u_{i0} = C_{i-1} + \left(C_{i+1} - C_{i-1}\right)\left(X_i - b_i\right).$$

(almost homogeneous deformation)

Multi-component high-contrast elastic rods



From solvability of next order for stiff components

$$\begin{cases} \Omega_{10}^2 = L_2^1\left(1-\dfrac{C_3}{C_1}\right), \\ \quad\vdots \\ \Omega_{i0}^2 = \left(L_{i-1}^i - L_{i+1}^i\right) - L_{i-1}^i\dfrac{C_{i-2}}{C_i} - L_{i+1}^i\dfrac{C_{i+2}}{C_i}, \\ \quad\vdots \\ \Omega_{n0}^2 = L_{n-1}^n\left(1-\dfrac{C_{n-2}}{C_n}\right); \qquad i = 1,3,5,\ldots,n. \end{cases}$$

⇒ **Polynomial** (not trig.!) equation for eigenfrequencies

- Leading order eigenform – piecewise linear (constant for stiff parts), relation to spring-mass system
- Next order – correction for frequencies, polynomial correction to the eigenform

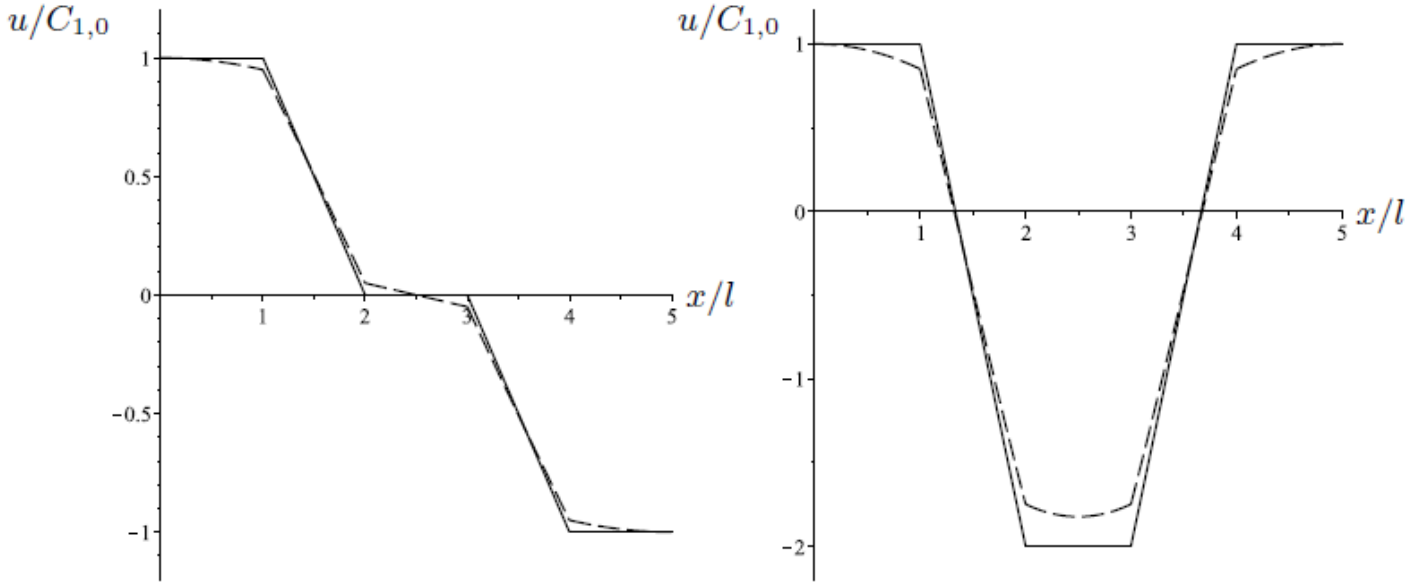
Example: Five-component rod *(free ends)*

Bi-cubic frequency equation

$$\Omega_{10}^6 + a_1\Omega_{10}^4 + a_2\Omega_{10}^2 = 0 \quad \Rightarrow \quad \Omega_{10}^2 = kL_2^1, \quad k = 0 \quad \text{or} \quad k = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_2}}{2L_2^1}.$$

Approximate polynomial eigenform

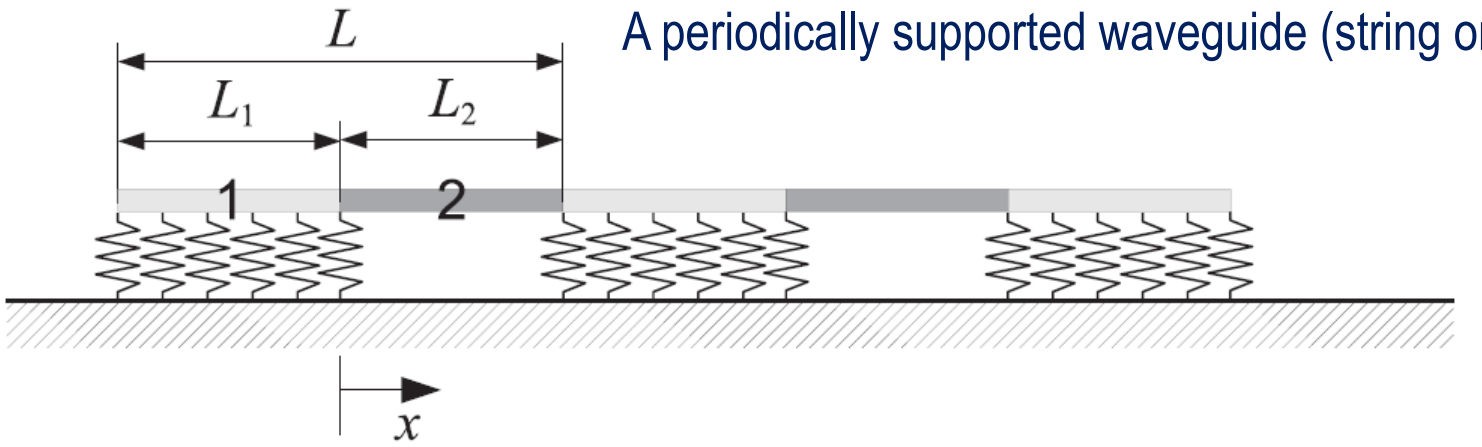
- For $k = 0$ the solution is exact – rigid body motion!



(a) $k = 1$ (b) $k = 3$

Further developments: periodic structures

- J.Kaplunov, A.Nobili. (2017). Multi-parametric analysis of strongly inhomogeneous periodic waveguides with internal cutoff frequencies. *Math.Met.Appl.Sci.*, *40*(9), 3381-3392.



A periodically supported waveguide (string or beam)

Microscale rigid body motion is retrieved at leading order, occurring near a non-zero cutoff frequency

A double cancellation:

- over the supported region, the inertia force balances the support reaction because of the proximity of global and internal cutoffs
- over the unsupported region, the inertia force vanishes due to high contrast

Microscale equations: string on a Winkler; beam on Pasternak foundations

2.2. Low-frequency vibrations of multi-span high-contrast elastic beams

O.Şahin, B.Erbaş, J.Kaplunov, T.Savšek, (2020). The lowest vibration modes of an elastic beam composed of alternating stiff and soft components. *Arch.Appl.Mech.*, *90*(2), 339-352.

O. Şahin, (2021). On the lowest frequencies of strongly inhomogeneous multicomponent beams. *Math.Met.Appl.Sci.*, *44*(11), 9302-9314.

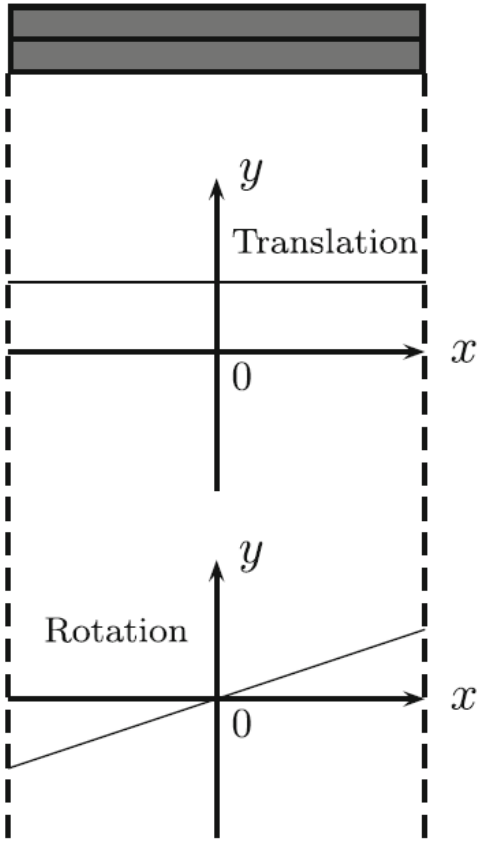
Equations of motion

$$\frac{d^4 u_i}{dx^4} - \frac{\omega^2}{a_m^2} u_i = 0, \; i = 1, 2, \ldots, n, \; m = 1, 2,$$

where

$$a_m = \sqrt{\frac{E_m I}{\rho_m A}}$$

Note: rigid body motions for a beam may include both translations and rotations



Continuity conditions

$$u_i|_{x=\chi_i} = u_{i+1}|_{x=\chi_i}, \qquad \left.\frac{du_i}{dx}\right|_{x=\chi_i} = \left.\frac{du_{i+1}}{dx}\right|_{x=\chi_i},$$

$$\left.\frac{d^2u_i}{dx^2}\right|_{x=\chi_i} = \varepsilon^j \left.\frac{d^2u_{i+1}}{dx^2}\right|_{x=\chi_i}, \qquad \left.\frac{d^3u_i}{dx^3}\right|_{x=\chi_i} = \varepsilon^j \left.\frac{d^3u_{i+1}}{dx^3}\right|_{x=\chi_i},$$

with

$$\varepsilon = \frac{E_2}{E_1} \ll 1 \quad \text{and} \quad \chi_i = \sum_{m=1}^{i} l_m, \; i = 1, 2 \; \ldots \; , n,$$

where $j = 1$ or $j = -1$ for the case of the ith component being stiff/soft.

Two types of boundary conditions

Free ends:

$$\left.\frac{d^2u_1}{dx^2}\right|_{x=0} = \left.\frac{d^2u_n}{dx^2}\right|_{x=\chi_n} = \left.\frac{d^3u_1}{dx^3}\right|_{x=0} = \left.\frac{d^3u_n}{dx^3}\right|_{x=\chi_n} = 0,$$

and simply supported ends:

$$u_1|_{x=0} = u_n|_{x=\chi_n} = \left.\frac{d^2u_1}{dx^2}\right|_{x=0} = \left.\frac{d^2u_n}{dx^2}\right|_{x=\chi_n} = 0.$$

- Note: *the range of boundary conditions, allowing global low-frequency, is less clear for the beam, than for a rod.*

Scaling

$$\xi_i = \frac{x}{l_i}, \quad h_i = \frac{\chi_{i-1}}{l_i}, \quad \lambda_i = l_i \sqrt{\frac{\omega,}{a_m}} \; i = 1, 2, \ldots, n, \; m = 1, 2.$$

Then,

$$h_i \le \xi_i \le h_i + 1, \qquad \lambda_i^4 = \left(L_j^i\right)^4 \lambda_j^4 \left(\frac{a_1}{a_2}\right)^{2p},$$

where $L_j^i = l_i / l_j$ and

$p = 1$ if i is odd, j is even, $\quad p = -1$ if i is even, j is odd, $\quad p = 0$ if $i + j$ is even.

Assume $\rho = \frac{\rho_1}{\rho_2} \sim \varepsilon, \qquad L_i^j \sim 1$.

- GLF perturbation procedure (free ends)

Asymptotic expansions

$$\lambda_i^4 = \varepsilon\left(\lambda_{i0}^4 + \varepsilon\lambda_{i1} + \varepsilon^2\lambda_{i2} + \ \dots\ \right), \qquad u_i = u_i^0 + \varepsilon u_i^1 + \varepsilon^2 u_i^2 + \ \dots\ .$$

Leading order problem for stiff components: $\dfrac{d^4u_i^0}{d\xi_i^4} = 0,$ subject to

$$\left.\frac{d^2u_i^0}{d\xi_i^2}\right|_{\xi_i=h_i} = \left.\frac{d^2u_i^0}{d\xi_i^2}\right|_{\xi_i=1+h_i} = \left.\frac{d^3u_i^0}{d\xi_i^3}\right|_{\xi_i=h_i} = \left.\frac{d^3u_i^0}{d\xi_i^3}\right|_{\xi_i=1+h_i} = 0,$$

$$i = 2k-1,\ k = 1, 2, \dots, n_1, \quad n_1 = (n+1)/2.$$

Solutions (rigid body motions) $u_i^0 = A_i\xi_i + B_i$

Leading order problem for soft components: $\dfrac{d^4 u_i^0}{d\xi_i^4} = 0,$ subject to

$$u_{i+1}^0\Big|_{\xi_{i+1}=h_{i+1}} = u_i^0\big|_{\xi_i=1+h_i}, \qquad u_{i+1}^0\Big|_{\xi_{i+1}=1+h_{i+1}} = u_{i+2}^0\Big|_{\xi_{i+2}=h_{i+2}},$$

$$\frac{du_{i+1}^0}{d\xi_{i+1}}\bigg|_{\xi_{i+1}=h_{i+1}} = L_i^{i+1}\,\frac{du_i^0}{d\xi_i}\bigg|_{\xi_i=1+h_i}, \qquad \frac{du_{i+1}^0}{d\xi_{i+1}}\bigg|_{\xi_{i+1}=1+h_{i+1}} = L_{i+2}^{i+1}\,\frac{du_{i+2}^0}{d\xi_{i+2}}\bigg|_{\xi_{i+2}=h_{i+2}},$$

where $i = 2k-1,\ k = 1, 2, \ldots, n_1 - 1.$

$$u_{i+1}^0 = A_{i+1}\xi_{i+1}^3 + B_{i+1}\xi_{i+1}^2 + C_{i+1}\xi_{i+1} + D_{i+1},\ i = 2k-1,\ k = 1, 2, \ldots, n_1 - 1.$$

The coefficients for soft components may be expressed through those for stiff ones as

$$\begin{pmatrix} A_{i+1} \\ B_{i+1} \\ C_{i+1} \\ D_{i+1} \end{pmatrix} = \mathbf{P_i} \begin{pmatrix} A_i \\ B_i \\ A_{i+2} \\ B_{i+2} \end{pmatrix}, \qquad i = 2k-1,\ k = 1, 2, \ \dots\ n_1 - 1.$$

Next order problem for stiff components $\dfrac{d^4 u_i^1}{d\xi_i^4} - \lambda_{i0}^4 u_i^0 = 0,$ subject to

$$\left.\frac{d^2 u_i^1}{d\xi_i^2}\right|_{\xi_i = 1+h_i} = \left(L_{i+1}^i\right)^2 \left.\frac{d^2 u_{i+1}^0}{d\xi_{i+1}^2}\right|_{\xi_{i+1} = h_{i+1}}, \qquad \left.\frac{d^2 u_{i+2}^1}{d\xi_{i+2}^2}\right|_{\xi_{i+2} = h_{i+2}} = \left(L_{i+1}^{i+2}\right)^2 \left.\frac{d^2 u_{i+1}^0}{d\xi_{i+1}^2}\right|_{\xi_{i+1} = 1+h_{i+1}},$$

$$\left.\frac{d^3 u_i^1}{d\xi_i^3}\right|_{\xi_i = 1+h_i} = \left(L_{i+1}^i\right)^3 \left.\frac{d^3 u_{i+1}^0}{d\xi_{i+1}^3}\right|_{\xi_{i+1} = h_{i+1}}, \qquad \left.\frac{d^3 u_{i+2}^1}{d\xi_{i+2}^3}\right|_{\xi_{i+2} = h_{i+2}} = \left(L_{i+1}^{i+2}\right)^3 \left.\frac{d^3 u_{i+1}^0}{d\xi_{i+1}^3}\right|_{\xi_{i+1} = 1+h_{i+1}},$$

and $$\left.\frac{d^2 u_1^1}{d\xi_1^2}\right|_{\xi_1 = h_1} = \left.\frac{d^2 u_n^0}{d\xi_n^2}\right|_{\xi_n = h_n + 1} = \left.\frac{d^3 u_1^1}{d\xi_1^3}\right|_{\xi_1 = h_1} = \left.\frac{d^3 u_n^0}{d\xi_n^3}\right|_{\xi_n = h_n + 1} = 0.$$

Integrating

$$\int_{h_i}^{h_i+1} \frac{d^4 u_i^1}{d\xi_i} d\xi_i - \lambda_{i0}^4 \left\{ A_i \int_{h_i}^{h_i+1} \xi_i d\xi_i + B_i \int_{h_i}^{h_i+1} d\xi_i \right\} = 0, \; i = 2k-1, k = 2, \ldots n_1 - 1,$$

$$\int_{h_i}^{h_i+1} \xi_i \frac{d^4 u_i^1}{d\xi_i} d\xi_i - \lambda_{i0}^4 \left\{ A_i \int_{h_i}^{h_i+1} \xi_i^2 d\xi_i + B_i \int_{h_i}^{h_i+1} \xi_i d\xi_i \right\} = 0,$$

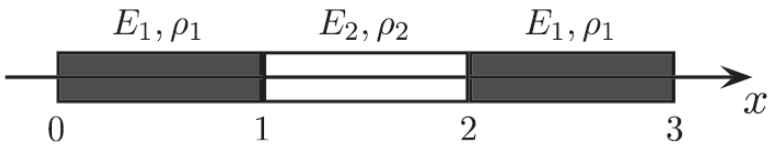
and treating similarly the near-end components, we deduce $2n_1 + 2$ equations for the constants A_i and B_i, associated with the stiff components.

Solvability of the latter gives the frequency polynomial equation.

Note 1: twice more equations compared to the rod.

Note 2: the case of simply supported ends is treated similarly.

Example: 3-component rod (free ends)

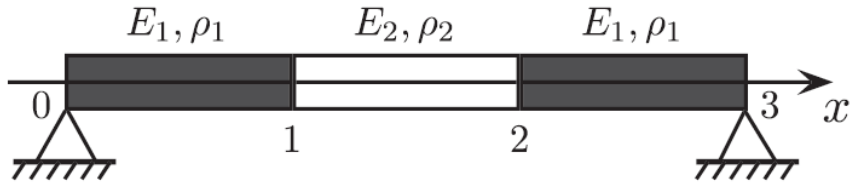


$$12(3A_1 + 2B_1 - 3A_3 - 2B_3) - (A_1 + 2B_1)\lambda_{10}^4 = 0$$
$$12(14A_1 + 9B_1 - 14A_3 - 9B_3) - (2A_1 + 3B_1)\lambda_{10}^4 = 0$$
$$12(3A_1 + 2B_1 - 3A_3 - 2B_3) - (5A_3 + 2B_3)\lambda_{30}^4 = 0$$
$$12(14A_1 + 9B_1 - 14A_3 - 9B_3) - (38A_3 + 15B_3)\lambda_{30}^4 = 0.$$

$$\lambda_{10}^4 \left(\lambda_{10}^4 - 312\right)\left(\lambda_{10}^4 - 24\right) = 0$$

- Zero eigenvalue corresponds to rigid body motion.

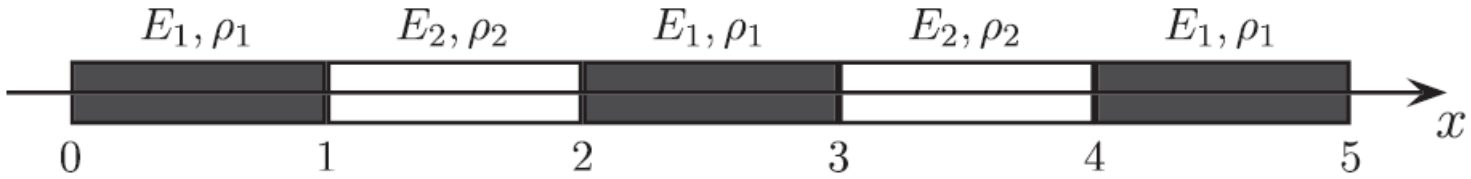
Example: 3-component rod (simply supported)



$$\left(\lambda_{10}^4 - 162\right)\left(\lambda_{10}^4 - 6\right) = 0.$$

- Rigid body motions are not supported.

