Time-dispersive behaviour of resonant media

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1 Physical context and motivation

Understanding the dependence of material properties of continuous media on frequency is a natural and practically relevant task, stemming from the theoretical and experimental studies of "metamaterials", *e.g.* materials that exhibit negative refraction of propagating wave packets. As was noted in the pioneering work [17], that negative refraction is only possible under the assumption of frequency dispersion, *i.e.* when the material parameters (permittivity and permeability in electromagnetism, elastic moduli and mass density in acoustics) are not only frequency-dependent, but also become negative in certain frequency bands.

Independently of the search for metamaterials, in the course of the development of the theory of electromagnetism, it has transpired in modern physics that the Maxwell equations need to be considered with time-nonlocal "memory" terms, see *e.g.* [9, Section 7.10] and also [1], [15]. The generalised system (in the absence of charges and currents in the domain of interest) has the form

$$\rho\partial_t u + \int_{-\infty}^{t} a(t-\tau)u(\tau)d\tau + \mathrm{i}Au = 0, \qquad A = \begin{pmatrix} 0 & \mathrm{i}\,\mathrm{curl} \\ -\mathrm{i}\,\mathrm{curl} & 0 \end{pmatrix}, \tag{1}$$

where u represents the (time-dependent) electromagnetic field $(H, E)^\top$, the matrix ρ depends on the electric permittivity and magnetic permeability, and a is a matrix-valued "susceptibility" operator, set to zero in the more basic form of the system.[1]

Applying the Fourier transform in time t to (1), an equation in the frequency domain is obtained:

$$\big(\mathrm{i}\omega\rho + \widehat{a}(\omega)\big)\widehat{u}(\cdot,\omega) + \mathrm{i}A\widehat{u}(\cdot,\omega) = 0, \tag{2}$$

where $\widehat{u}$ is the Fourier transform of u, and ω is the frequency. Equation (2) is often interpreted as a "non-classical" version of Maxwell's system of equations, where the permittivity and/or permeability are frequency-dependent. The existence of such media (commonly known as Lorentz materials) and the analysis of their properties go back a few decades in time and has also attracted considerable interest quite recently, *e.g.* in the study of plasma in tokamaks, see [5] and references therein.

It is reasonable to ask the question of whether frequency dispersion laws such as pertaining to (2), which in turn may provide one with metamaterial behaviour in appropriate frequency intervals [17], can be derived by some process of homogenisation of composite media with contrast (or, more generally, any other miscroscopic degeneracies resonating with the macroscopic wavefields, see e.g. Fig. 1).

It is instructive to point out that the results of [2] establish a thrilling relationship between the analysis of thin structures and the homogenisation theory of high-contrast composites. Namely, the paper [2] deals with the case of the so-called superlattices [16] with high contrast, see Fig. 2. It can be shown that the asymptotic model for this system is precisely the one derived in [11, 12, 6] in the case of a resonant thin structure converging to a chain-graph, see Fig. 1. As we shall argue in the present article, such superlattices (and the corresponding chain-graphs) offer a simple prototype for a metamaterial, via the mathematical approach outlined above.

As a particular realistic example of a thin network with high contrast, consider the problem of modelling acoustic wave propagation in a system of channels $\Omega^{\varepsilon,\delta}$, ε-periodic in one direction, of thickness $\delta \ll \varepsilon$, and with contrasting material properties (cf. Fig. 3). To simplify the presentation, we assume the antiplane shear

[1] From the rigorous operator-theoretic point of view, A in (1) is treated as a self-adjoint operator in a Hilbert space $\mathbb{H}$ of functions of $x \in \Omega$, for example $\mathbb{H} = L^2(\Omega; \mathbb{R}^6)$, where Ω is the part of the space occupied by the medium.

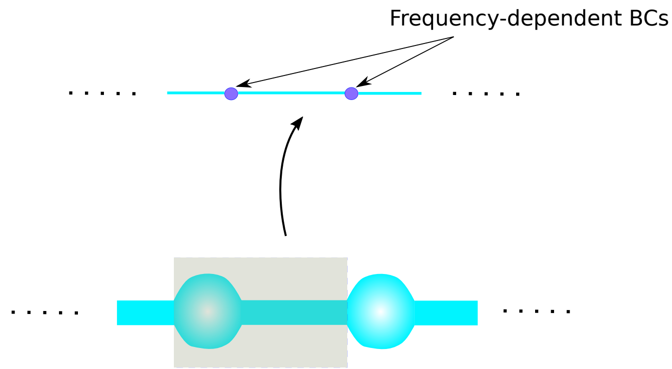

Figure 1: AN EXAMPLE OF A RESONANT THIN NETWORK. Edge volumes are asymptotically of the same order as vertex volumes. The stiffness of the material of the structure is of the order period-squared. In the spectral analysis of such "thin" periodic structures converging to metric graphs, impedance-type problems are obtained [11, 12, 6], whiich in the periodic context lead to frequency/time dispersion.

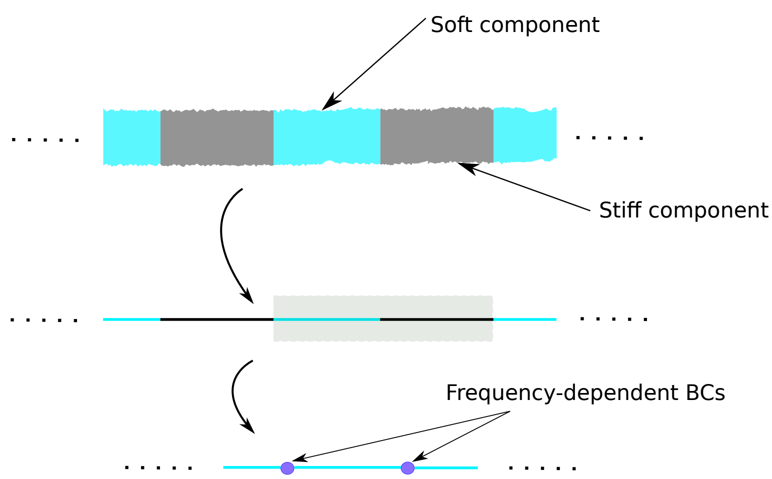

Figure 2: HIGH-CONTRAST SUPERLATTICE. The problem for a superlattice is reduced to a one-dimensional high-contrast problem. This is asymptotically equivalent to an impedance-type problem on the soft component. Note that the stiff component reduces to an array of vertices.

wave polarisation (the so called S-waves), which leads to a scalar wave equation for the only non-vanishing component W, of the form

$$W_{tt} - \nabla_x \cdot (a^\varepsilon(x)\nabla_x W) = 0, \qquad W = W(x,t), \quad x, t \in \mathbb{R},$$

where the coefficient a^ε takes values 1 and ε^2 in different channels of the ε-periodic structure. Looking for time-harmonic solutions $W(x,t) = U(x)\exp(\mathrm{i}\omega t)$, $\omega > 0$, one arrives at the spectral problem

$$-\nabla \cdot (a^\varepsilon \nabla U) = \omega^2 U. \tag{3}$$

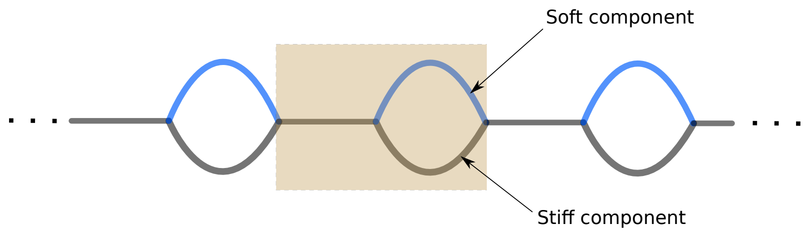

Figure 3: THIN NETWORK. An example of a high-contrast periodic network. Stiff channels are in grey, soft channels are in blue.

As we argue below, the behaviour of (3) is close, in a quantitatively controlled way as $\varepsilon \to 0$, to that of an "effective medium" on $\mathbb{R}$ described by an equation of the form

$$-U'' = \beta(\omega)U, \tag{4}$$

for an appropriate function $\beta = \beta(\omega)$, explicitly given in terms of the material parameters a^ε and the topology of the original system of channels.

Our goal: To derive an explicit formula for the function β in (4), in terms of the topology of the graph representing the original domain of wave propagation.