Example: 5-component rod (free ends)



$$12\,(3A_1 + 2B_1 - 3A_3 - 2B_3) - (A_1 + 2B_1)\,\lambda_{10}^4 = 0$$

$$12\,(14A_1 + 9B_1 - 14A_3 - 9B_3) - (2A_1 + 3B_1)\,\lambda_{10}^4 = 0$$

$$12\,(3A_1 + 2B_1 - 10A_3 - 4B_3 + 7A_5 + 2B_5) + (5A_3 + 2B_3)\,\lambda_{30}^4 = 0$$

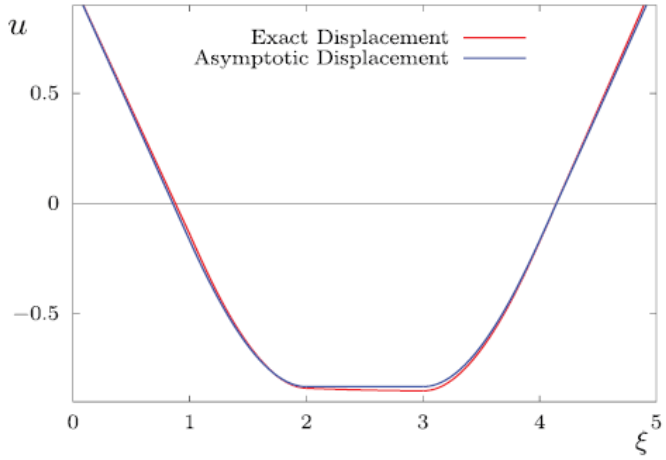
$$12\,(14A_1 + 9B_1 - 88A_3 - 30B_3 + 74A_5 + 21B_5) + (38A_3 + 15B_3)\,\lambda_{30}^4 = 0$$

$$12\,(7A_3 + 2B_3 - 7A_5 - 2B_5) + (9A_5 + 2B_5)\,\lambda_{50}^4 = 0$$

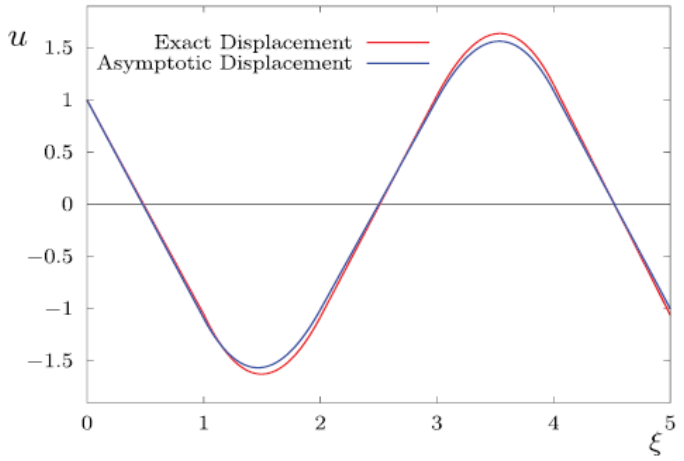
$$12\,(74A_3 + 21B_3 - 74A_5 - 21B_5) + (122A_5 + 27B_5)\,\lambda_{50}^4 = 0.$$

$$\lambda_{10}^4 \left(\lambda_{10}^8 - 480\lambda_{10}^4 + 14256\right)\left(\lambda_{10}^8 - 192\lambda_{10}^4 + 432\right) = 0$$

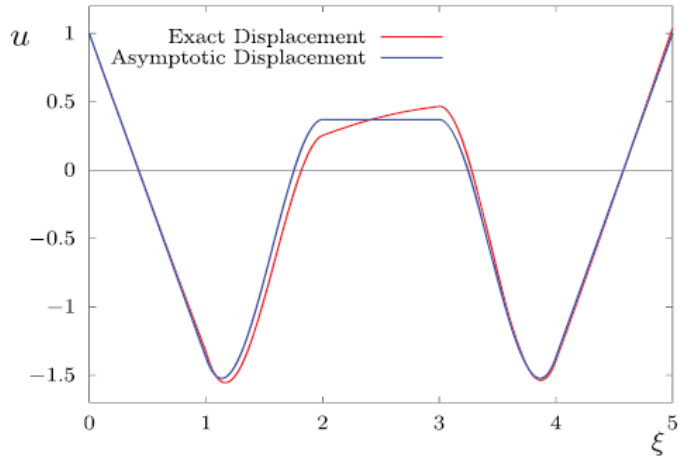
Numerical illustrations: comparison of “exact” and approximate eigenmodes



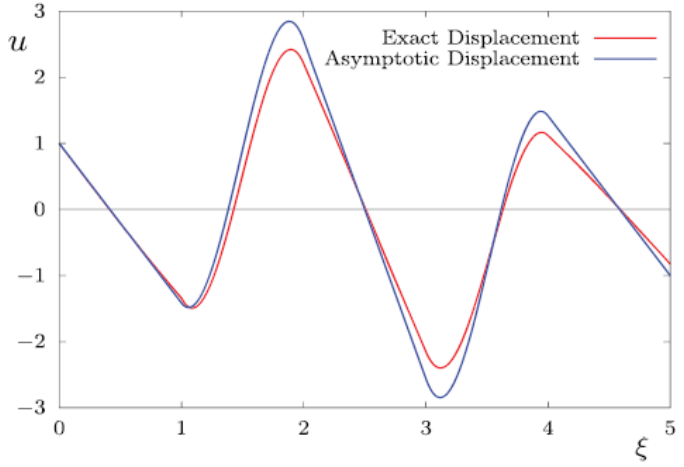
(A) $\lambda_{10}^4 = 2.2770$



(B) $\lambda_{10}^4 = 31.8078$



(C) $\lambda_{10}^4 = 189.7230$



(D) $\lambda_{10}^4 = 448.192$

Five-component beam with free ends.

Contrast $\varepsilon = 0.01$

Remarks

- T. Fang, I. Elishakoff, C. Jiang. (2020). Random response of multi-segment beam may exceed response of homogeneous counterparts by order of magnitude. *Int.J.Struct.Stab.Dyn.*, *20*(13), 2041006.

- LLF regime may be studied

- Effect of elastic foundation may be incorporated

- Other types of boundary conditions may be studied

3. Antiplane problems in elasticity for strongly inhomogeneous solids

J.Kaplunov, D.A.Prikazchikov, O.Sergushova, (2017). Lowest vibration modes of strongly inhomogeneous elastic structures. In Mechanics for Materials and Technologies (pp. 265-277). Springer, Cham.

$$\Delta u_i + \frac{\omega^2}{c_i^2} u_i = 0, \qquad i = 1,2, \qquad c_i = \sqrt{\mu_i/\rho_i}$$

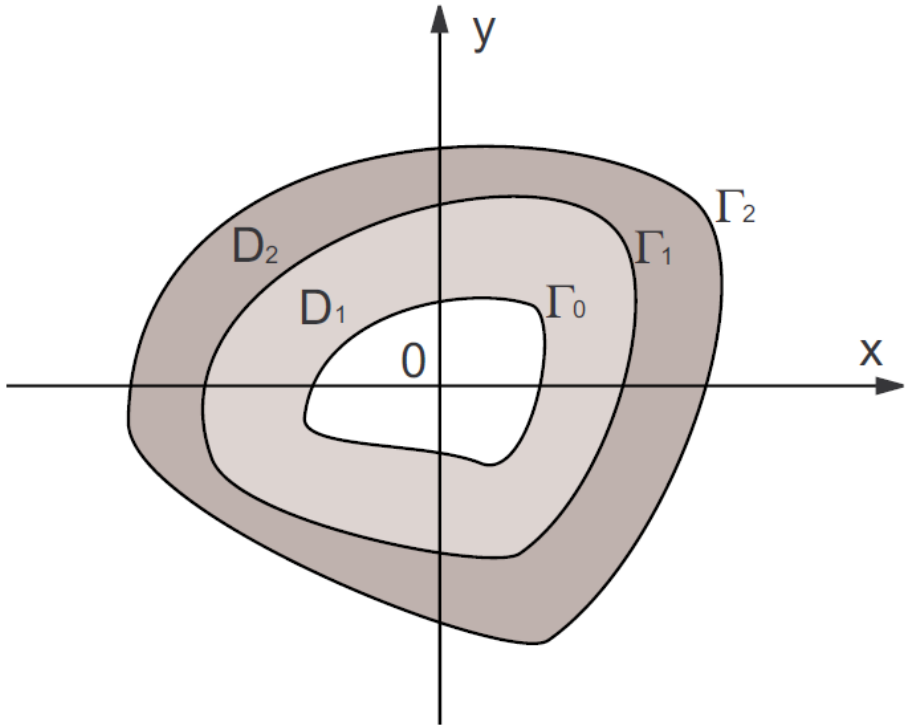
Scaling

$$\mu = \frac{\mu_1}{\mu_2}, \quad l = \frac{l_1}{l_2}, \quad \rho = \frac{\rho_1}{\rho_2}, \quad \lambda_i = \frac{\omega l_i}{c_i}$$

$$X_i = \frac{x}{l_i}, \quad Y_i = \frac{y}{l_i}, \quad i = 1,2.$$

$$\Delta u_i + \lambda_i^2 u_i = 0, \qquad i = 1,2, \qquad (u_1 - u_2)|_{\Gamma_1} = 0,$$

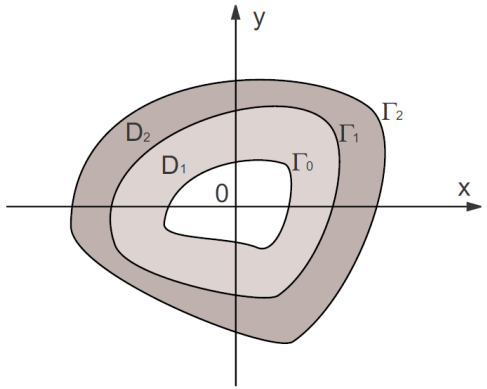
$$\left(\frac{\mu}{l} \frac{\partial u_1}{\partial n} - \frac{\partial u_2}{\partial n} \right)\Big|_{\Gamma_1} = 0, \qquad u_1|_{\Gamma_0} = \left. \frac{\partial u_2}{\partial n} \right|_{\Gamma_2} = 0.$$



A two-layered cylindrical body

Small parameter $\varepsilon = \frac{\mu}{l} \ll 1$

GLF perturbation scheme



$$\mu \sim \rho \sim \varepsilon, \quad l \sim 1, \quad \lambda_1^2 \sim \lambda_2^2 \sim \varepsilon.$$

$$u_i = u_i^{(0)} + \varepsilon u_i^{(1)} + \varepsilon^2 u_i^{(2)} + \ldots,$$

$$\lambda_i^2 = \varepsilon(\lambda_{i0}^2 + \varepsilon\lambda_{i1}^2 + \varepsilon^2\lambda_{i2}^2 + \ldots).$$

Leading order problem for stiff component $\Delta u_2^{(0)} = 0,$ $\left.\frac{\partial u_2^{(0)}}{\partial n}\right|_{\Gamma_1} = \left.\frac{\partial u_2^{(0)}}{\partial n}\right|_{\Gamma_2} = 0.$

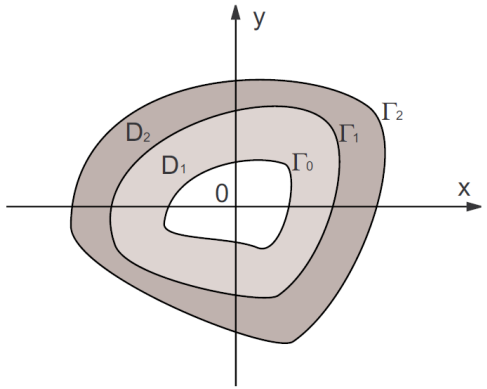
→ $u_2^{(0)} = C = \text{const}$ (rigid body motion)

Leading order problem for soft component $\Delta u_1^{(0)} = 0, \quad u_1^{(0)}|_{\Gamma_0} = 0, \quad u_1^{(0)}|_{\Gamma_1} = C.$

→ $u_1^{(0)} = CH_1(X_1, Y_1), \quad H_1(X_1, Y_1)|_{\Gamma_0} = 0, \quad H_1(X_1, Y_1)|_{\Gamma_1} = 1.$

Next order analysis for a stiff component

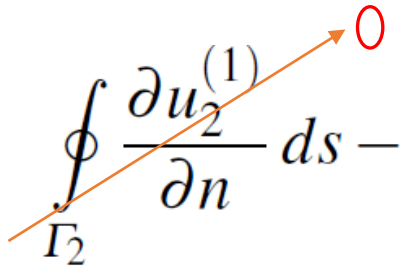
$$\Delta u_2^{(1)} = -\lambda_{20}^2 u_2^{(0)}, \qquad \left.\frac{\partial u_2^{(1)}}{\partial n}\right|_{\Gamma_1} = C \left.\frac{\partial H_1}{\partial n}\right|_{\Gamma_1}, \qquad \left.\frac{\partial u_2^{(1)}}{\partial n}\right|_{\Gamma_2} = 0.$$



Integrating over the outer domain

$$\iint_{D_2} \Delta u_2^{(1)} dX_2 dY_2 = -\lambda_{20}^2 C \iint_{D_2} dX_2 dY_2.$$

Employing the first Green's identity



$$\oint_{\Gamma_2} \frac{\partial u_2^{(1)}}{\partial n} ds - \oint_{\Gamma_1} \frac{\partial u_2^{(1)}}{\partial n} ds = -\lambda_{20}^2 C A_2$$

$$\lambda_2^2 \sim \frac{\varepsilon}{A_2} \oint_{\Gamma_1} \frac{\partial H_1}{\partial n} ds.$$

3.1. Example: a square cylinder with a circular annular inclusion

Equations of motion

$$\Delta u_i + \frac{\omega^2}{c_i^2} u_i = 0, \qquad i = 1,2, \qquad c_i = \sqrt{\mu_i/\rho_i}$$

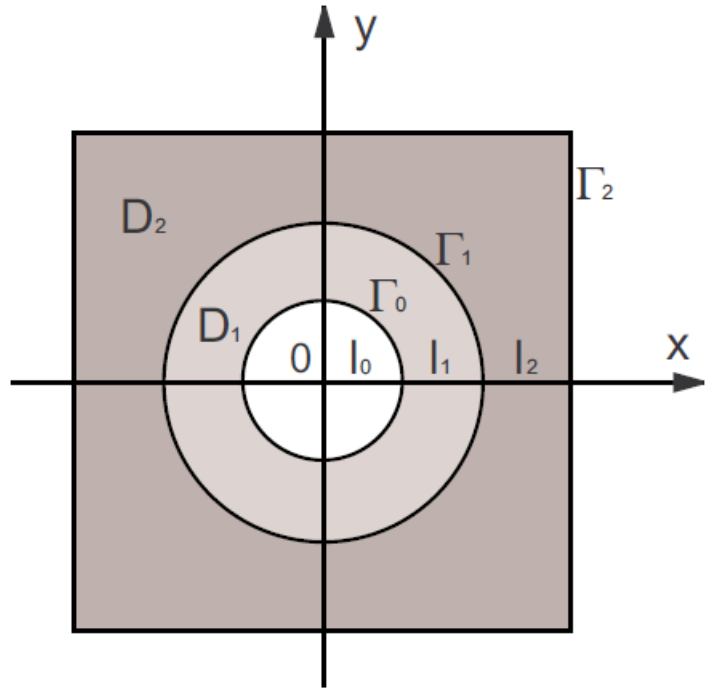
Boundary and continuity conditions

$$u_1\big|_{R_1=L_1} = \left.\frac{\partial u_2}{\partial n}\right|_{\Gamma_2} = 0,$$

$$u_1\big|_{R_1=L_1+1} = u_2\big|_{R_2=L_2}, \qquad \varepsilon \left.\frac{du_1}{dR_1}\right|_{R_1=L_1+1} = \left.\frac{du_2}{dR_2}\right|_{R_2=L_2},$$

where

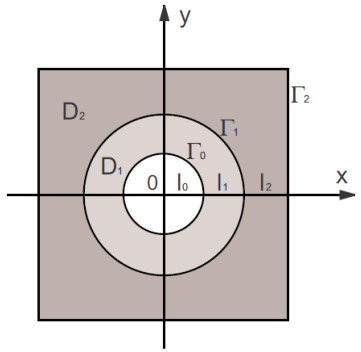
$$L_1 = \frac{l_0}{l_1}, \quad L_2 = \frac{l_0+l_1}{l_2}, \qquad R_i = r/l_i, \quad i = 1,2, \qquad \varepsilon = \frac{\mu}{l} \ll 1$$



Cross-sectional domain

Leading order displacements

$$u = \begin{cases} H_1(R_1), & \text{in } D_1; \\ 1, & \text{in } D_2, \end{cases} \qquad H_1|_{R_1=L_1} = 0, \quad H_1|_{R_1=L_1+1} = 1 \quad \Rightarrow \quad H_1(R_1) = \frac{\ln R_1 - \ln L_1}{\ln(1 + L_1^{-1})}.$$

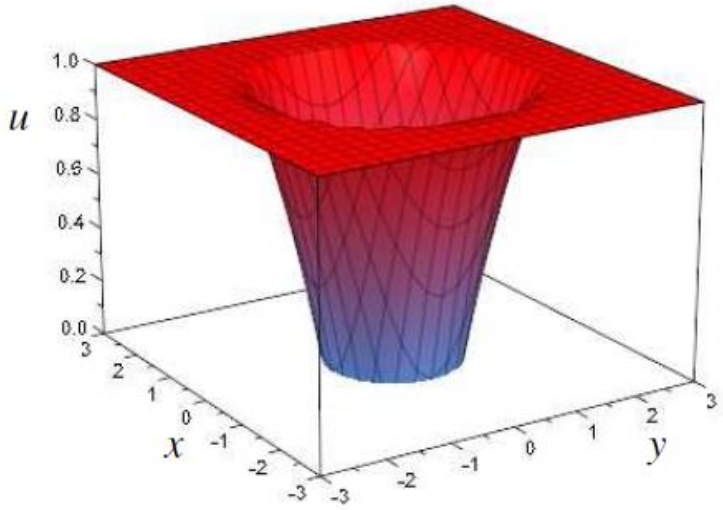


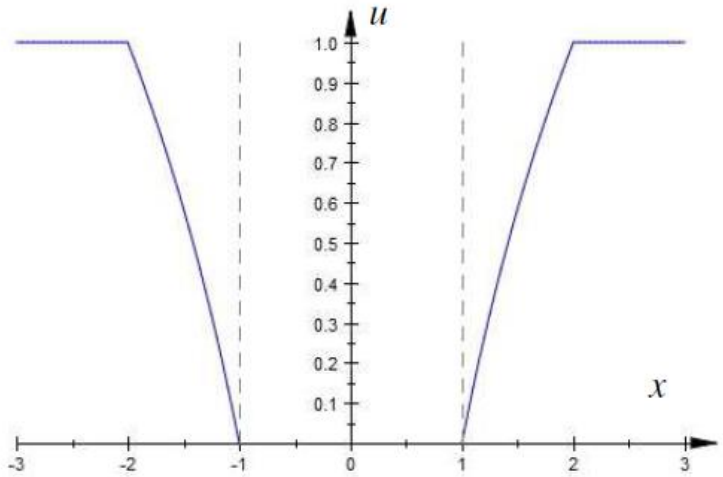
Frequency $$\lambda_2^2 = \frac{2\pi\varepsilon l}{\ln(L_1^{-1}+1)(4(L_2+1)^2 - \pi L_2^2)}.$$