2 Infinite-graph setup

Consider a graph $\mathbb{G}_\infty$, periodic in one direction, so that $\mathbb{G}_\infty + \ell = \mathbb{G}_\infty$, where ℓ is a fixed vector, which defines the graph axis. Let the periodicity cell $\mathbb{G}$ be a finite compact graph of total length $\varepsilon \in (0,1)$, and denote by e_j, $j = 1, 2, \dots n$, $n \in \mathbb{N}$, its edges. For each $j = 1, 2, \dots, n$, we identify e_j with the interval $[0, \varepsilon l_j]$, where εl_j is the length of e_j. We associate with the graph $\mathbb{G}_\infty$ the Hilbert space

$$L_2(\mathbb{G}_\infty) := \bigoplus_{\mathbb{Z}} \bigoplus_{j=1}^{n} L_2(0, \varepsilon l_j).$$

Consider a sequence of operators A^ε, $\varepsilon > 0$, in $L_2(\mathbb{G}_\infty)$, generated by second-order differential expressions

$$-\frac{d}{dx}\left(\left(a^\varepsilon\right)^2 \frac{d}{dx}\right),$$

with positive $\mathbb{G}$-periodic coefficients $(a^\varepsilon)^2$ defined on $\mathbb{G}_\infty$, with the domain $\operatorname{dom}(A^\varepsilon)$ that describes the coupling conditions at the vertices of $\mathbb{G}_\infty$:

$$\operatorname{dom}(A^\varepsilon) = \left\{ u \in \bigoplus_{e \in \mathbb{G}_\infty} W^{2,2}(e) \,\middle|\, u \text{ continuous}, \sum_{e \ni V} \sigma_e (a^\varepsilon)^2 u'(V) = 0 \ \forall\, V \in \mathbb{G}_\infty \right\}, \tag{5}$$

In the formula (5) the summation is carried out over the edges e sharing the vertex V, the coefficient $(a^\varepsilon)^2$ in the vertex condition is calculated on the edge e, and $\sigma_e = -1$ or $\sigma_e = 1$ for e incoming or outgoing for V, respectively. The matching conditions (5) represent the combined conditions of continuity of the function and of vanishing sums of its co-normal derivatives at all vertices (*i.e.* the so-called Kirchhoff conditions).

3 Gelfand transform

We seek to apply the one-dimensional Gelfand transform

$$v(x)=\sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}}u(x+\varepsilon n)\mathrm{e}^{-\mathrm{i}t(x+\varepsilon n)}. \tag{6}$$

to the operator A^ε defined on $\mathbb{G}_\infty$ in order to obtain the direct fibre integral for the operator A^ε :

$$A^\varepsilon\cong\oplus\int_{-\pi/\varepsilon}^{\pi/\varepsilon}A_t^\varepsilon dt.$$

In order to do achieve this goal, we first note that the geometry of $\mathbb{G}_\infty$ is encoded in the matching conditions (5) *only*. This opens up the possibility to embed the graph $\mathbb{G}_\infty$ into $\mathbb{R}$ by rearranging its edges as consecutive segments of the real line (leading to a one-dimensional ε-periodic chain graph). The embedding leads to rather complex non-local matching conditions, but, on the positive side, allows us to use the Gelfand transform (6). The Gelfand transform leads to periodic conditions on the boundary of the cell $\mathbb{G}$ and thus in our case identifies the "left" boundary vertices of the graph $\mathbb{G}$ with their translations by ℓ, which results in a modified graph $\widehat{\mathbb{G}}$. Due to the non-locality of matching conditions for the embedding of $\mathbb{G}_\infty$ into $\mathbb{R}$, the Kirchhoff matching conditions at the vertices of $\mathbb{G}$ are replaced by by suitable "weighted Kirchhoff" conditions.

The image of the Gelfand transform is described as follows. There exists a unimodular list $\{w_V(e)\}_{e\ni V}$, *cf.* [3], defined at each vertex V of $\widehat{\mathbb{G}}$ as a finite collection of values corresponding to the edges adjacent to V. For each $t\in[-\pi/\varepsilon,\pi/\varepsilon)$, the fibre operator A_t^ε is generated by the differential expression

$$\left(\frac{1}{\mathrm{i}}\frac{d}{dx}+t\right)(a^\varepsilon)^2\left(\frac{1}{\mathrm{i}}\frac{d}{dx}+t\right) \tag{7}$$

on the domain

$$\begin{aligned}\operatorname{dom}(A_t^\varepsilon)=\biggl\{v\in\bigoplus_{e\in\mathbb{G}}W^{2,2}(e)\ \bigg|\ & w_V(e)v|_e(V)=w_V(e')v|_{e'}(V)\text{ for all } e,e' \text{ adjacent to } V,\\ &\sum_{e\ni V}\partial^{(t)}v(V)=0\quad\text{for each vertex } V\biggr\},\end{aligned} \tag{8}$$

where $\partial^{(t)}v(V)$ is the weighted "co-derivative" $\sigma_e w_V(e)(a^\varepsilon)^2(v'+\mathrm{i}tv)$ of the function v on the edge e, calculated at V.