Eigenform

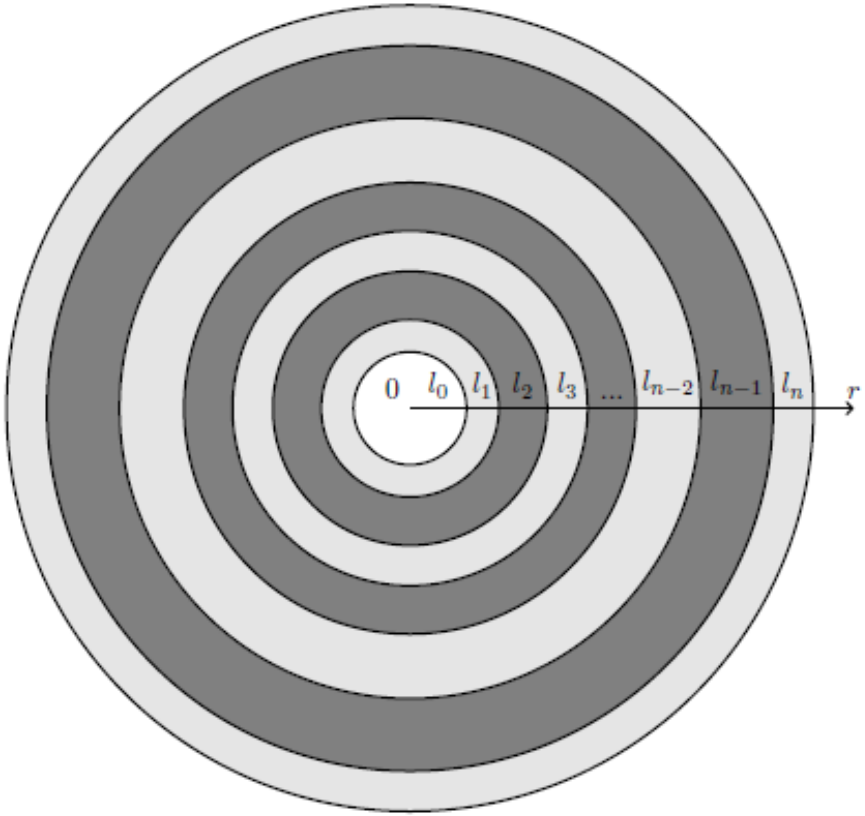
$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

$$u_{20} = 1$$





3.2. Antiplane motion of layered circular cylinders



Problem parameters

$$\varepsilon = \frac{\mu_1}{\mu_2} << 1, \quad c_1 \sim c_2, \quad L_i^j = \frac{l_j}{l_i} \sim l, \quad c_m^2 = \frac{\mu_m}{\rho_m}, \quad m = 1,2.$$

$$R_i = \frac{r_i}{l_i}, \quad \Omega_i = \frac{\omega l_i}{c_m}, \quad b_i \le R_i \le b_i + 1, \quad b_i = l_i^{-1} \sum_{k=0}^{i-1} l_k, \quad i = \overline{1,n}.$$

Equations of motion

$$R_i \frac{d^2 u_i}{dR_i^2} + \frac{du_i}{dR_i} + R_i \Omega_i^2 u_i = 0.$$

Boundary conditions

$$u_1\big|_{R_1 = b_1} = u_n\big|_{R_n = b_n + 1} = 0.$$

Continuity

$$u_i\big|_{R_i = b_i + 1} = u_{i+1}\big|_{R_{i+1} = b_{i+1}}, \qquad \left.\frac{du_i}{dR_i}\right|_{R_i = b_i + 1} = \varepsilon^j L_{i+1}^i \left.\frac{du_{i+1}}{dR_{i+1}}\right|_{R_{i+1} = b_{i+1}}.$$

($j = 1$ or -1 for i^{th} component being stiff or soft, respectively)

Antiplane motion of concentric circular cylinders

Summary of the approach

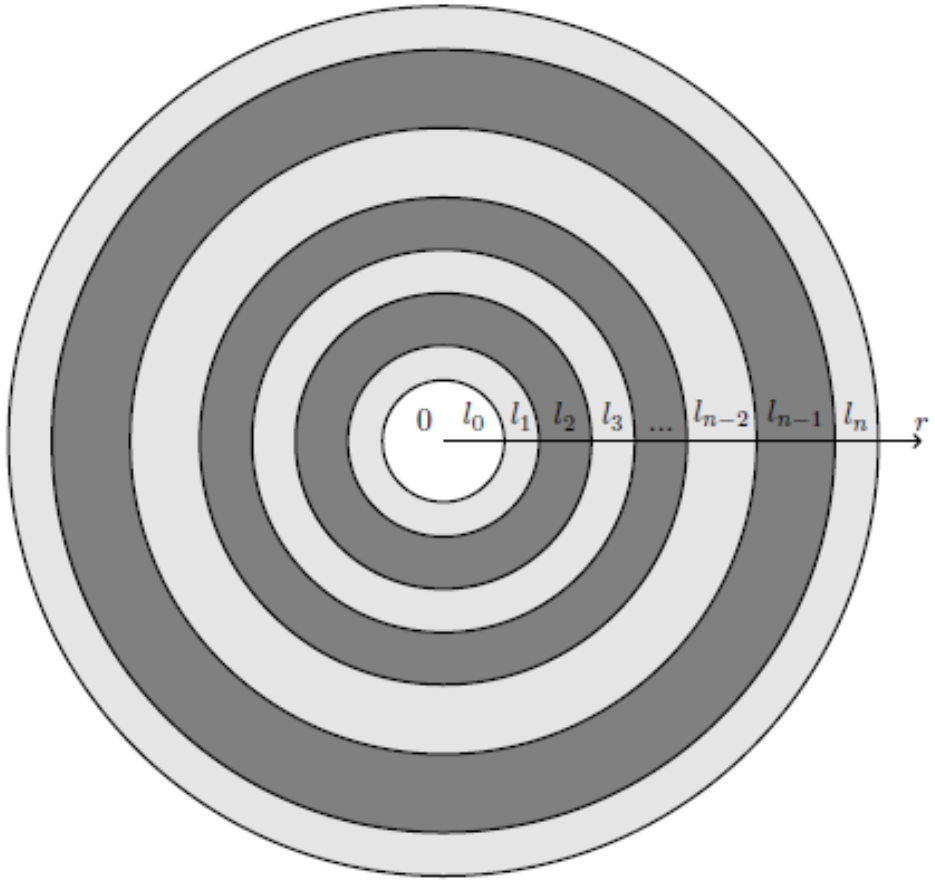
- Global low-frequency perturbation

- Rigid body motions of stiffer components
  (at leading order)

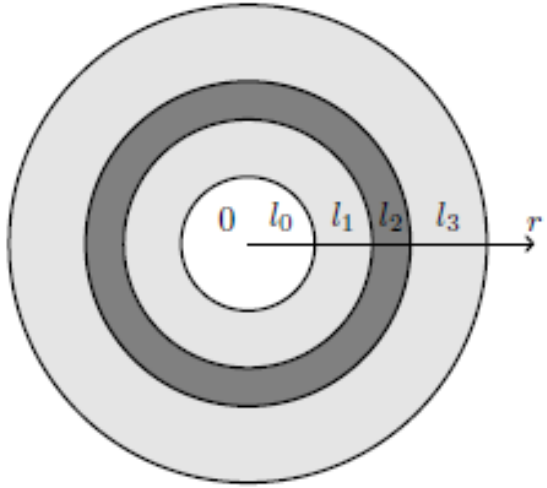
- Leading order solution for softer components, involving logarithmic functions (no homogeneous deformation)

- Solvability of the next order problem
  for stiffer components

Polynomial equation for frequency



Example: Three-layered cylinder



Frequency

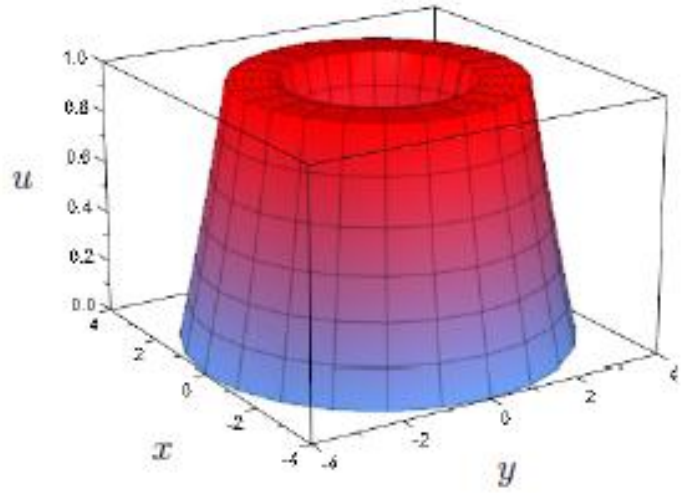
$$\Omega_{20}^2 = \frac{2\left(\delta_1 + \delta_3\right)}{2b_2 + 1}, \qquad \delta_i = \frac{1}{\ln\left(1 + b_i^{-1}\right)}.$$

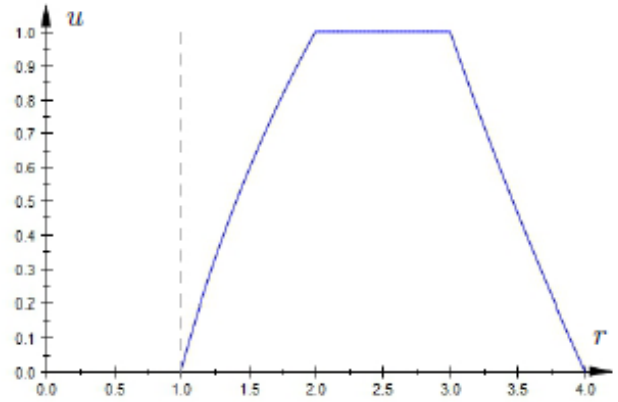
Eigenform

$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

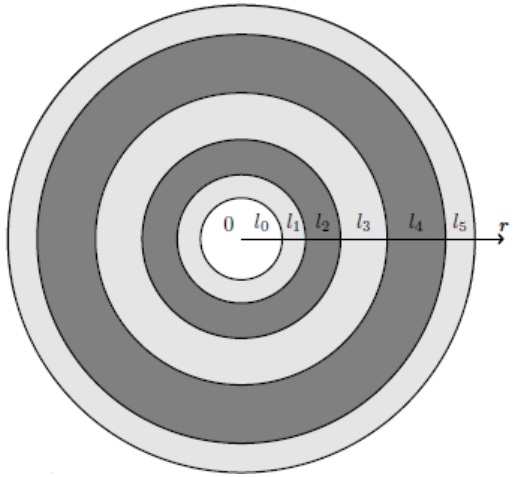
$$u_{20} = 1,$$

$$u_{30} = \delta_3 \ln\left(\frac{b_3 + 1}{R_3}\right).$$

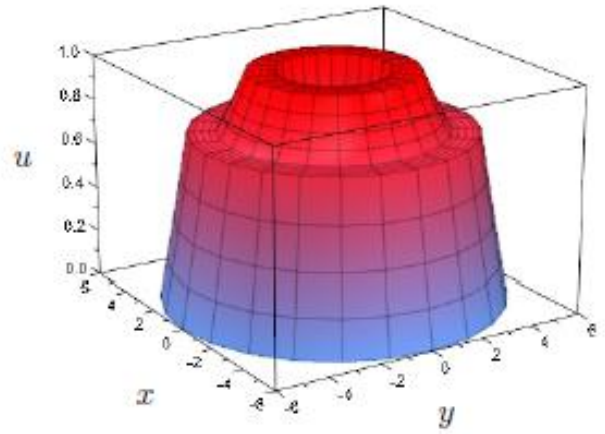


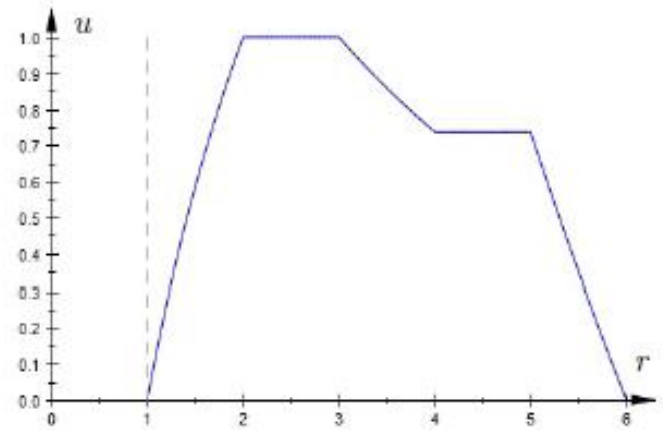


Example: Five-layered cylinder



Frequency $\Omega_{20}^2 = \dfrac{2}{2b_2+1}\left(\delta_1 + (1-k)\delta_3\right)$ (two roots for k)





Eigenform

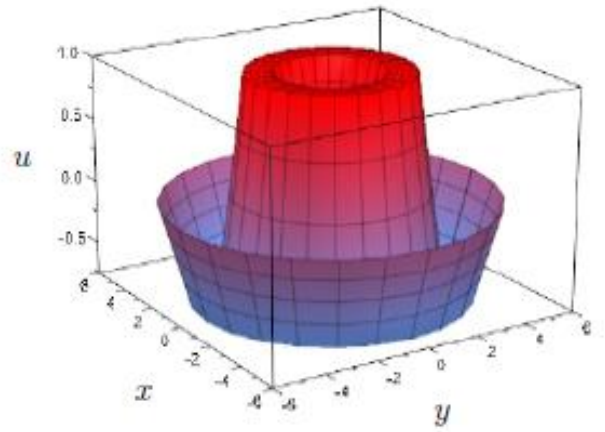
$$u_{10} = \delta_1 \ln\left(\frac{R_1}{b_1}\right),$$

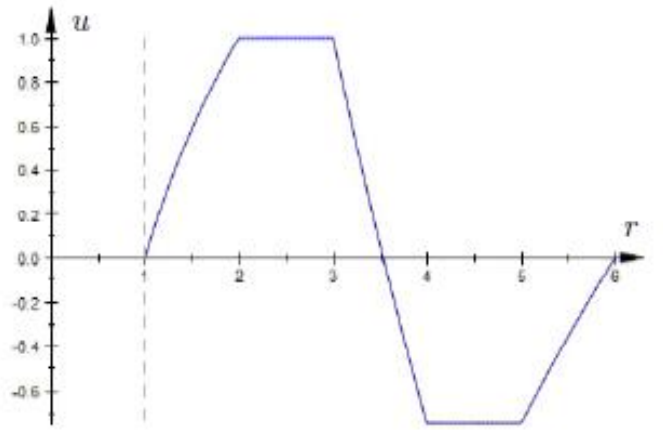
$$u_{20} = 1,$$

$$u_{30} = -\delta_3 \left\{ \ln\left(\frac{R_3}{b_3+1}\right) - k \ln\left(\frac{R_3}{b_3}\right) \right\},$$

$$u_{40} = k,$$

$$u_{50} = k\delta_5 \ln\left(\frac{b_5+1}{R_5}\right).$$





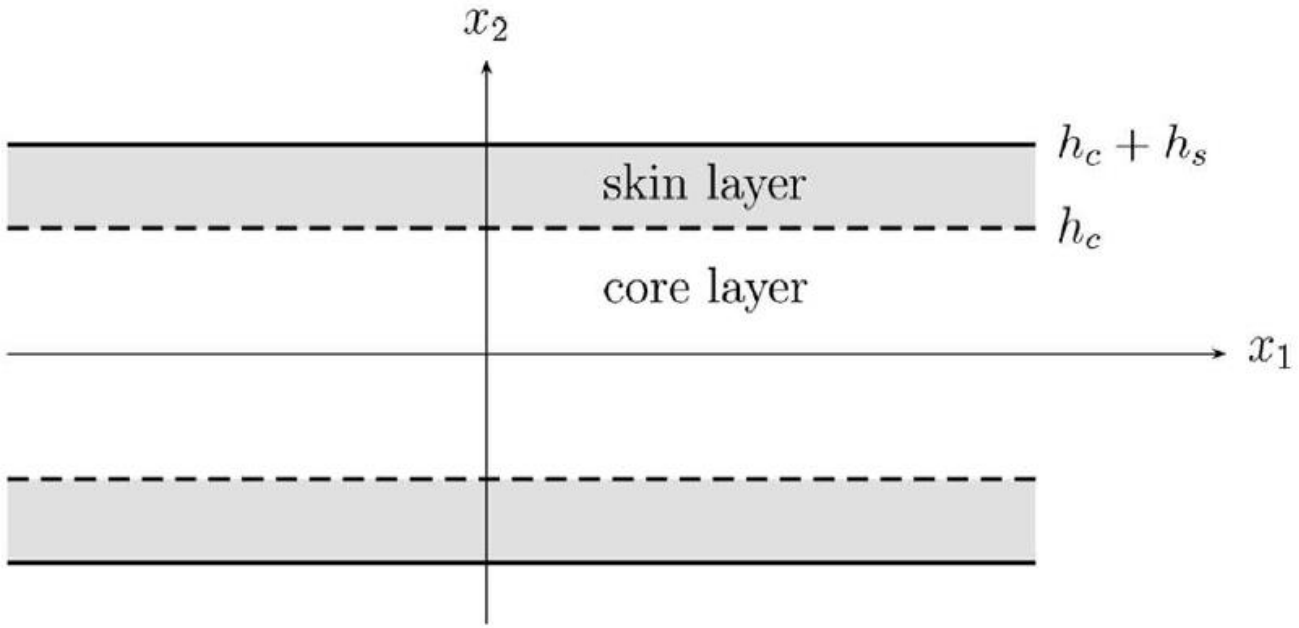
3.2 Antiplane shear motion of a symmetric 3-layered laminate

L.Prikazchikova, Y.Ece Aydın, B.Erbaş, J.Kaplunov. (2020). Asymptotic analysis of an anti-plane dynamic problem for a three-layered strongly inhomogeneous laminate. *Mathematics and Mechanics of Solids*, *25*(1), 3-16.

Governing equations

$$\frac{\partial \sigma_{13}^q}{\partial x_1} + \frac{\partial \sigma_{23}^q}{\partial x_2} - \rho_q \frac{\partial^2 u_q}{\partial t^2} = 0, \quad q = \mathrm{c}, \mathrm{s},$$

$$\sigma_{i3}^q = \mu_q \frac{\partial u_q}{\partial x_i}, \quad i = 1, 2,$$

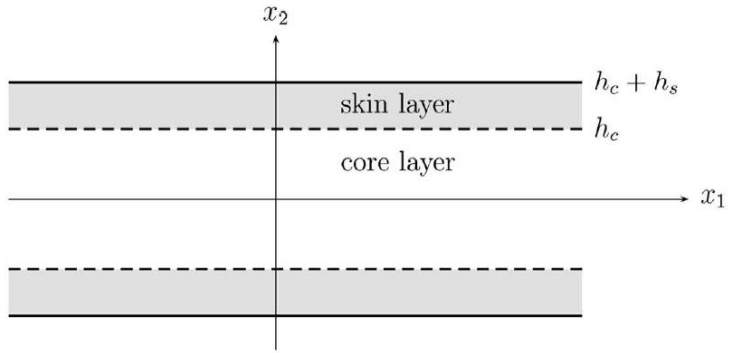


Boundary and continuity conditions

$$\sigma_{23}^{\mathrm{c}} = \sigma_{23}^{\mathrm{s}} \quad \text{and} \quad u_{\mathrm{c}} = u_{\mathrm{s}} \quad \text{at} \quad x_2 = \pm h_{\mathrm{c}},$$

$$\sigma_{23}^{\mathrm{s}} = 0 \quad \text{at} \quad x_2 = \pm(h_{\mathrm{c}} + h_{\mathrm{s}}).$$

Dispersion relation



$$\mu\alpha_1 \cosh(\alpha_1)\cosh(\alpha_2 h) + \alpha_2 \sinh(\alpha_1)\sinh(\alpha_2 h) = 0,$$

where $\alpha_1 = \sqrt{K^2 - \Omega^2}$, $\alpha_2 = \sqrt{K^2 - \frac{\mu}{\rho}\Omega^2}$, $\Omega = \frac{\omega h_c}{c_{2c}}$, $K = kh_c$,

with $h = \frac{h_s}{h_c}$, $\mu = \frac{\mu_c}{\mu_s}$, $\rho = \frac{\rho_c}{\rho_s}$.