4 Boundary triples

Definition 4.1 ([7, 10, 4])**.** *Suppose:*

- $A_{\max}$ *is the adjoint to a densely defined symmetric operator on a separable Hilbert space* H *("physical region space")*
- Γ_0, Γ_1 *be linear mappings of* $\operatorname{dom}(A_{\max})\subset H$ *to a separable Hilbert space* $\mathcal{H}$ *("boundary space").*

A. The triple $(\mathcal{H},\Gamma_0,\Gamma_1)$ *is called a* **boundary triple** *for the operator* $A_{\max}$ *if the following two conditions hold:*

1. *For all* $u,v\in\operatorname{dom}(A_{\max})$ *one has the second Green's identity*

$$\langle A_{\max}u,v\rangle_H-\langle u,A_{\max}v\rangle_H=\langle\Gamma_1u,\Gamma_0v\rangle_{\mathcal{H}}-\langle\Gamma_0u,\Gamma_1v\rangle_{\mathcal{H}}.$$

2. The mapping $\mathrm{dom}(A_{\max}) \ni u \longmapsto (\Gamma_0 u, \Gamma_1 u) \in \mathcal{H} \oplus \mathcal{H}$ *is onto.*

B. The operator-valued Herglotz[2] *function* $M = M(z)$*, defined by*

$$M(z)\Gamma_0 u_z = \Gamma_1 u_z, \quad u_z \in \ker(A_{\max} - z), \quad z \in \mathbb{C}_+ \cup \mathbb{C}_-,$$

is called the M**-function** *of the operator* $A_{\max}$ *with respect to the triple* $(\mathcal{H}, \Gamma_0, \Gamma_1)$*.*

Consider the restriction A_B of $A_{\max}$ given by

$$\mathrm{dom}(A_B) = \bigl\{u \in \mathrm{dom}(A_{\max}) : \ \Gamma_1 u = B\Gamma_0 u\bigr\}.$$

Key observation: $u \in \mathrm{dom}(A_B)$ is an eigenvector of A_B with eigenvalue z_0 if and only if[3]

$$\bigl(M(z_0) - B\bigr)\Gamma_0 u = 0.$$

In what follows we use the triple $(\mathbb{C}^m, \Gamma_0, \Gamma_1)$, where m is the number of vertices in the graph $\widehat{\mathbb{G}}$, and[4]

$$\Gamma_0 v = \bigl\{v(V)\bigr\}_V, \qquad \Gamma_1 v = \Bigl\{\sum_{e \ni V} \partial^{(t)} v(V)\Bigr\}_V, \qquad v \in \mathrm{dom}(A_{\max}),$$

where $v(V)$ is the common value of $w_V(e)v|_e(V)$ for all edges e adjacent to V, and $\partial^{(t)}v(V)$ is defined at the end of Section 3, see also (9) below.

Exercise 4.2. *By the definition of the* M*-matrix one has* $\Gamma_1 v = M(z)\Gamma_0 v$*, for functions* $v \in \ker(A_{\max} - z)$*, which have the form*

$$v(x) = \exp(-\mathrm{i}xt)\biggl\{A_e \exp\Bigl(-\frac{\mathrm{i}kx}{a^\varepsilon}\Bigr) + B_e \exp\Bigl(\frac{\mathrm{i}kx}{a^\varepsilon}\Bigr)\biggr\}, \quad x \in e, \quad A_e, B_e \in \mathbb{C},$$

where $k := \sqrt{z}$*, and the co-derivative is given by*

$$(a^\varepsilon)^2(v'(x) + \mathrm{i}tv(x)) = \mathrm{i}ka^\varepsilon \exp(-\mathrm{i}xt)\biggl\{-A_e \exp\Bigl(-\frac{\mathrm{i}kx}{a^\varepsilon}\Bigr) + B_e \exp\Bigl(\frac{\mathrm{i}kx}{a^\varepsilon}\Bigr)\biggr\}, \quad x \in e, \tag{9}$$

For the vertex V *and for every "Dirichlet data" vector* $\Gamma_0 v$ *one of whose entries is unity and the other entries vanish, the "Neumann data" vector* $\Gamma_1 v$ *gives the column of the* M*-matrix corresponding to* V*. The elements of* $\Gamma_1 v$ *corresponding to diagonal and off-diagonal entries of* $M(z)$ *are, respectively,*

$$-\sum_{e \in V} ka^\varepsilon \cot\left(\frac{k\varepsilon l_e}{a^\varepsilon}\right), \qquad \sum_{e \in V} ka^\varepsilon \widetilde{w}_V(e)\left(\sin\frac{k\varepsilon l_e}{a^\varepsilon}\right)^{-1},$$

where $\{\widetilde{w}_V(e)\}_{e \ni V}$ *is a unimodular list uniquely determined by the list* $\{w_V(e)\}_{e \ni V}$*. The resulting* M*-matrix is constructed from these columns over all vertices* V*.*

5 Examples

(0) Both components disconnected

Consider the high-contrast one-dimensional periodic medium (cf. [2]), as shown in Fig. 4. In this case, the graph $\mathbb{G}_{\mathrm{per}}$ is an infinite periodic chain-graph.

[2] For a definition and properties of Herglotz functions, see *e.g.* [13].

[3] Assuming z_0 is not in the spectrum of the Dirichlet restriction of $A_{\max}$.

[4] The maximal operator $A_{\max} = A^*_{\min}$ is defined by the differential expression (7) on the domain

$$\mathrm{dom}(A_{\max}) = \Bigl\{v \in \bigoplus_{e \in \widehat{\mathbb{G}}} W^{2,2}(e) \ \Big| \ w_V(e)v|_e(V) = w_V(e')v|_{e'}(V) \quad \text{for all } e, e' \text{ adjacent to } V, \quad \forall V \in \widehat{\mathbb{G}}\Bigr\}.$$

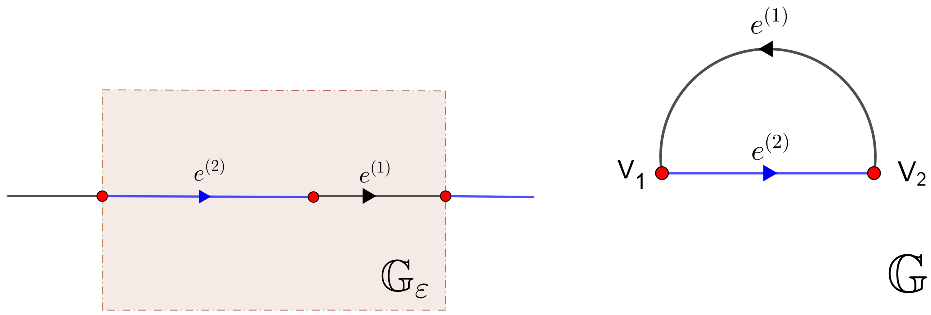

Figure 4: Example (0). $\mathbb{G}_{\rm per}$ with $\mathbb{G}_\varepsilon$ outlined on the left; the graph $\mathbb{G}$ after Gelfand transform on the right. The soft component is drawn in blue.

The boundary space $\mathcal{H}$ pertaining to the graph $\mathbb{G}$ is chosen as $\mathcal{H} = \mathbb{C}^2$. The unimodular list functions w_{V_1} and w_{V_2} are chosen as follows:

$$\{w_{V_1}(e^{(j)})\}_{j=1}^2 = \{1,1\}, \qquad \{w_{V_2}(e^{(j)})\}_{j=1}^2 = \{1,1\}.$$

Note: The weights in this example are trivial due to the fact that no flattening was applied to $\mathbb{G}_\varepsilon$.

The M-matrix for the period graph is the sum of

$$M_\varepsilon^{(\tau),\text{stiff}} = \frac{k}{\varepsilon}\begin{pmatrix} -a_1 \cot\dfrac{k\varepsilon l_1}{a_1} & a_1 e^{-il_1\tau}\csc\dfrac{k\varepsilon l_1}{a_1} \\ a_1 e^{il_1\tau}\csc\dfrac{k\varepsilon l_1}{a_1} & -a_1\cot\dfrac{k\varepsilon l_1}{a_1}\end{pmatrix}, \quad M_\varepsilon^{(\tau),\text{soft}} = k\begin{pmatrix} -a_2\cot\dfrac{kl_2}{a_2} & a_2 e^{il_2\tau}\csc\dfrac{kl_2}{a_2} \\ a_2 e^{-il_2\tau}\csc\dfrac{kl_2}{a_2} & -a_2\cot\dfrac{kl_2}{a_2}\end{pmatrix}.$$