Two scenarios: $(a)\ \mu \ll 1,\ h \sim 1,\ \rho \sim \mu,$

$(b)\ \mu \ll 1,\ h \sim \mu,\ \rho \sim \mu^2.$

Displacements and stresses

$$u_c = h_c \frac{\sinh(\alpha_1 \xi_{2c})}{\alpha_1},$$

$$\sigma_{13}^c = \mathrm{i}\mu_c K \frac{\sinh(\alpha_1 \xi_{2c})}{\alpha_1},$$

$$\sigma_{23}^c = \mu_c \cosh(\alpha_1 \xi_{2c}),$$

$$u_s = h_c \beta \left(\cosh\left[\alpha_2(h\xi_{2s}+1)\right] - \tanh\left[\alpha_2(h+1)\right]\sinh\left[\alpha_2(h\xi_{2s}+1)\right]\right),$$

$$\sigma_{13}^s = \mathrm{i}\mu_s K \beta \left(\cosh\left[\alpha_2(h\xi_{2s}+1)\right] - \tanh\left[\alpha_2(h+1)\right]\sinh\left[\alpha_2(h\xi_{2s}+1)\right]\right),$$

$$\sigma_{23}^s = \mu_s \alpha_2 \beta \left(\sinh\left[\alpha_2(h\xi_{2s}+1)\right] - \tanh\left[\alpha_2(h+1)\right]\cosh\left[\alpha_2(h\xi_{2s}+1)\right]\right),$$

where

$$\beta = \frac{\sinh \alpha_1}{\alpha_1\left(\cosh \alpha_2 - \sinh \alpha_2 \tanh[\alpha_2(h+1)]\right)},$$

and

$$\xi_{2c} = \frac{x_2}{h_c}, \qquad 0 \le x_2 \le h_c,$$

$$\xi_{2s} = \frac{x_2 - h_c}{h_s}, \qquad h_c \le x_2 \le h_c + h_s.$$

Cut-off frequencies: set $K = 0$ ⟹ $\tan(\Omega)\tan\left(h\sqrt{\frac{\mu}{\rho}}\Omega\right) = \sqrt{\mu\rho}.$

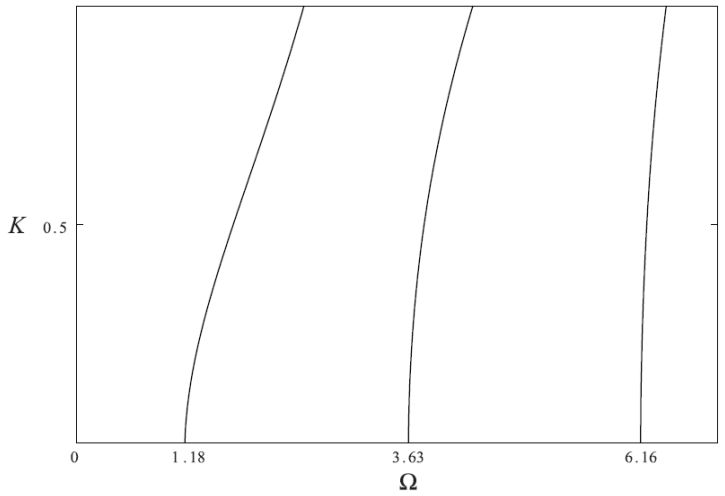
Low-frequency (GLF) regime

$$\Omega \approx \left(\frac{\rho}{h}\right)^{\frac{1}{2}} \ll 1,$$

provided that

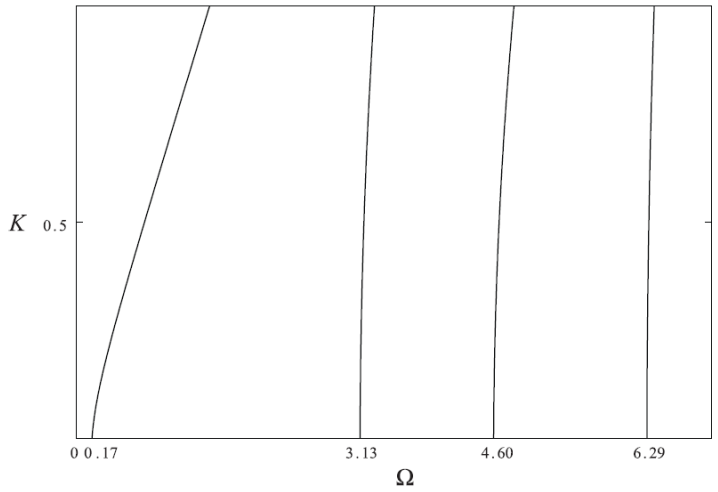
$$\rho \ll h \ll \mu^{-1}.$$

(coincides with results of plane-strain, but does not support a fundamental mode)



$\mu = 0.232$, $\rho = 3.0$ and $h = 1.0$.

high contrast



$\mu = 0.014$, $\rho = 0.03$ and $h = 1.0$.

Expand the dispersion relation (long-wave low-frequency)

$$\mu + \gamma_1 K^2 + \gamma_2 K^4 + \gamma_3 K^2 \Omega^2 + \gamma_4 \Omega^2 + \gamma_5 \Omega^4 + \cdots = 0,$$

where

$$\gamma_1 = \frac{\mu}{2}\left(1 + h^2\right) + h,$$

$$\gamma_2 = \frac{\mu}{24}\left(1 + 6h^2 + h^4\right) + \frac{h}{6}(1 + h^2),$$

$$\gamma_3 = -\frac{\mu}{12}(1 + 3h^2) - \frac{h}{6} - \frac{\mu h}{12\rho}(2 + 3\mu h) - \frac{\mu h^3}{12\rho}(4 + \mu h),$$

$$\gamma_4 = -\frac{\mu}{2} - \frac{\mu h}{\rho}\left(1 + \frac{\mu h}{2}\right),$$

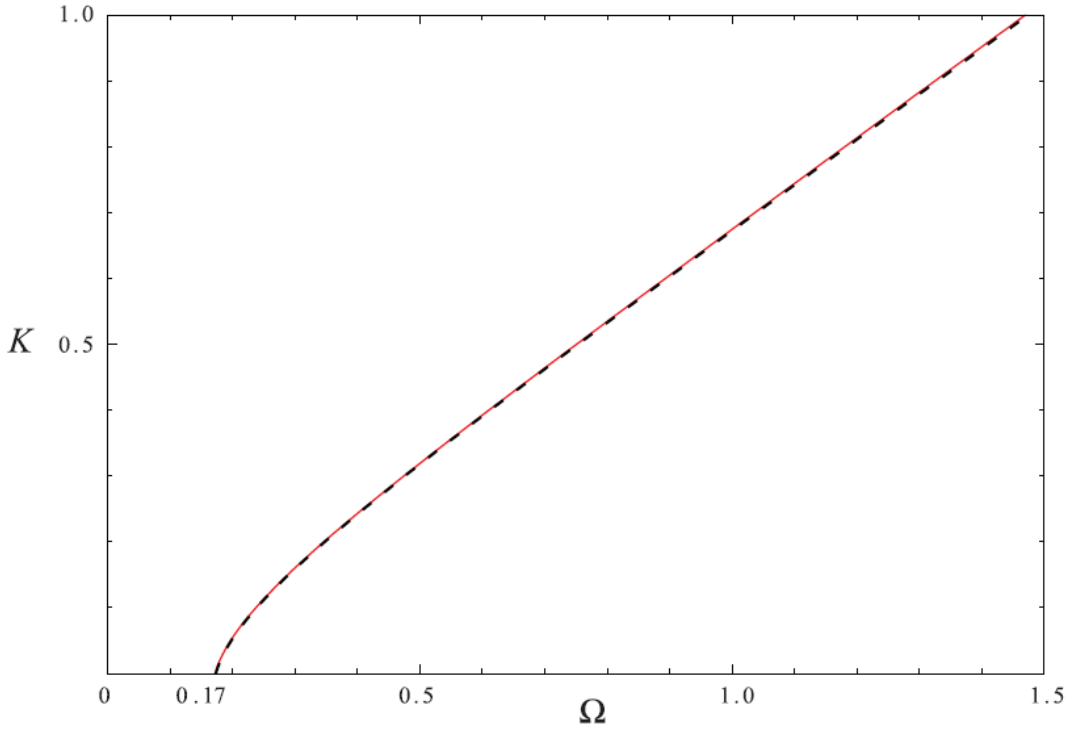
$$\gamma_5 = \frac{\mu}{24} + \frac{\mu h}{12\rho}(2 + 3\mu h) + \frac{\mu^2 h^3}{24\rho^2}(4 + \mu h).$$

$(a)\ \mu \ll 1,\ h \sim 1,\ \rho \sim \mu,$

For case a) $\gamma_1 \sim \gamma_2 \sim \gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$

At leading order

$$\gamma_1 = h\,,$$
$$\gamma_2 = \frac{h(1+h^2)}{6}\,,$$
$$\gamma_3 = -\frac{h}{6\rho_\mu}\left(\rho_\mu + 1 + 2h^2\right)\,,$$
$$\gamma_4 = -\frac{h}{\rho_\mu}\,,$$
$$\gamma_5 = \frac{h}{6\rho_\mu}\left(1 + \frac{h^2}{\rho_\mu}\right)\,,$$



$$\frac{\mu}{h} + K^2 - \frac{1}{\rho_\mu}\Omega^2 = 0.$$

(shortened dispersion relation)

where $\rho_\mu = \rho/\mu \sim 1.$

For case b) $\gamma_1 \sim \gamma_2 \sim \mu$ and $\gamma_3 \sim \gamma_4 \sim \gamma_5 \sim 1.$

$(b)\ \mu \ll 1,\ h \sim \mu,\ \rho \sim \mu^2.$

Expansions

$$\gamma_1 = \mu\left(\frac{1}{2} + h_\mu\right),$$

$$\gamma_2 = \frac{\mu}{6}\left(\frac{1}{4} + h_\mu\right)$$

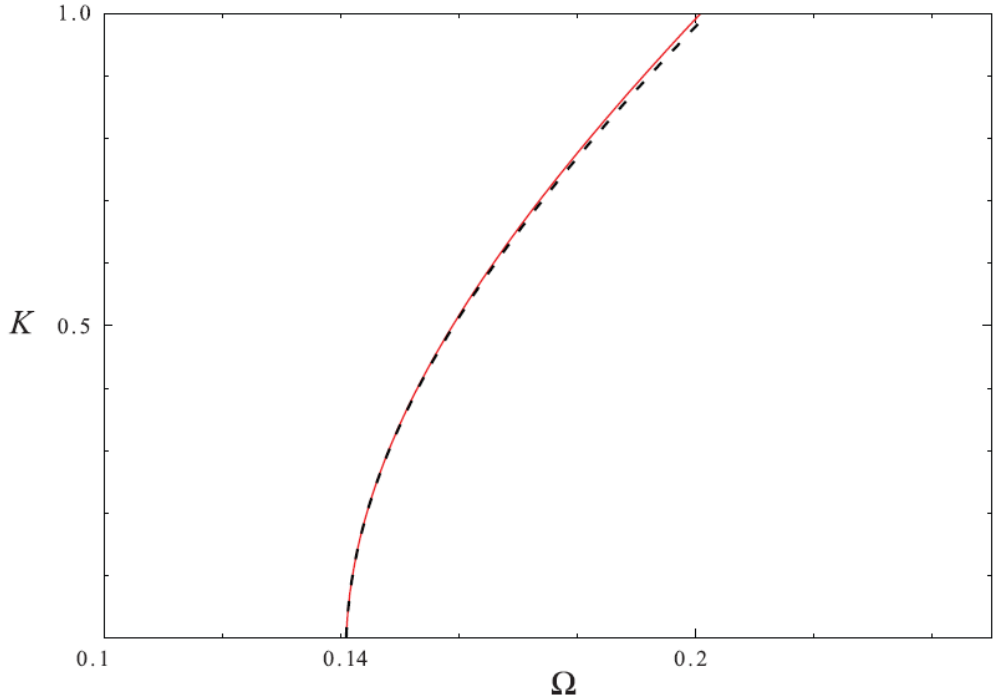
$$\gamma_3 = -\frac{h_\mu}{6\rho_\mu},$$

$$\gamma_4 = -\frac{h_\mu}{\rho_\mu} - \frac{\mu}{2},$$

$$\gamma_5 = \frac{h_\mu}{6\rho_\mu},$$

where $\rho_\mu = \rho/\mu^2 \sim 1$

and $h_\mu = h/\mu \sim 1.$



$\mu = 0.014,\ \rho = 0.03$ and $h = 1.0.$

(near cut-off expansion)

$$\left(h_\mu + \frac{1}{3}\right)K^2 - \frac{1}{\mu}\frac{h_\mu}{\rho_\mu}\Omega^2 + \left(1 - \frac{\mu\rho_\mu}{3h_\mu}\right) = 0.$$

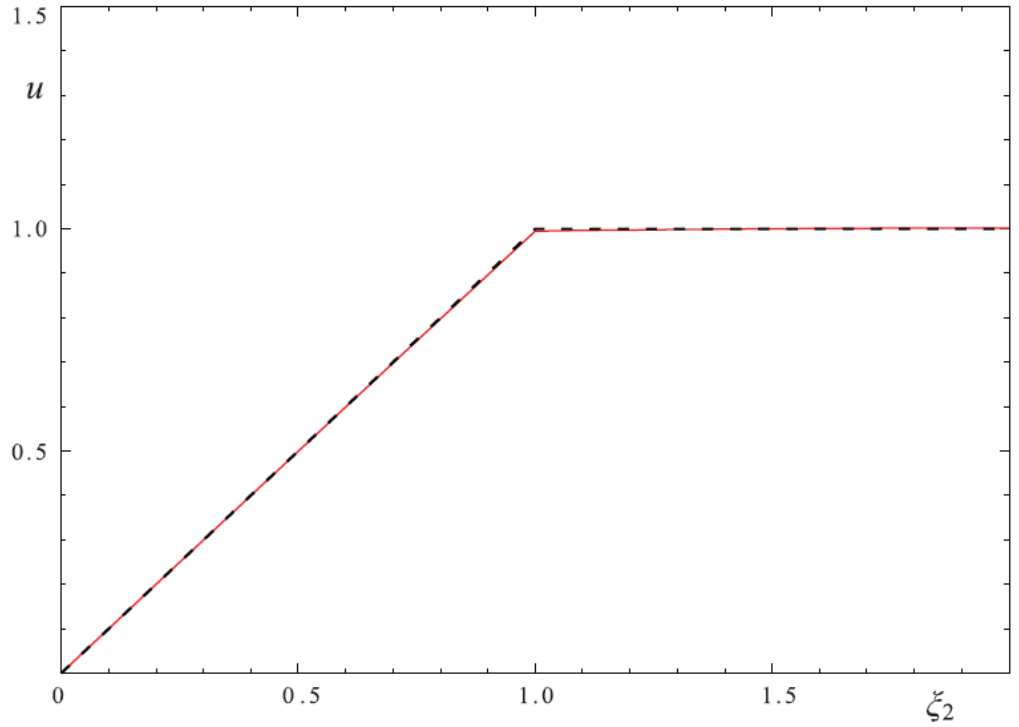
Leading order displacements and stresses

$$
\begin{aligned}
u_{\mathrm{c}} &= h_{\mathrm{c}}\xi_{2\mathrm{c}}, \\
\sigma_{13}^{\mathrm{c}} &= \mathrm{i}\mu_{\mathrm{c}}\sqrt{\mu}\,K_{*}\xi_{2\mathrm{c}}, \qquad \text{and} \\
\sigma_{23}^{\mathrm{c}} &= \mu_{\mathrm{c}},
\end{aligned}
$$

$$
\begin{aligned}
u_{\mathrm{s}} &= h_{\mathrm{c}}, \\
\sigma_{13}^{\mathrm{s}} &= \mathrm{i}\mu_{\mathrm{s}}\sqrt{\mu}K_{*}, \\
\sigma_{23}^{\mathrm{s}} &= \mu_{\mathrm{c}}h\left(K_{*}^{2} - \frac{\Omega_{*}^{2}}{\rho_{*}}\right)(\xi_{2\mathrm{s}} - 1)
\end{aligned}
$$

where

$$K^2 = \mu K_*^2 \quad \text{and} \quad \Omega^2 = \mu\Omega_*^2 \,.$$



Displacement profile

$(a)\ \mu \ll 1,\ h \sim 1,\ \rho \sim \mu,$

Asymptotic derivation of the approximate equation of motion (case a))

Scaling

$$x_1 = \frac{h_c}{\sqrt{\mu}}\xi_1, \quad t = \frac{h_c}{c_{2c}\sqrt{\mu}}\tau, \quad u^q = h_c v^q, \quad \sigma_{13}^q = \mu_q\sqrt{\mu}S_{13}^q, \quad \sigma_{23}^q = \mu_c S_{23}^q, \quad q = c, s,$$

Governing equations

$$\mu\frac{\partial S_{13}^c}{\partial \xi_1} + \frac{\partial S_{23}^c}{\partial \xi_{2c}} - \mu\frac{\partial^2 v^c}{\partial \tau^2} = 0,$$

$$S_{13}^c = \frac{\partial v^c}{\partial \xi_1}, \qquad S_{23}^c = \frac{\partial v^c}{\partial \xi_{2c}}$$

and

$$\frac{\partial S_{13}^s}{\partial \xi_1} + \frac{1}{h}\frac{\partial S_{23}^s}{\partial \xi_{2s}} - \frac{1}{\rho_\mu}\frac{\partial^2 v^s}{\partial \tau^2} = 0,$$

$$S_{13}^s = \frac{\partial v^s}{\partial \xi_1}, \qquad \mu h S_{23}^s = \frac{\partial v^s}{\partial \xi_{2s}},$$

where $\rho_\mu = \rho/\mu \sim 1.$

Continuity and boundary conditions

$$v^{\mathrm{c}}\big|_{\xi_{2\mathrm{c}}=1} = v^{\mathrm{s}}\big|_{\xi_{2\mathrm{s}}=0}, \qquad S^{\mathrm{c}}_{23}\big|_{\xi_{2\mathrm{c}}=1} = S^{\mathrm{s}}_{23}\big|_{\xi_{2\mathrm{s}}=0} \quad \text{and} \quad S^{\mathrm{s}}_{23}\big|_{\xi_{2\mathrm{s}}=1} = 0.$$

Expanding displacements and stresses

$$v^q = v^q_0 + \mu v^q_1 + \cdots,$$
$$S^q_{j3} = S^q_{j3,0} + \mu S^q_{j3,1} + \cdots, \qquad q = \mathrm{c}, \mathrm{s} \qquad \text{and} \qquad j = 1, 2.$$

Leading order problem

$$S^{\mathrm{c}}_{13,0} = \frac{\partial v^{\mathrm{c}}_0}{\partial \xi_1}, \qquad \frac{\partial S^{\mathrm{c}}_{23,0}}{\partial \xi_{2\mathrm{c}}} = 0, \qquad S^{\mathrm{c}}_{23,0} = \frac{\partial v^{\mathrm{c}}_0}{\partial \xi_{2\mathrm{c}}},$$

$$\frac{\partial S^{\mathrm{s}}_{13,0}}{\partial \xi_1} + \frac{1}{h}\frac{\partial S^{\mathrm{s}}_{23,0}}{\partial \xi_{2\mathrm{s}}} - \frac{1}{\rho_\mu}\frac{\partial^2 v^{\mathrm{s}}_0}{\partial \tau^2} = 0,$$

$$S^{\mathrm{s}}_{13,0} = \frac{\partial v^{\mathrm{s}}_0}{\partial \xi_1}, \qquad \frac{\partial v^{\mathrm{s}}_0}{\partial \xi_{2\mathrm{s}}} = 0 \quad \longrightarrow \quad v^{\mathrm{s}}_0 = w(\xi_1, \tau)$$

together with

$$v^{\mathrm{c}}_0\Big|_{\xi_{2\mathrm{c}}=1} = v^{\mathrm{s}}_0\Big|_{\xi_{2\mathrm{s}}=0}, \qquad S^{\mathrm{c}}_{23,0}\Big|_{\xi_{2\mathrm{c}}=1} = S^{\mathrm{s}}_{23,0}\Big|_{\xi_{2\mathrm{s}}=0} \quad \text{and} \quad S^{\mathrm{s}}_{23,0}\Big|_{\xi_{2\mathrm{s}}=1} = 0\,.$$

The remaining quantities are expressed in terms of w

$$S^{\mathrm{c}}_{13,0} = \xi_{2\mathrm{c}} \frac{\partial w}{\partial \xi_1}, \qquad S^{\mathrm{c}}_{23,0} = w, \qquad v^{\mathrm{c}}_0 = \xi_{2\mathrm{c}} w,$$

$$S^{\mathrm{s}}_{13,0} = \frac{\partial w}{\partial \xi_1}, \qquad S^{\mathrm{s}}_{23,0} = w(1 - \xi_{2\mathrm{s}}),$$

with the displacement itself satisfying

$$\frac{\partial^2 w}{\partial \xi_1^2} - \frac{1}{\rho_\mu} \frac{\partial^2 w}{\partial \tau^2} - \frac{1}{h} w = 0,$$

or, in terms of the original variables,

$$\frac{\partial^2 u_{\mathrm{s}}}{\partial x_1^2} - \frac{\rho_{\mathrm{s}}}{\mu_{\mathrm{s}}} \frac{\partial^2 u_{\mathrm{s}}}{\partial t^2} - \frac{\mu_{\mathrm{c}}}{\mu_{\mathrm{s}} h_{\mathrm{c}} h_{\mathrm{s}}} u_{\mathrm{s}} = 0,$$

$$\frac{\mu}{h} + K^2 - \frac{1}{\rho_\mu} \Omega^2 = 0.$$

(shortened dispersion relation)

$$u_{\mathrm{s}}(x_1, t) \approx w(x_1, t).$$

3.3 Asymmetric laminate

J.Kaplunov, L.Prikazchikova, M.Alkinidri. (2021). Antiplane shear of an asymmetric sandwich plate. *Continuum Mechanics and Thermodynamics*, *33*(4), 1247-1262.

Governing relations

$$\frac{\partial \sigma_{13}^q}{\partial x_1} + \frac{\partial \sigma_{23}^q}{\partial x_2} - \rho_q \frac{\partial^2 u_q}{\partial t^2} = 0, \quad q = 1, 2, 3,$$

$$\sigma_{i3}^q = \mu_q \frac{\partial u_q}{\partial x_i}, \quad i = 1, 2,$$

with $\mu_3 = \mu_1$ and $\rho_3 = \rho_1$.

Continuity and boundary conditions

$$u_1 = u_2, \quad \sigma_{23}^1 = \sigma_{23}^2 \quad \text{at} \quad x_2 = \frac{h_2}{2}, \qquad u_2 = u_3, \quad \sigma_{23}^2 = \sigma_{23}^3 \quad \text{at} \quad x_2 = -\frac{h_2}{2},$$

$$\sigma_{23}^1 = F_1 \quad \text{at} \quad x_2 = \frac{h_2}{2} + h_1, \qquad \sigma_{23}^3 = F_3 \quad \text{at} \quad x_2 = -\frac{h_2}{2} - h_3.$$

Dispersion relation (in absence of load on the faces)

$$\mu\alpha_1\big(\tanh(h_{12}\alpha_1)+\tanh(h_{32}\alpha_1)\big)+\mu^2\alpha_2\tanh(\alpha_2)$$
$$+\alpha_1{}^2\tanh(h_{12}\alpha_1)\tanh(h_{32}\alpha_1)\frac{\tanh(\alpha_2)}{\alpha_2}=0,$$

where $\alpha_1=\sqrt{K^2-\frac{\mu}{\rho}\Omega^2}, \quad \alpha_2=\sqrt{K^2-\Omega^2},$

and $K=kh_2, \quad \Omega=\frac{\omega h_2}{c_{22}}, \quad \mu=\frac{\mu_2}{\mu_1}, \quad \rho=\frac{\rho_2}{\rho_1}, \quad h_{12}=\frac{h_1}{h_2}, \quad h_{32}=\frac{h_3}{h_2},$

with $c_{22}=\sqrt{\mu_2/\rho_2}$.

Assume $\mu\ll 1, \quad \rho\sim\mu, \quad h_{12}\sim 1, \quad h_{32}\sim 1.$

- For cut-off frequencies we set $K = 0$

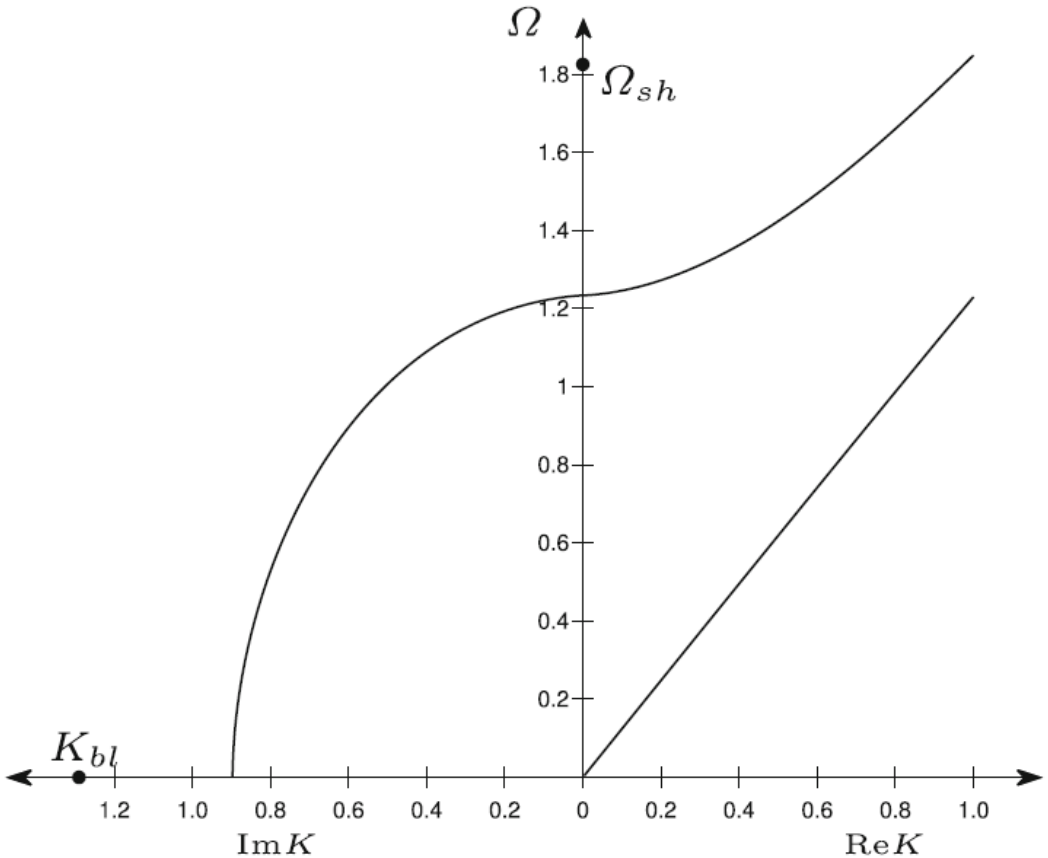
$$\sqrt{\mu\rho}\left(\tan\left(h_{12}\sqrt{\frac{\mu}{\rho}}\Omega\right)+\tan\left(h_{32}\sqrt{\frac{\mu}{\rho}}\Omega\right)\right)+\mu\rho\tan(\Omega)$$
$$-\tan\left(h_{12}\sqrt{\frac{\mu}{\rho}}\Omega\right)\tan\left(h_{32}\sqrt{\frac{\mu}{\rho}}\Omega\right)\tan(\Omega)=0.$$

Two solutions: $\Omega = 0$ and $\Omega^2 = \Omega_{sh}^2 \approx \dfrac{h_{12}+h_{32}}{h_{12}h_{32}}\rho \sim \mu \ll 1.$

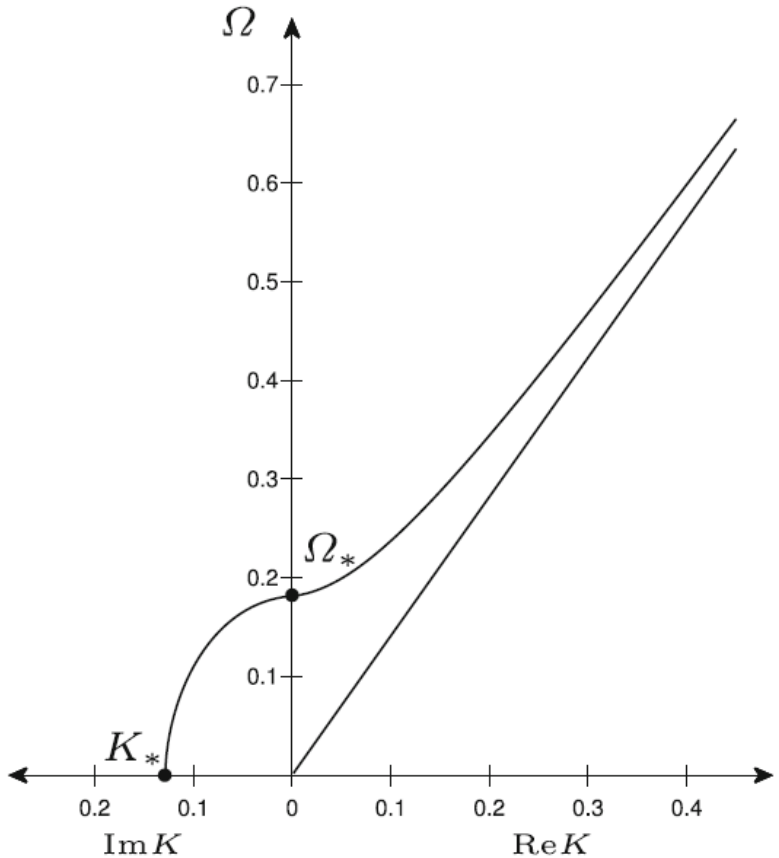
- Now we set for statics $\Omega = 0$

$$K^2\Big(\mu\big(\tanh(h_{12}K)+\tanh(h_{32}K)\big)+\mu^2\tanh(\alpha_2)$$
$$+\tanh(h_{12}K)\tanh(h_{32}K)\tanh(K)\Big)=0.$$

Two solutions: $K = 0$ and $K^2 = K_{bl}^2 \approx -\dfrac{h_{12}+h_{32}}{h_{12}h_{32}}\mu \sim \mu \ll 1,$ slowly decaying BL.



(a) no contrast

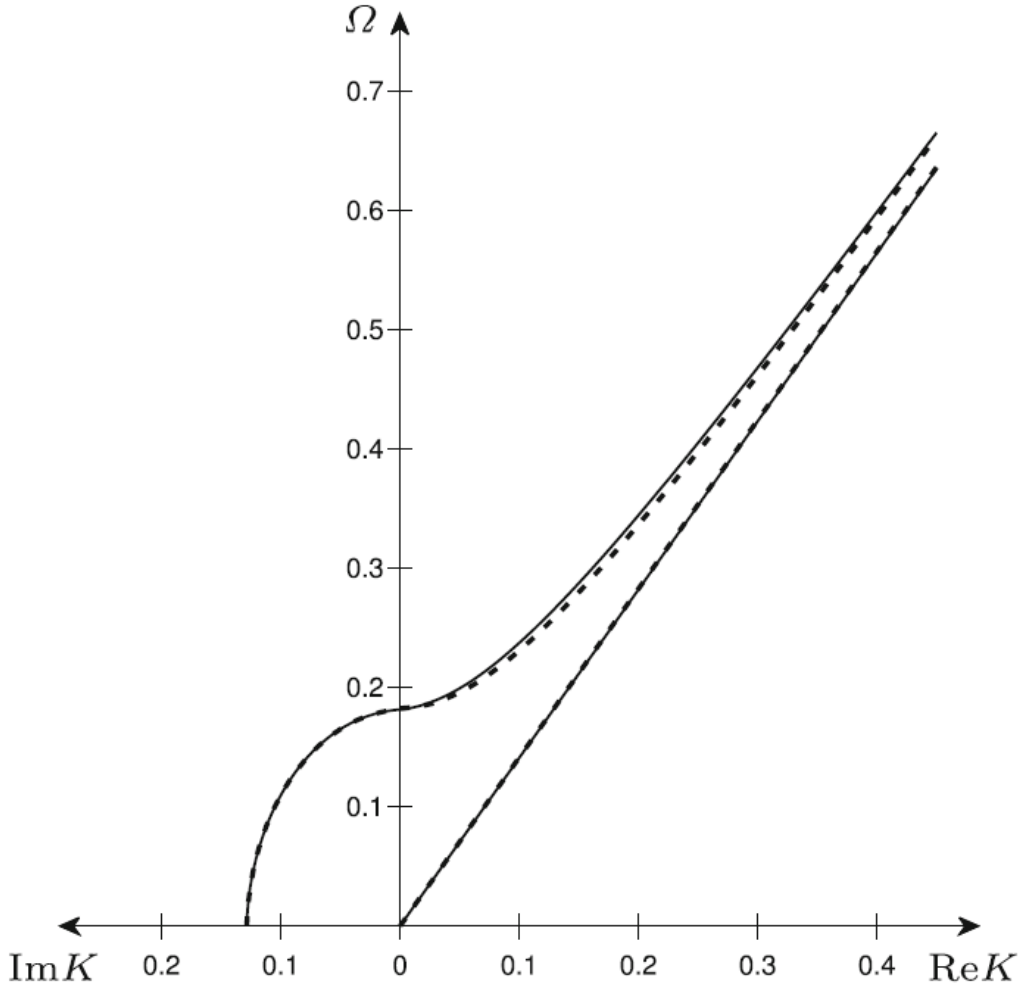


(b) high contrast

$h_{12} = 1.0$, $h_{32} = 1.5$ and **a** $\mu = 1.0$ and $\rho = 2.0$, **b** $\mu = 0.01$ and $\rho = 0.02$

Shortened dispersion relation

$$\left(K^2\rho_0 - \Omega^2\right)\left\{h_{12}h_{32}\left(K^2\rho_0 - \Omega^2\right) + \mu\rho_0(h_{12} + h_{32})\right\} = 0.$$



- Asymptotic equations of motion

$$\mu_1(h_1+h_3)\frac{\partial^2 U}{\partial x_1^2}-\rho_1(h_1+h_3)\frac{\partial^2 U}{\partial t^2}=F_3-F_1,$$

$$\mu_1 h_2\frac{\partial^2\phi}{\partial x_1^2}-\mu_2\left(\frac{1}{h_1}+\frac{1}{h_3}\right)\phi-\rho_1 h_2\frac{\partial^2\phi}{\partial t^2}=-\left(\frac{F_3}{h_3}+\frac{F_1}{h_1}\right).$$

Here U is average displacement and ϕ is the angle of rotation

$$U=\frac{h_1u_1+h_3u_3}{h_1+h_3}\approx\frac{h_2(h_1w_1+h_3w_3)}{h_1+h_3},\quad \phi=\frac{u_1-u_3}{h_2}\approx w_2.$$

$$\left(v_0^1=w_1(\xi_1,\tau),\quad v_0^3=w_3(\xi_1,\tau),\quad v_0^2=w_2\xi_{22}+w_3\right)$$

3.4 Boundary conditions

Consider equilibrium of a semi-strip $(0 \leqslant x_1 < +\infty, -h_3 - h_2/2 \leqslant x_2 \leqslant h_2/2 + h_1)$

with traction free faces and a prescribed load at the edge $\sigma_{13}^q\big|_{x_1=0} = P(x_2, t), \quad q = 1, 2, 3.$

Boundary conditions in terms of stress resultant N for stiff layer and stress couple G for the soft layer

$$N = \int_{-h_3-h_2/2}^{h_2/2+h_1} P \mathrm{d}x_2,$$

$$G = \frac{\mu h_2^2 (h_1 + h_3)}{24 h_1 h_3} \left(\frac{2}{h_2} \int_{-h_2/2}^{h_2/2} x_2 P \mathrm{d}x_2 + \int_{h_2/2}^{h_2/2+h_1} P dx_2 - \int_{-h_2/2-h_3}^{-h_2/2} P \mathrm{d}x_2 - \frac{h_1 - h_3}{h_1 + h_3} \int_{-h_2/2-h_3}^{h_2/2+h_1} P \mathrm{d}x_2 \right).$$

$$N = \int_{h_2/2}^{h_2/2+h_1} \sigma_{13}^1 \mathrm{d}x_2 + \int_{-h_2/2-h_3}^{-h_2/2} \sigma_{13}^3 \mathrm{d}x_2$$

$$G = \int_{-h_2/2}^{h_2/2} \sigma_{13}^2 x_2 \mathrm{d}x_2$$

J.Kaplunov, L.Prikazchikova, M.Alkinidri. (2021). Antiplane shear of an asymmetric sandwich plate. *Continuum Mechanics and Thermodynamics*, *33*(4), 1247-1262.

4. High-contrast three-layered plates (*antisymmetric motion*)

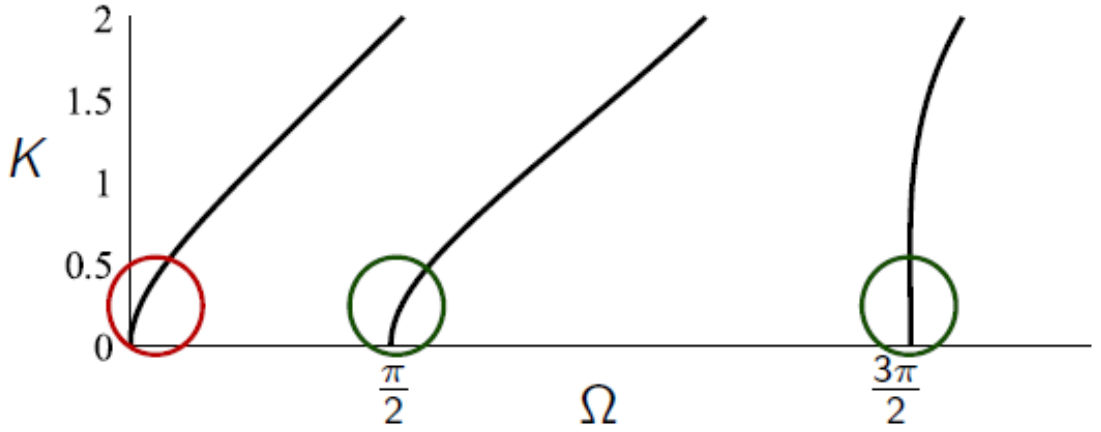
Preliminary remarks

- Rayleigh-Lamb dispersion relation for a single-layered plate

$$\gamma^4 \frac{\sinh\alpha}{\alpha}\cosh\beta - \beta^2 K^2 \cosh\alpha \frac{\sinh\beta}{\beta} = 0,$$

$$\alpha^2 = K^2 - \varkappa^2\Omega^2, \quad \beta^2 = K^2 - \Omega^2, \quad \gamma^2 = K^2 - \frac{1}{2}\Omega^2, \quad \varkappa = \frac{c_2}{c_1}$$

$$K = kh, \quad \Omega = \frac{\omega h}{c_2}$$



NO CHANCE OF TWO-MODE APPROXIMATIONS!