Exercise 5.1. *By considering the leading-order term of the left-hand side of the equation*

$$\det\bigl(M_\varepsilon^{(\tau),\text{stiff}} + M_\varepsilon^{(\tau),\text{soft}}\bigr) = 0$$

show that the limit spectrum as $\varepsilon \to 0$ consists of the values k^2 such that

$$\cos\left(\frac{kl_2}{a_2}\right) - \frac{kl_1}{2a_2}\sin\left(\frac{kl_2}{a_2}\right) = \cos\tau \tag{10}$$

for some $\tau \in [-\pi,\pi)$. By varying τ we obtain (for $l_1/a_2 = l_2/a_2 = 1/2$) the set shown in Fig. 5.

(1) Connected stiff component

The periodic graph considered, its periodicity cell and the result of Gelfand transform is shown in Fig. 6. The boundary space $\mathcal{H}$ pertaining to the graph $\mathbb{G}$ is chosen as $\mathcal{H} = \mathbb{C}^2$. The unimodular list functions w_{V_1} and w_{V_2} are chosen as follows:

$$\{w_{V_1}(e^{(j)})\}_{j=1}^3 = \{1,1,e^{i\tau(l_2+l_3)}\}, \quad \{w_{V_2}(e^{(j)})\}_{j=1}^3 = \{e^{i\tau l_3},1,1\} \tag{11}$$

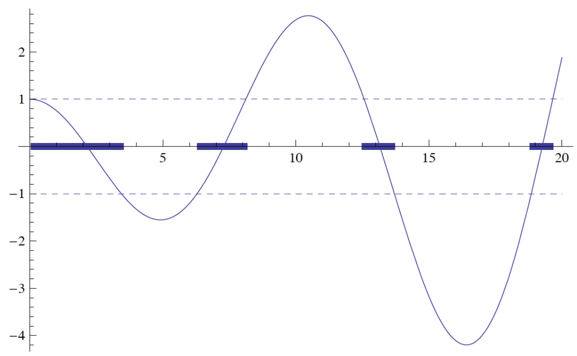

Figure 5: The square root of the limit Bloch spectrum in Example (0). The oscillating solid line is the graph of the function $f(k) = \cos(k/2) - k\sin(k/2)/4$. The square root of the spectrum is the union of the intervals indicated by bold lines, that correspond to $k \in \mathbb{R}^+$ such that $-1 \le f(k) \le 1$ ("pass bands").

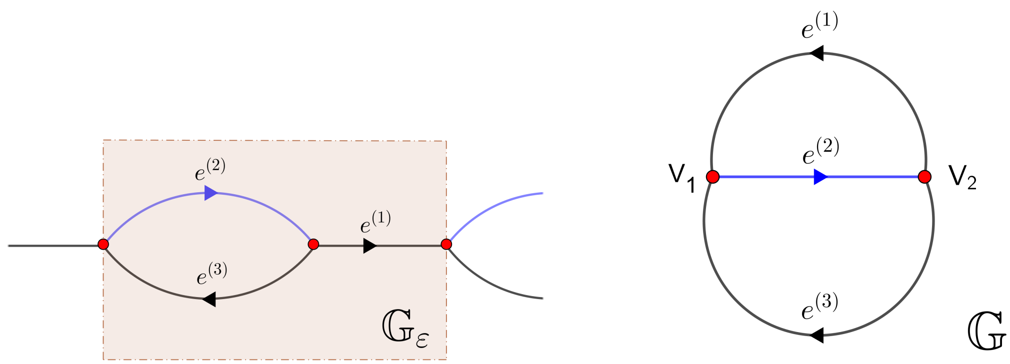

Figure 6: Example (1). $\mathbb{G}_{\rm per}$ with $\mathbb{G}_\varepsilon$ outlined on the left; the graph $\mathbb{G}$ after Gelfand transform on the right. The soft component is drawn in blue.

The M-matrix for the period graph is the sum of

$$M_\varepsilon^{(\tau),\mathrm{stiff}} = \frac{k}{\varepsilon}\begin{pmatrix} -a_1\cot\dfrac{k\varepsilon l_1}{a_1} - a_3\cot\dfrac{k\varepsilon l_3}{a_3} & a_1 e^{-i(l_1+l_3)\tau}\csc\dfrac{k\varepsilon l_1}{a_1} + a_3 e^{il_2\tau}\csc\dfrac{k\varepsilon l_3}{a_3} \\ a_1 e^{i(l_1+l_3)\tau}\csc\dfrac{k\varepsilon l_1}{a_1} + a_3 e^{-il_2\tau}\csc\dfrac{k\varepsilon l_3}{a_3} & -a_1\cot\dfrac{k\varepsilon l_1}{a_1} - a_3\cot\dfrac{k\varepsilon l_3}{a_3} \end{pmatrix},$$

$$M_\varepsilon^{(\tau),\mathrm{soft}} = k\begin{pmatrix} -a_2\cot\dfrac{kl_2}{a_2} & a_2 e^{il_2\tau}\csc\dfrac{kl_2}{a_2} \\ a_2 e^{-il_2\tau}\csc\dfrac{kl_2}{a_2} & -a_2\cot\dfrac{kl_2}{a_2} \end{pmatrix}$$

(2) Connected soft component

The periodic graph considered, its periodicity cell and the result of Gelfand transform is shown in Fig. 7. It represents a “dual” situation to the one of Example (1), exhibiting a globally connected soft component. The boundary space $\mathcal{H}$ pertaining to the graph $\mathbb{G}$ is chosen as $\mathcal{H} = \mathbb{C}^2$. The unimodular list functions w_{V_1}

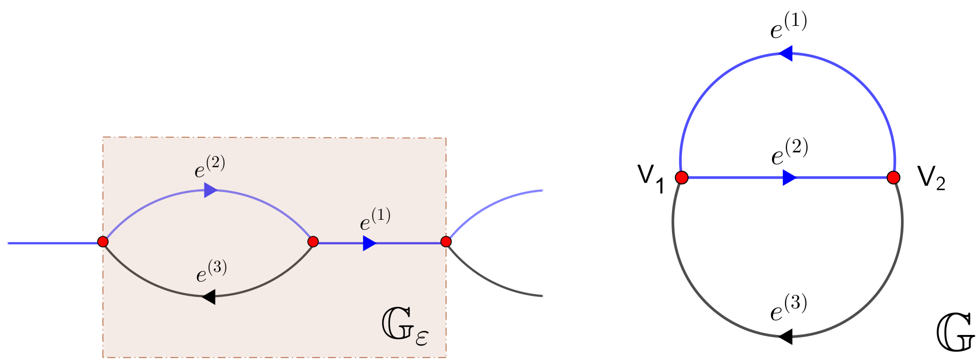

Figure 7: Example (2). $\mathbb{G}_{\text{per}}$ with $\mathbb{G}_\varepsilon$ outlined on the left; the graph $\mathbb{G}$ after Gelfand transform on the right. The soft component is drawn in blue.

and w_{V_2} are chosen as in (11).

The M-matrix for the period graph is the sum of

$$M_\varepsilon^{(\tau),\text{stiff}} = \frac{k}{\varepsilon}\begin{pmatrix} -a_3 \cot \dfrac{k\varepsilon l_3}{a_3} & a_3 e^{il_2\tau} \csc \dfrac{k\varepsilon l_3}{a_3} \\ a_3 e^{-il_2\tau} \csc \dfrac{k\varepsilon l_3}{a_3} & -a_3 \cot \dfrac{k\varepsilon l_3}{a_3} \end{pmatrix},$$

$$M_\varepsilon^{(\tau),\text{soft}} = k\begin{pmatrix} -a_1 \cot \dfrac{kl_1}{a_1} - a_2 \cot \dfrac{kl_2}{a_2} & a_1 e^{-i(l_1+l_3)\tau} \csc \dfrac{kl_1}{a_1} + a_2 e^{il_2\tau} \csc \dfrac{kl_2}{a_2} \\ a_1 e^{i(l_1+l_3)\tau} \csc \dfrac{kl_1}{a_1} + a_2 e^{-il_2\tau} \csc \dfrac{kl_2}{a_2} & -a_1 \cot \dfrac{kl_1}{a_1} - a_2 \cot \dfrac{kl_2}{a_2} \end{pmatrix},$$

6 “Homogenised” operators

(0) Both components disconnected

Assume (without loss of generality) that $a_2 = 1$. For all $\tau \in [-\pi, \pi)$, consider an operator $\mathcal{A}_{\text{hom}}^{(\tau)}$ on $L^2(0, l_2) \oplus \mathbb{C}$ defined as follows. The domain $\operatorname{dom} \mathcal{A}_{\text{hom}}^{(\tau)}$ is

$$\operatorname{dom} \mathcal{A}_{\text{hom}}^{(\tau)} = \Big\{(u, \beta)^\top : \ u \in W^{2,2}(0, l_2), \ u(0) = \overline{\xi}^{(\tau)} u(l_2) = \beta/\sqrt{l_1}\Big\}.$$