Long-wave approximations: $(K \ll 1)$

◯ Low-frequency $(\Omega \ll 1)$

At the leading order $\Omega^2 \sim K^4$

Kirchhoff plate theory

◯ High-frequency approximations near cut-off frequencies $\Omega_* \sim 1$ $(|\Omega - \Omega_*| \ll 1)$

At the leading order $K^2 \sim \Omega^2 - \Omega_*^2$

J. Kaplunov et al. *Dynamics of thin walled elastic bodies, Academic Press, 1998*

Composite (non-uniform) plate theories

Originate from Timoshenko-Reissner-Mindlin *ad hoc* theories.

$$\overbrace{D_a \frac{d^4 W}{d\xi^4}}^{\text{low-frequency (Kirchhoff)}} \underbrace{- \Omega^2 W + B_a \Omega^2 \frac{d^2 W}{d\xi^2} + C_a \Omega^4 W}_{\text{high-frequency}} = 0,$$

D_a, B_a, C_a are constants of order unity

Contributions for composite plate and shells theories

V.L. Berdichevsky. Variational principles of continuum mechanics: I. Fundamentals. Springer Science and Business Media, 2009

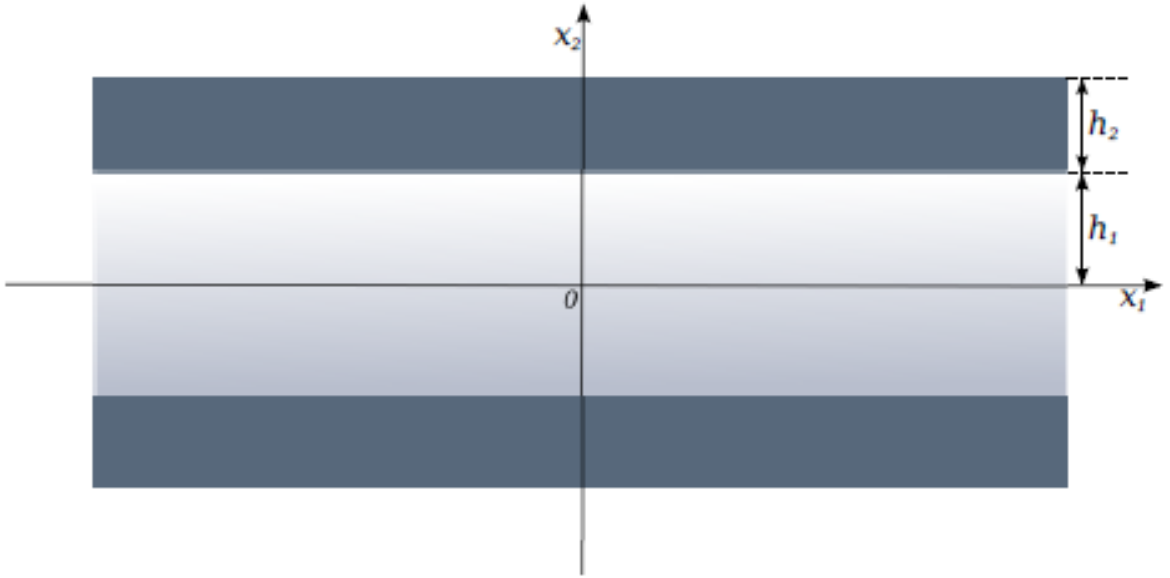
K.C. Le. Vibrations of shells and rods. Springer Science and Business Media, 2012

I.V. Andrianov, J. Awrejcewicz, L.I. Manevitch. Asymptotical mechanics of thin-walled structures. Springer Science and Business Media, 2013

Low-frequency vibrations of high-contrast three-layered plates

J.Kaplunov, D.A.Prikazchikov, L.A. Prikazchikova. (2017). Dispersion of elastic waves in a strongly inhomogeneous three-layered plate. *International Journal of Solids and Structures*, *113*, 169-179.

Statement of the problem



Equations of motion

$$\sigma_{ji,j} = \rho \ddot{u}_i, \quad i = 1, 2 \quad \text{for layers I and II}$$

Boundary and continuity conditions

$$\sigma_{12}^{\mathrm{II}} = 0, \quad \sigma_{22}^{\mathrm{II}} = 0 \quad \text{at} \quad x_2 = h_1 + h_2$$

$$\sigma_{12}^{\mathrm{I}} = \sigma_{12}^{\mathrm{II}}, \quad \sigma_{22}^{\mathrm{I}} = \sigma_{22}^{\mathrm{II}} \quad \text{and} \quad u_1^{\mathrm{I}} = u_1^{\mathrm{II}}, \quad u_2^{\mathrm{I}} = u_2^{\mathrm{II}} \quad \text{at} \quad x_2 = h_1$$

Dispersion relation

Ustinov, Doklady Physics (1976); Lee, Chang, Journal of Elasticity (1979)

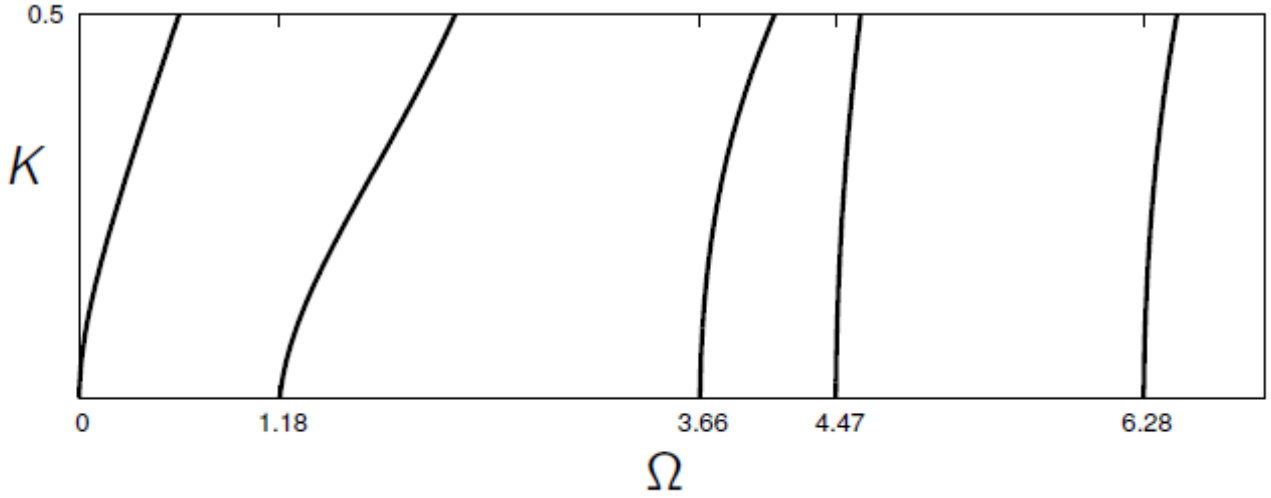
$$
\begin{aligned}
&4K^2h^3\alpha_2\beta_2F_4\left[F_1F_2C_{\beta_1}S_{\alpha_1}-2\alpha_1\beta_1(\varepsilon-1)F_3C_{\alpha_1}S_{\beta_1}\right]+\\
&h\alpha_2\beta_2C_{\alpha_2}C_{\beta_2}\left[4\alpha_1\beta_1K^2\left(h^4F_3^2+{F_4}^2(\varepsilon-1)^2\right)C_{\alpha_1}S_{\beta_1}-\right.\\
&\qquad\left.\left(4K^4h^4F_2^2+{F_4}^2F_1^2\right)S_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\beta_2}S_{\alpha_2}\varepsilon\beta_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_2^2\beta_1K^2h^2S_{\alpha_1}S_{\beta_1}-{F_4}^2\alpha_1C_{\alpha_1}C_{\beta_1}\right]+\\
&C_{\alpha_2}S_{\beta_2}\varepsilon\alpha_2(\beta_2^2-K^2h^2)(\beta_1^2-K^2)\left[4\alpha_1\beta_2^2K^2h^2C_{\alpha_1}C_{\beta_1}-{F_4}^2\beta_1S_{\alpha_1}S_{\beta_1}\right]+\\
&h^3S_{\alpha_2}S_{\beta_2}\left[\left(4\alpha_2^2\beta_2^2K^2F_1^2+K^2{F_4}^2F_2^2\right)C_{\beta_1}S_{\alpha_1}-\right.\\
&\qquad\left.\alpha_1\beta_1\left(16\alpha_2^2\beta_2^2(\varepsilon-1)^2K^4+{F_4}^2F_3^2\right)C_{\alpha_1}S_{\beta_1}\right]=0
\end{aligned}
$$

$\Omega=\dfrac{\omega h_1}{c_2^{\mathrm{I}}}$, $K=kh_1$, $C_{\alpha_j}, C_{\beta_j}, S_{\alpha_j}, S_{\beta_j}$ - hyperbolic functions

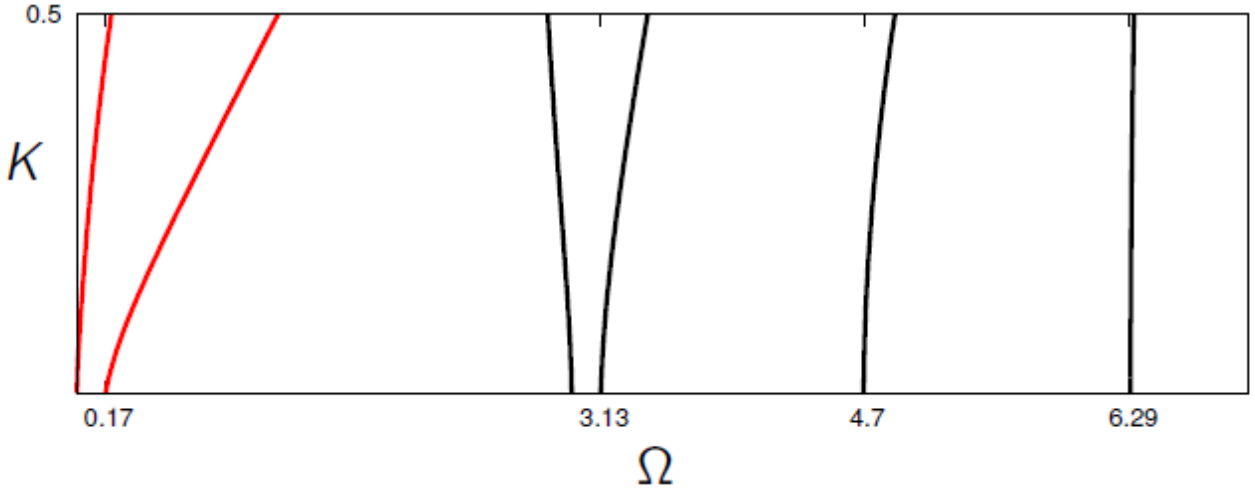
F_i, $i=1..4$, α_j,β_j, $j=1,2$ - functions of Ω and K, $\varepsilon=\dfrac{\mu_1}{\mu_2}$

Dispersion curves

No contrast



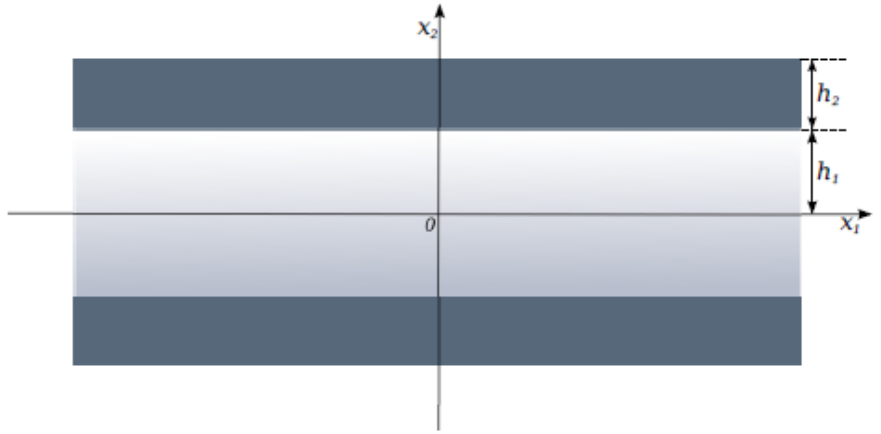
Effect of contrast



NEED OF TWO-MODE MODELS!

1D eigenvalue problem for shear cut-off

Flexural motion $\dfrac{\partial}{\partial x_1} = 0, \qquad u_2 = 0$



Frequency equation

$$\tan(\Omega)\tan\left(\sqrt{\frac{\varepsilon}{r}}h\,\Omega\right) = \sqrt{\varepsilon r}$$

Condition for a first shear cut-off frequency to be small

$$\boxed{r \ll h \ll \varepsilon^{-1}}$$

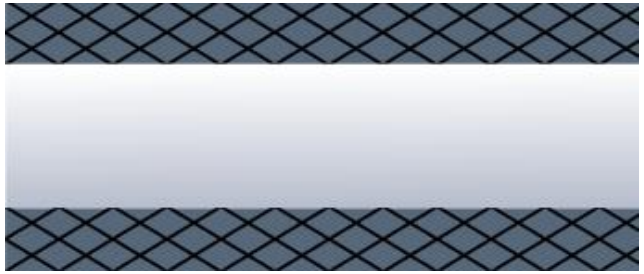
Frequency $\Omega^2 \sim \dfrac{r}{h}$

where $r = \dfrac{\rho_1}{\rho_2}, \quad h = \dfrac{h_2}{h_1}, \quad \varepsilon = \dfrac{\mu_1}{\mu_2}$

Some three-layered structures satisfying the condition $r \ll h \ll \varepsilon^{-1}$

A) Photovoltaic panels

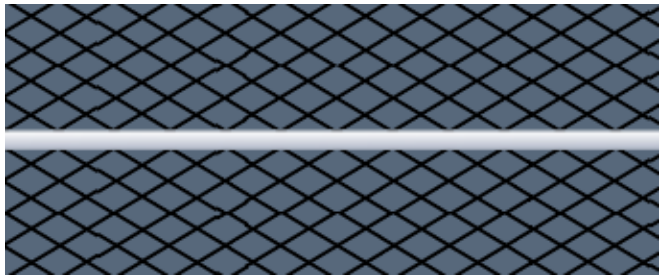
$\varepsilon << 1,\ h \sim 1,\ \rho \sim \varepsilon$



(stiff skin layers and light core layer)

B) Laminated glass

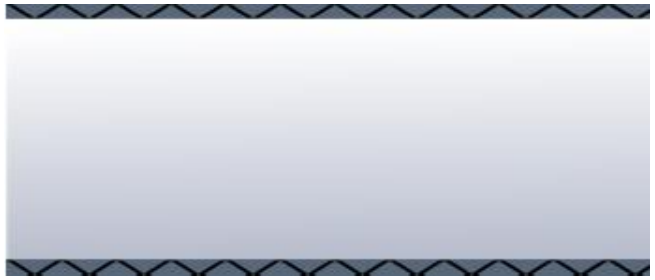
$\varepsilon << 1,\ h \sim \varepsilon^{-1/4},\ \rho \sim 1$



(stiff skin layers and light thin core layer)

C) Sandwich structure

$\varepsilon << 1,\ h \sim \varepsilon,\ \rho \sim \varepsilon^{2}$



(stiff thin skin layers and light core layer)

UNEXPECTEDLY LOW FIRST SHEAR CUT-OFF FREQUENCIES!

Long-wave low-frequency asymptotic approximation of the dispersion relation

For $K \ll 1$ and $\Omega \ll 1$

$$\gamma_1\Omega^2 + \gamma_2 K^4 + \gamma_3 K^2\Omega^2 + \gamma_4 K^6 + \gamma_5\Omega^4 + \gamma_6 K^4\Omega^2 + \gamma_7 K^8 + \gamma_8 K^2\Omega^4 + \gamma_9 K^2\Omega^6 + \gamma_{10}\Omega^6 + \ldots = 0$$

Multi-parametric analysis

$$\varepsilon \ll 1, \quad h \sim \varepsilon^a, \quad r \sim \varepsilon^b$$

Expanding coefficients

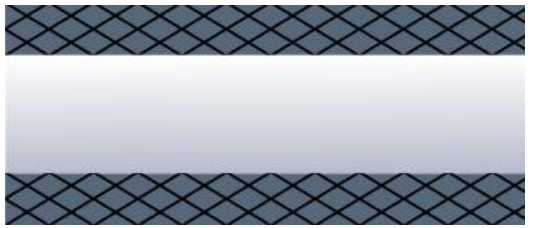
$$\gamma_i \to G_i \varepsilon^c$$

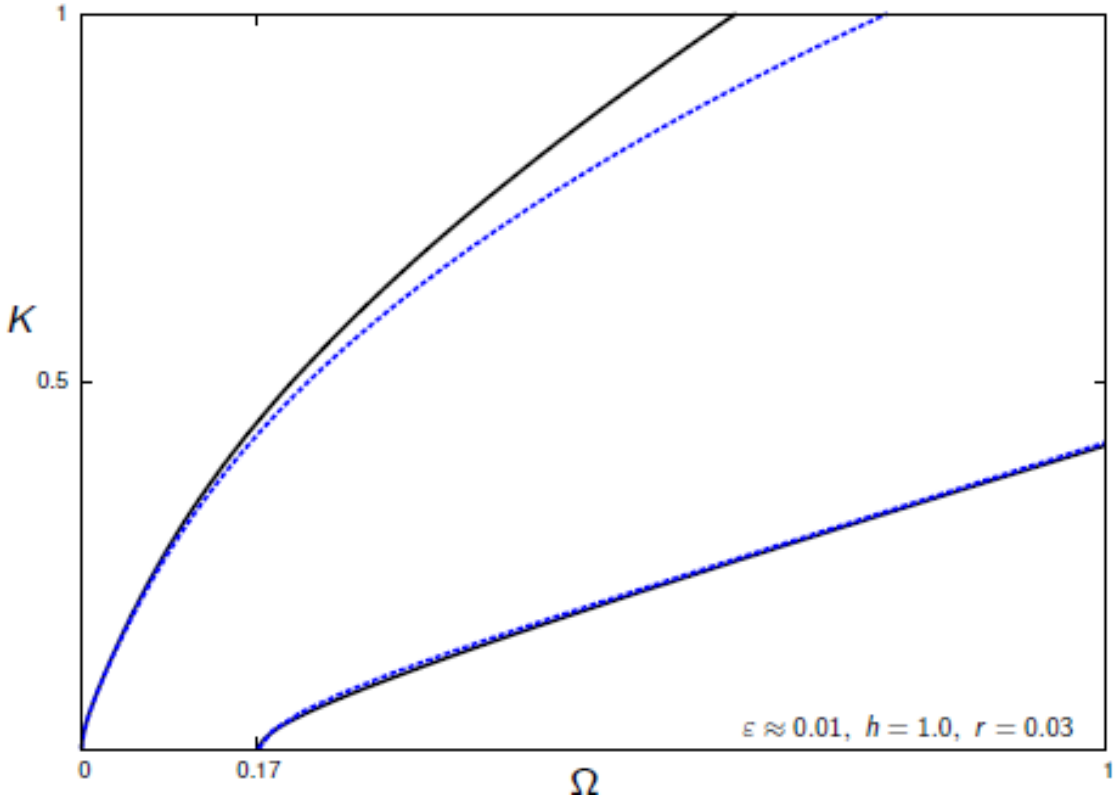
Low-frequency dispersion behaviour

A) Photovoltaic panels

(stiff skin layers and light core layer)

$$\varepsilon \ll 1, \quad h \sim 1, \quad r \sim \varepsilon$$





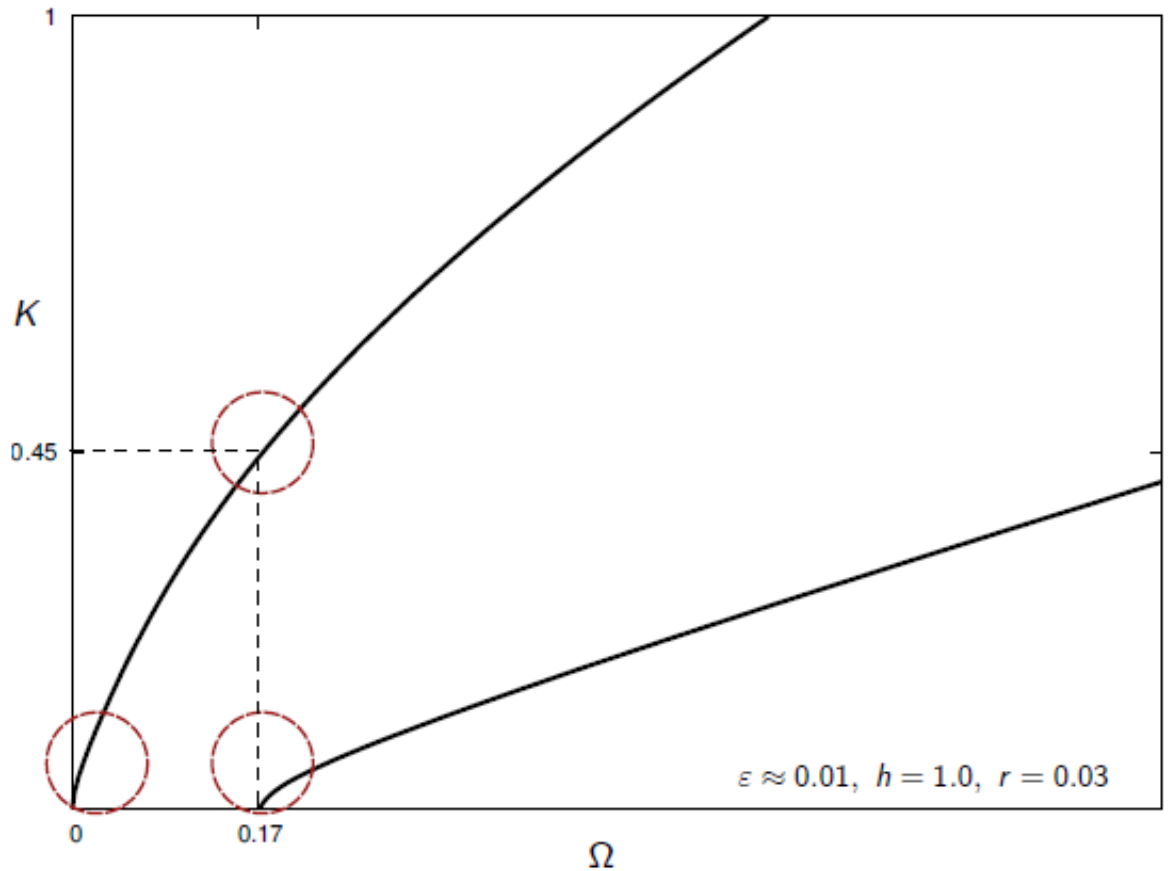
Retain leading order terms for both:
(i) fundamental mode ($\Omega \sim K^2$)
(ii) shear mode cut-off ($\Omega_{sh} \sim \sqrt{\varepsilon}$)

UNIFORM TWO-MODE APPROXIMATIONS

$$G_1\varepsilon\Omega^2 + G_2\varepsilon K^4 + G_3K^2\Omega^2 + G_4K^6 + G_5\Omega^4 = 0$$

Local approximations

Three local approximations

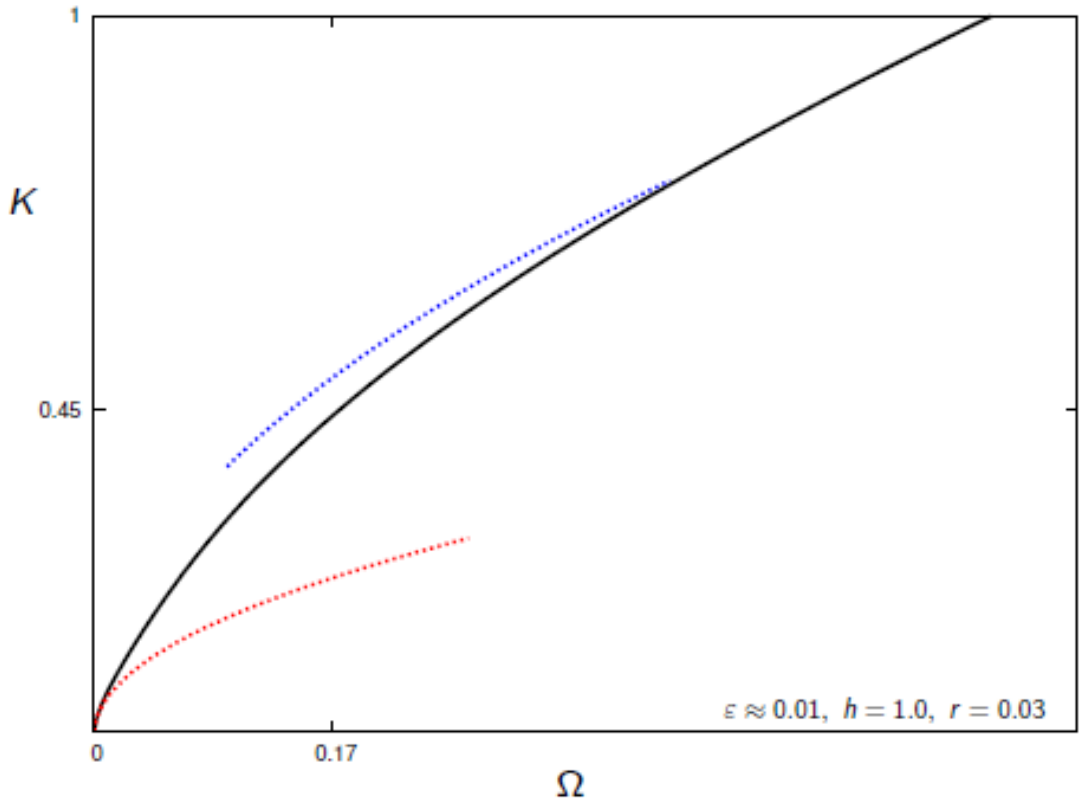


In the vicinity of zero frequency

$$G_1\Omega^2 + G_2K^4 = 0, \quad 0 < K \ll \sqrt{\varepsilon}, \quad \Omega \ll \varepsilon$$

At higher frequencies, including the vicinity of shear cut-off

$$G_3\Omega^2 + G_4K^4 = 0, \quad \sqrt{\varepsilon} \ll K \ll 1, \quad \varepsilon \ll \Omega \ll 1$$

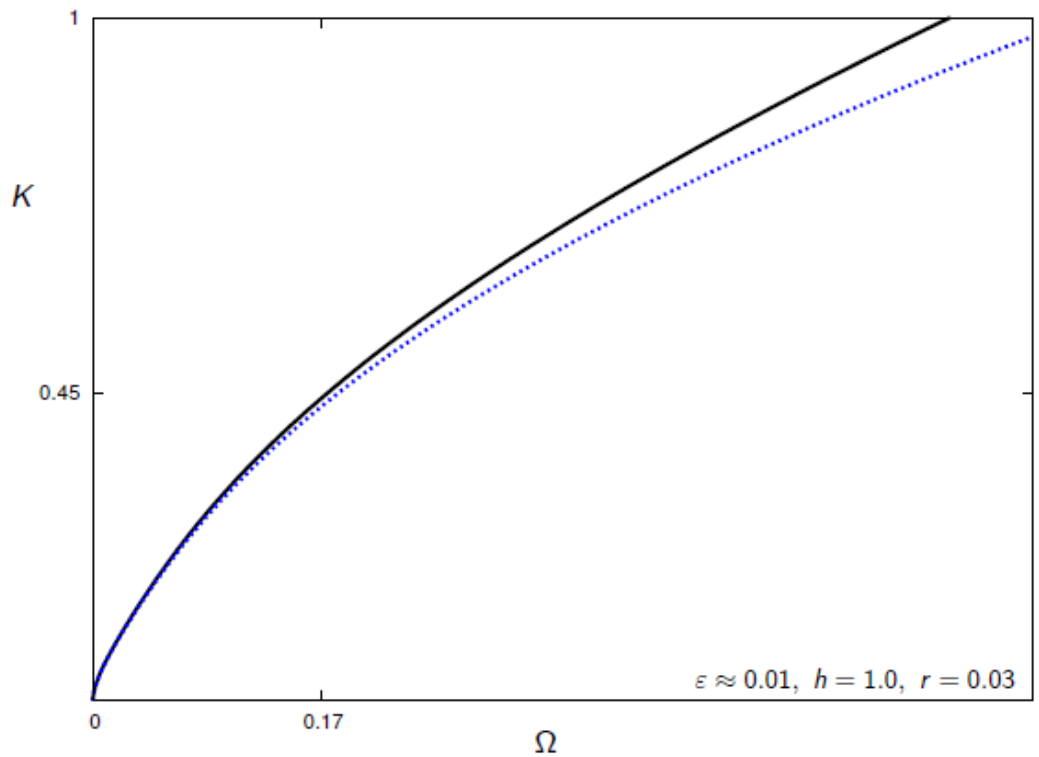


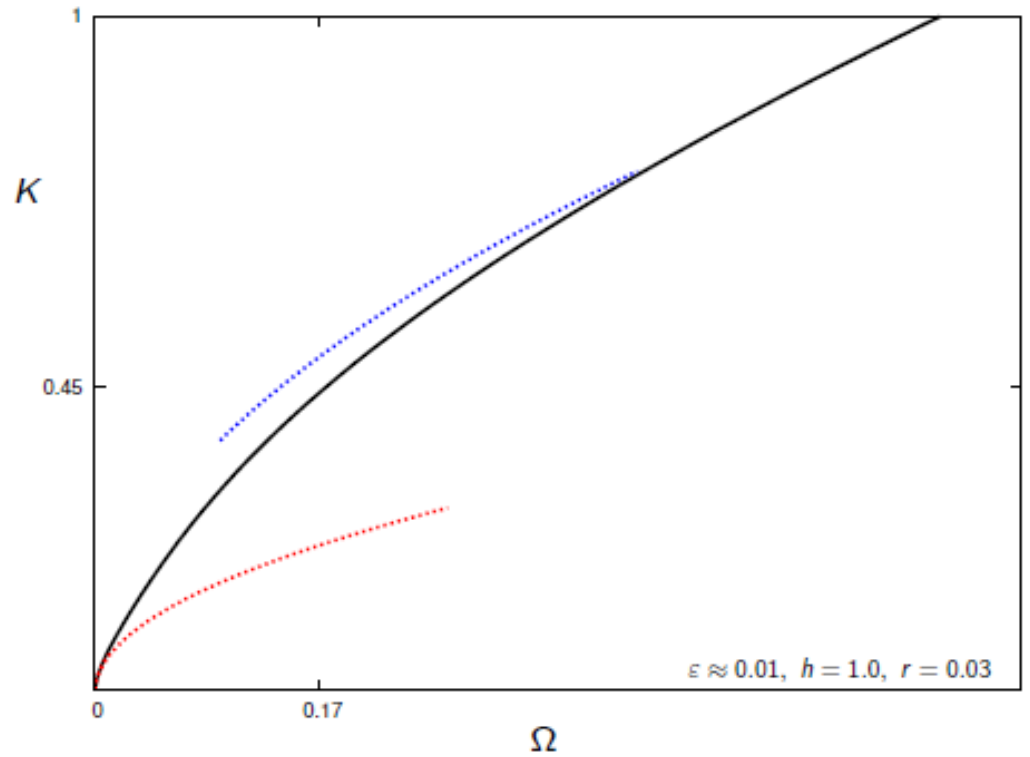
CLASSICAL KIRCHHOFF-TYPE THEORY IS NOT APPLICABLE!

Uniform approximation for the fundamental mode

Taking both local approximations we derive a uniform one:

$$G_1\varepsilon\Omega^2 + G_2\varepsilon K^4 + G_3 K^2\Omega^2 + G_4 K^6 = 0$$



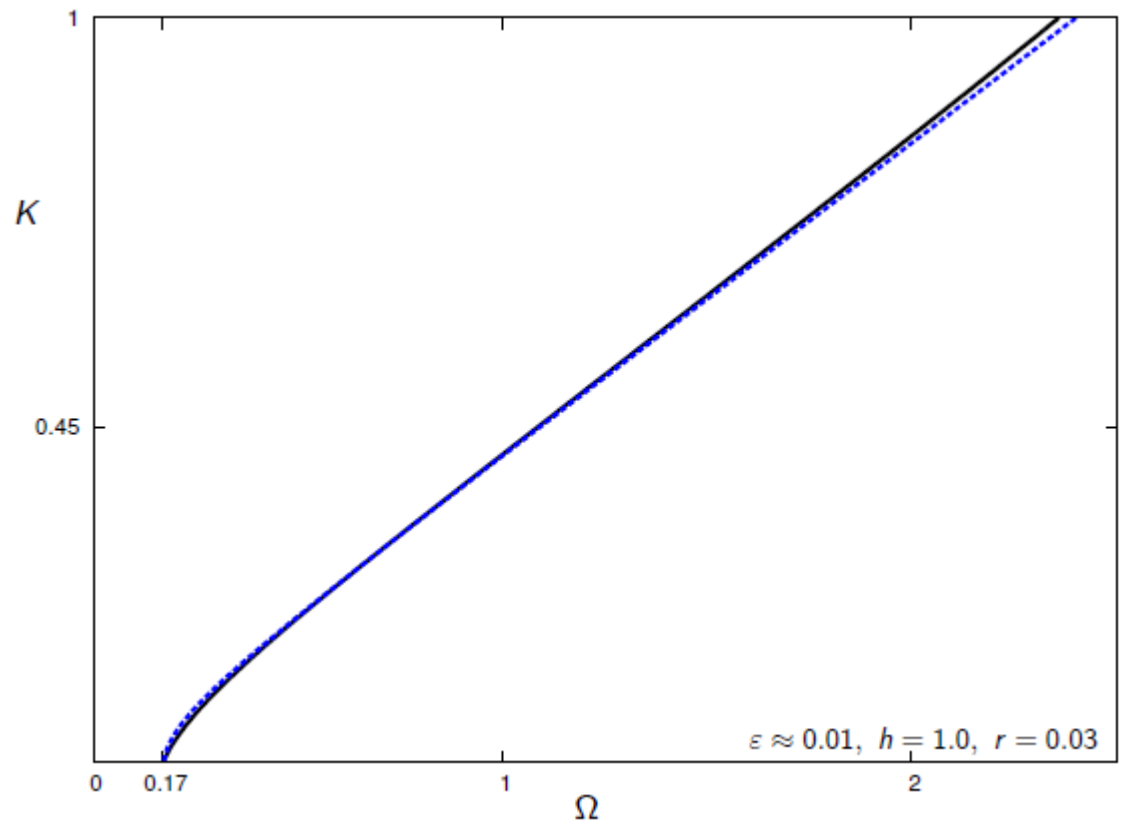


Also valid in the transition region $\Omega \sim \varepsilon, K \sim \sqrt{\varepsilon}$

Near shear cut-off approximation

For $\Omega \sim \sqrt{\varepsilon}$, $K \ll 1$

$$G_1\varepsilon + G_3K^2 + G_5\Omega^2 = 0$$



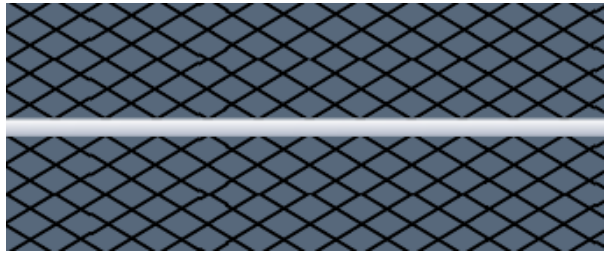
Low-frequency dispersion behaviour

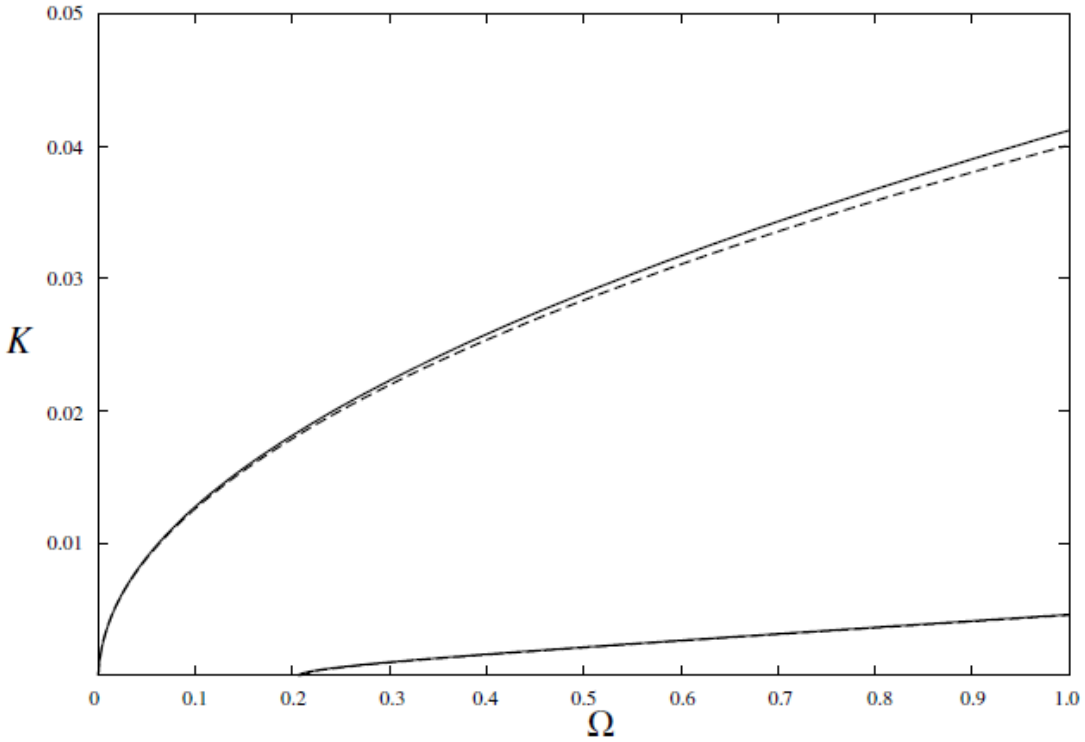
J. Kaplunov et al. *Proc. Eng.* 199 (2017): 1489-1494

B) Laminated glass

(stiff skin layers and light core layer)

$$\varepsilon << 1,\ h \sim \varepsilon^{-1/4},\ \rho \sim 1$$





UNIFORM TWO-MODE APPROXIMATIONS

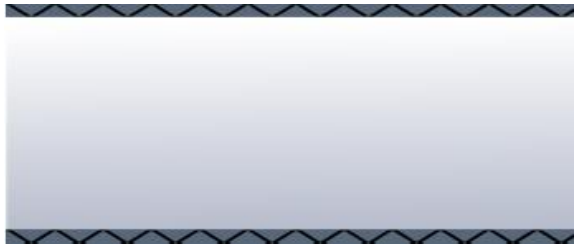
$$\varepsilon^{11/4} G_1 \Omega^2 + \varepsilon^{5/4} G_2 K^4 + \varepsilon^{3/2} G_3 K^2 \Omega^2 + G_4 K^6 + \varepsilon^{5/2} G_5 \Omega^4 = 0$$

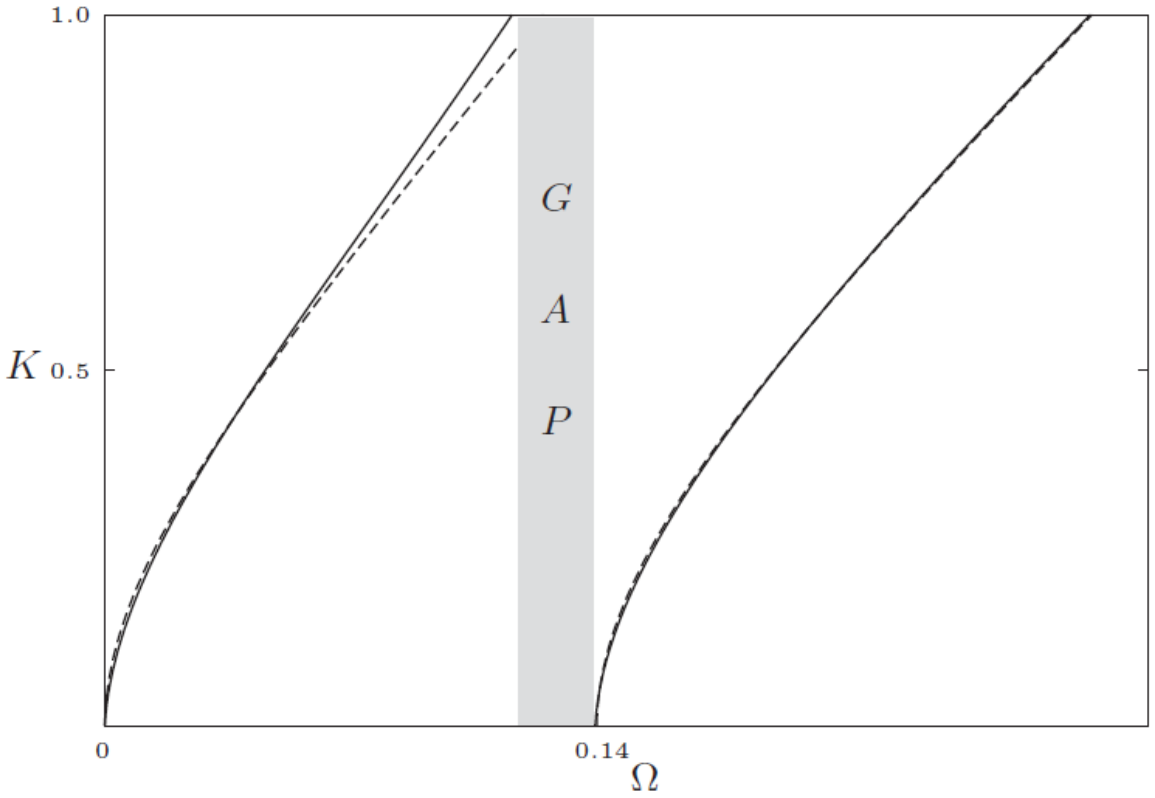
Low-frequency dispersion behaviour

C) Sandwich structure

$$\varepsilon \ll 1, \quad h \sim \varepsilon, \quad r \sim \varepsilon^2$$

(stiff thin skin layers and light core layer)



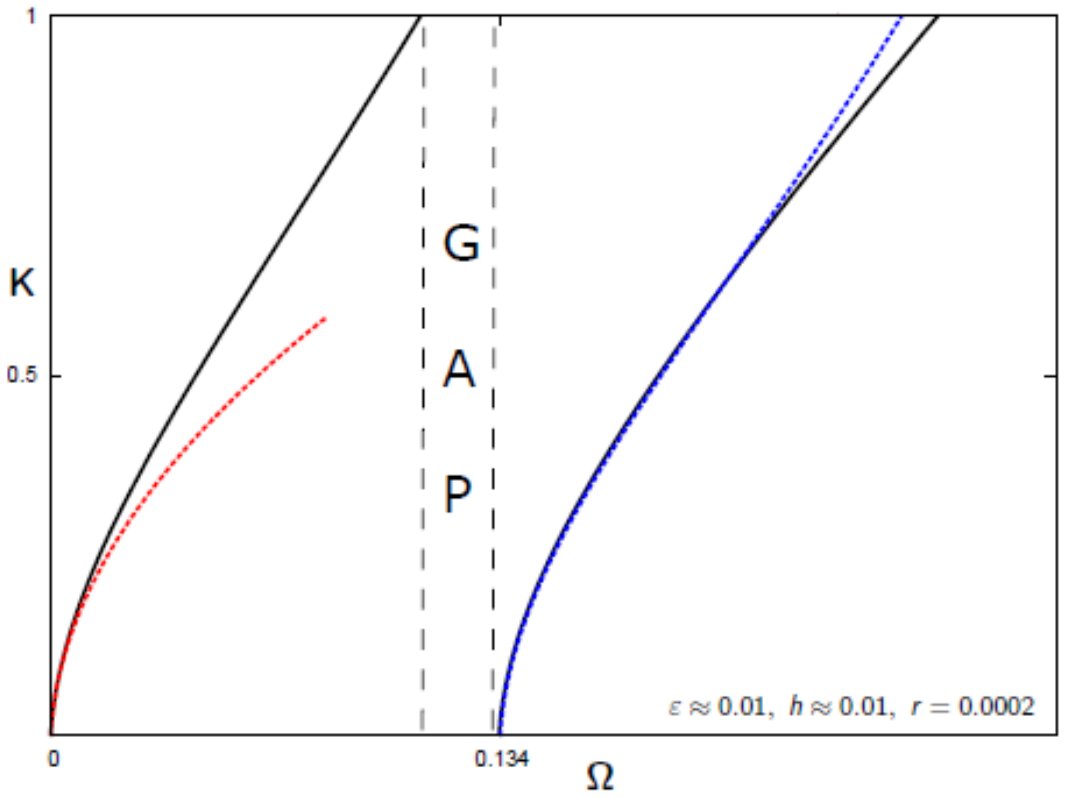


COMPOSITE APPROXIMATIONS

$$G_1\varepsilon\Omega^2 + G_2\varepsilon^2K^4$$

$$+ \varepsilon K^2\Omega^2\left(G_3 + \frac{r_0}{h_0}G_8\right) + G_5\Omega^4 = 0$$

Local approximations



NO OVERLAP REGION!

Fundamental mode

$$G_1\Omega^2 + G_2\varepsilon K^4 = 0,$$

$$K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}K^2 \ll \sqrt{\varepsilon}$$

Shear mode

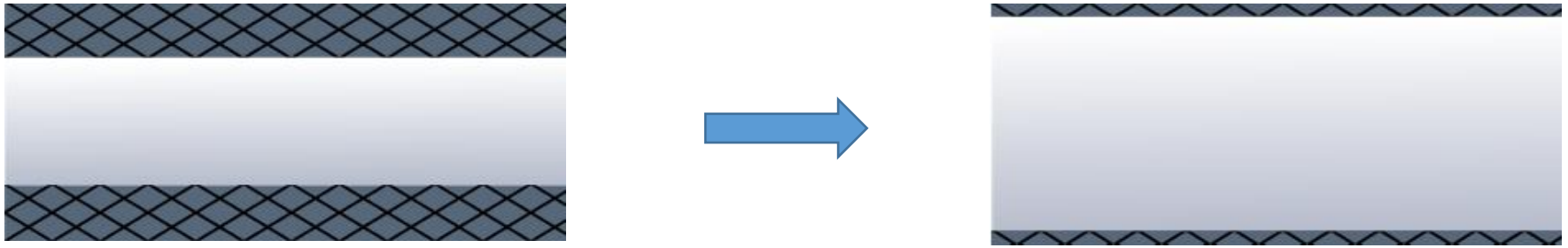
$$G_1\varepsilon + G_3\varepsilon K^2 + G_5\Omega^2 + G_8K^2\Omega^2 = 0,$$

$$\varepsilon^{\frac{1}{2}} \ll K \ll 1, \quad \Omega \sim \sqrt{\varepsilon}$$

Lightweight structures

J. Kaplunov et al., (2019) IOP Conference Series: Materials Science and Engineering 683 (1), 012014.

Where is transition from uniform approximation to a composite one?



$$\rho = \frac{\rho_c}{\rho_s} \ll 1, \quad h = \frac{h_s}{h_c} \sim \rho^a, \quad 0 \le a < 1 \qquad \mu = \frac{\mu_c}{\mu_s} \sim \rho.$$

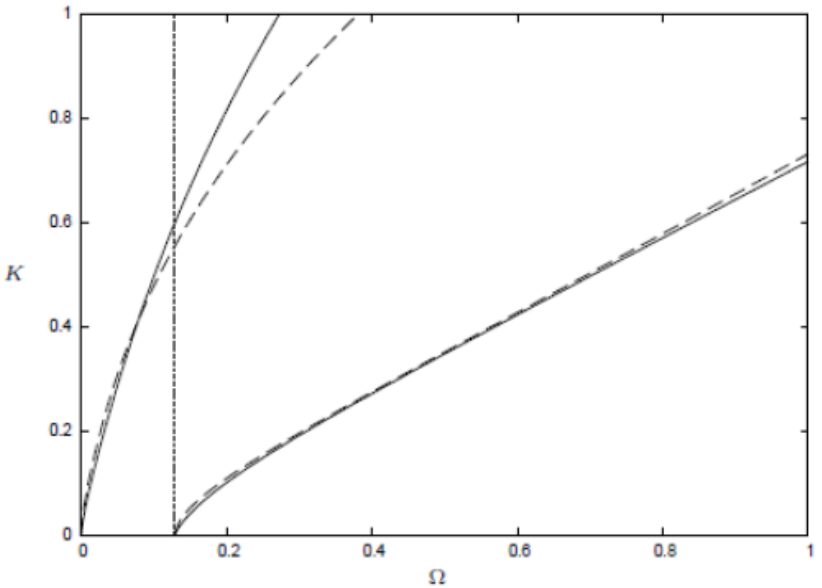
Low thickness shear cut-off frequency

$$\Omega_{sh} \approx \left(\frac{\rho}{h}\right)^{1/2} \sim \rho^{(1-a)/2} \ll 1$$

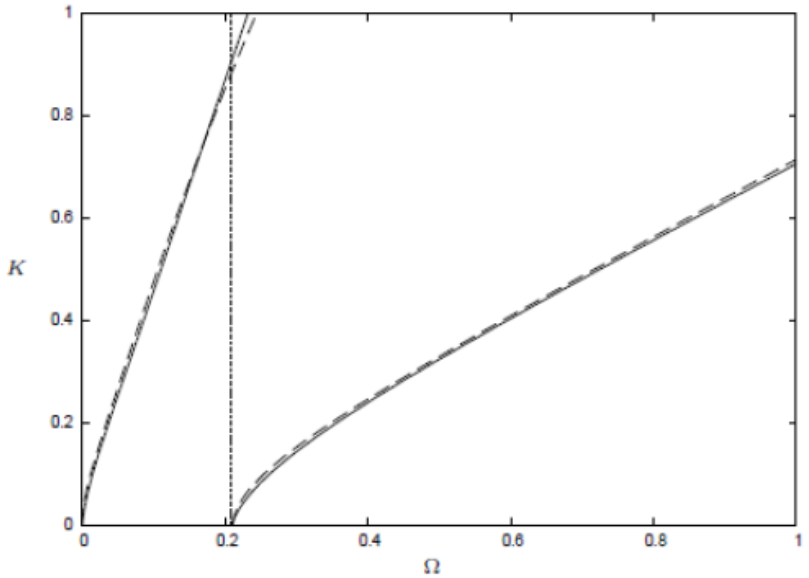
Uniform approximation $\left(0 \le a < \frac{1}{3}\right)$

Limiting case

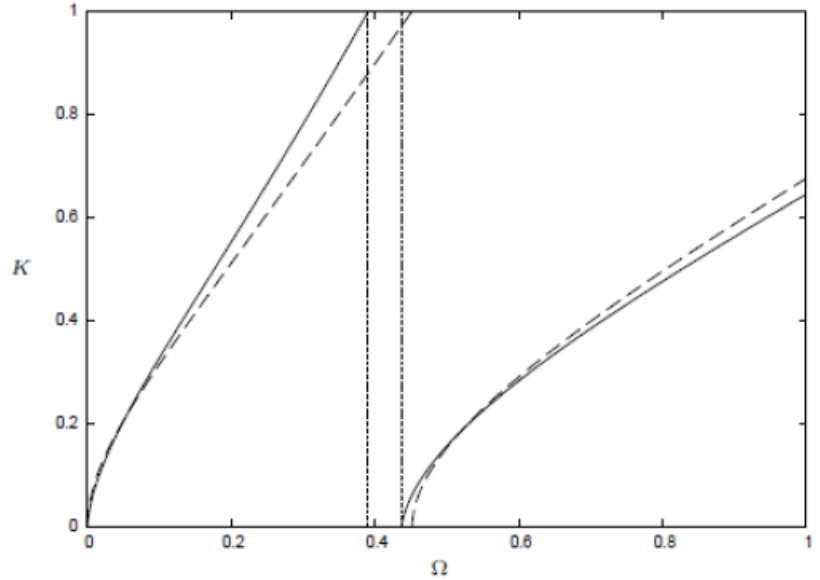
Composite approximation $\left(\frac{1}{3} < a < 1\right)$



$\left(a = \frac{1}{9}\right)$



$\left(a = \frac{1}{3}\right)$

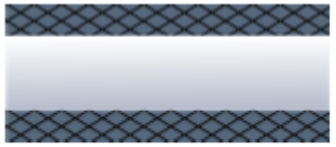


$\left(a = \frac{2}{3}\right)$

- **Two-mode uniform** $\rho^{1-a}G_1\Omega^2 + \rho^{1-a}G_2K^4 + G_3K^2\Omega^2 + \frac{1}{3}\rho^{2a}G_2K^6 + G_5\Omega^4 = 0$
- **Composite** $\rho^{1-a}G_1\Omega^2 + \rho^{1-a}G_2K^4 + G_3K^2\Omega^2 + G_5\Omega^4 = 0$

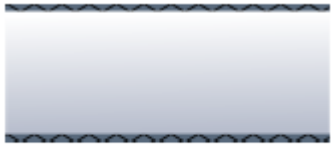
Remarks:

- Asymptotic models for strongly inhomogeneous plates



uniformly asymptotic

$$G_1\varepsilon u_{tt} + G_2\varepsilon\Delta^2 u + G_3\Delta u_{tt} - G_4\Delta^3 u + G_5 u_{tttt} = 0$$



composite

$$G_1\varepsilon u_{tt} + G_2\varepsilon^2\Delta^2 u + G_5 u_{tttt} + \varepsilon\left(G_3 + \frac{r_0}{h_0}G_8\right)\Delta u_{tt} = 0$$

5. Surface waves on a coated half-space with clamped surface

J.Kaplunov, D.Prikazchikov, L.Sultanova (2019). Rayleigh-type waves on a coated elastic half-space with a clamped surface. *Philosophical Transactions of the Royal Society A*, 377(2156), 20190111.

- *Surface waves on a coated half-space known for Neumann-type boundary conditions*
- *Clamped surface does not allow wave propagation* (*unless the layer is very soft compared to substrate*!)

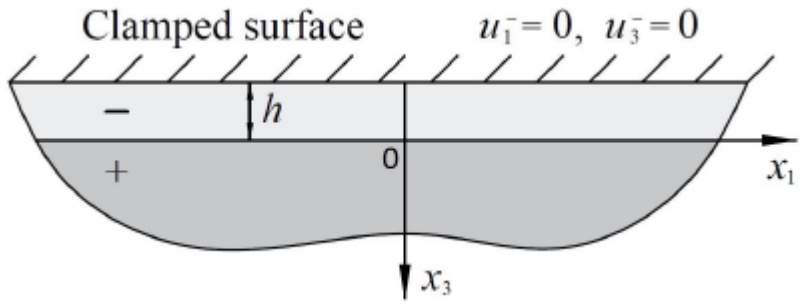
K. Cherednichenko & S. Cooper, Proc. Roy. Soc. A, 2015

Equations of motion

$$\sigma^{\pm}_{i1,1} + \sigma^{\pm}_{i3,3} - \rho^{\pm} u^{\pm}_{i,tt} = 0, \qquad i = 1, 3.$$

Constitutive relations

$$\sigma^{\pm}_{ij} = \lambda^{\pm}\delta_{ij}(u^{\pm}_{1,1} + u^{\pm}_{2,2} + u^{\pm}_{3,3}) + \mu^{\pm}(u^{\pm}_{i,j} + u^{\pm}_{j,i}), \qquad j = 1, 3.$$



Boundary conditions at $x_3 = -h$

$$u_i^- = 0.$$

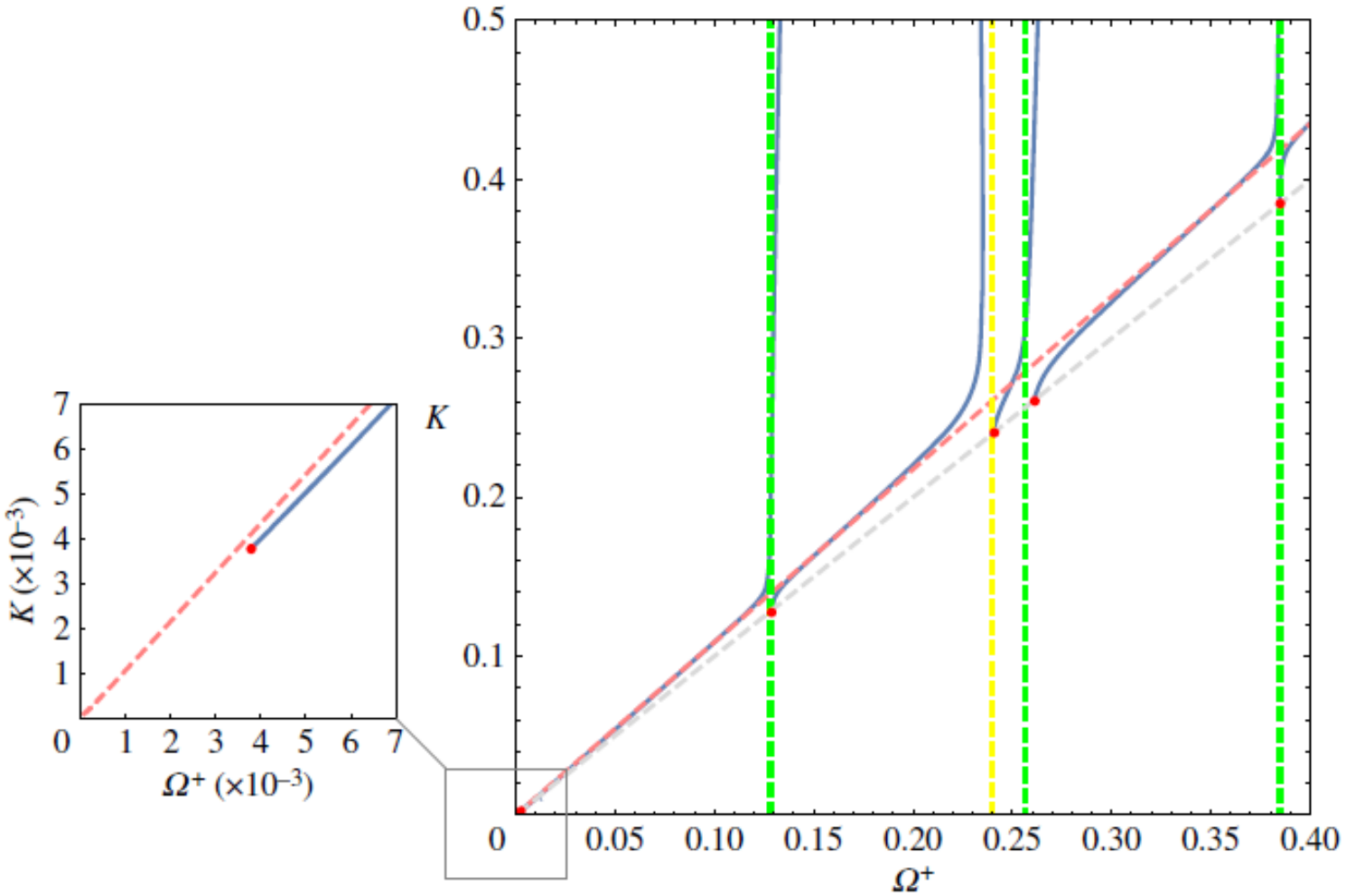
Continuity conditions at $x_3 = 0$

$$u_i^- = u_i^+, \qquad \sigma_{i3}^- = \sigma_{i3}^+.$$

Dispersion relation

High contrast (soft layer, stiff half-space)

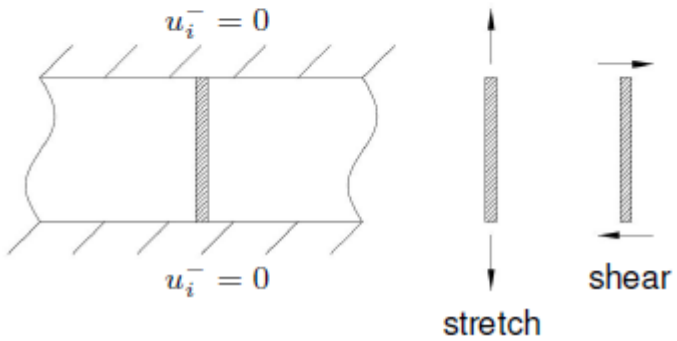
$\mu = 0.001$



The thickness resonances are eigenvalues of the problems

$$u^-_{3,33} + \left(\frac{\Omega^-}{\kappa^-}\right)^2 u^-_3 = 0,$$
$$u^-_{1,33} + (\Omega^-)^2 u^-_1 = 0$$

with $u^-_i = 0$, $i = 1, 3$ at the faces.



- *Further developments*

✓ *Classification of asymptotic regimes in static problems for a coated elastic half-space* (*J. Kaplunov, D.A.Prikazchikov, L.Sultanova, ZAMP, 2018,2019*)

✓ *Boundary conditions for vector problems in elasticity*

✓ *Account for viscosity in soft layers*

✓ *Analysis of local low-frequency behaviour*

✓ *Extension to strongly inhomogeneous shells*

Conclusions

- ✓ High contrast supports specific low-frequency motions which are not the feature of non-contrast layered and multi-component structures
- ✓ Multi-parametric analysis is crucial to cover all the diversity of observed setups
- ✓ Similarity in long-wave behaviour of thin and periodic structures increases the value of layered plate analysis

CISM advanced course

https://www.cism.it/en/activities/courses/C2212/

https://imechanica.org/node/26061