On $\operatorname{dom} \mathcal{A}_{\text{hom}}^{(\tau)}$ the action of the operator is set by

$$\mathcal{A}_{\text{hom}}^{(\tau)} \begin{pmatrix} u \\ \beta \end{pmatrix} = \begin{pmatrix} \left(\dfrac{1}{\mathrm{i}} \dfrac{d}{dx} + \tau\right)^2 \\ -\dfrac{1}{\sqrt{l_1}} \left(\partial^{(\tau)} u\big|_0 - \overline{\xi}^{(\tau)} \partial^{(\tau)} u\big|_{l_2}\right) \end{pmatrix},$$

where

$$\xi^{(\tau)} := \exp(\mathrm{i} l_1 \tau), \qquad \partial^{(\tau)} u := \left(\frac{d}{dx} + \mathrm{i}\tau\right) u.$$

Theorem 6.1. *The resolvent $(A_\varepsilon^{(\tau)}-z)^{-1}$ admits the following estimate in the uniform operator norm topology:*

$$(A_\varepsilon^{(\tau)}-z)^{-1}-\Psi^*(\mathcal{A}_{\rm hom}^{(\tau)}-z)^{-1}\Psi=O(\varepsilon^2),$$

where Ψ is a partial isometry from H to $H_{\rm hom}$. This estimate is uniform in $\tau\in[-\pi,\pi)$.

(1) Connected stiff component

Assume (without loss of generality) that $a_2=1$. For all $\tau\in[-\pi,\pi)$, consider an operator $\mathcal{A}_{\rm hom}^{(\tau)}$ on $L^2(0,l_2)\oplus\mathbb{C}$, defined as follows. The domain $\operatorname{dom}\mathcal{A}_{\rm hom}^{(\tau)}$ is

$$\operatorname{dom}\mathcal{A}_{\rm hom}^{(\tau)}=\biggl\{(u,\beta)^\top\in H_{\rm hom}:\ u\in W^{2,2}(0,l_2),u|_0=-\frac{\overline{\xi}^{(\tau)}}{|\xi^{(\tau)}|}u|_{l_2}=\frac{\beta}{\sqrt{l_1+l_3}}\biggr\}.$$

On $\operatorname{dom}\mathcal{A}_{\rm hom}^{(\tau)}$ the action of the operator is set by

$$\mathcal{A}_{\rm hom}^{(\tau)}\begin{pmatrix}u\\ \beta\end{pmatrix}=\begin{pmatrix}\biggl(\dfrac{1}{\mathrm{i}}\dfrac{d}{dx}+\tau\biggr)^2\\ -\dfrac{1}{\sqrt{l_1+l_3}}\biggl(\partial^{(\tau)}u\Big|_0+\dfrac{\overline{\xi}^{(\tau)}}{|\xi^{(\tau)}|}u\Big|_{l_2}\biggr)+\bigl(l_1+l_3\bigr)^{-1}\biggl(\dfrac{l_1}{a_1^2}+\dfrac{l_3}{a_3^2}\biggr)^{-1}\biggl(\dfrac{\tau}{\varepsilon}\biggr)^2\beta\end{pmatrix},$$

where

$$\xi^{(\tau)}=-\frac{a_1^2}{l_1}\exp\bigl(\mathrm{i}\tau(l_1+l_3)\bigr)-\frac{a_3^2}{l_3}\exp(-\mathrm{i}\tau l_2),\qquad \partial^{(\tau)}u:=\biggl(\frac{d}{dx}+i\tau\biggr)u.$$

Theorem 6.2. *The resolvent $(A_\varepsilon^{(\tau)}-z)^{-1}$ admits the following estimate in the uniform operator norm topology:*

$$(A_\varepsilon^{(\tau)}-z)^{-1}-\Psi^*(\mathcal{A}_{\rm hom}^{(\tau)}-z)^{-1}\Psi=O(\varepsilon^2),$$

where Ψ is a partial isometry from H to $H_{\rm hom}$. This estimate is uniform in $\tau\in[-\pi,\pi)$.

(2) Connected soft component

For all $\tau\in[-\pi,\pi)$, consider an operator $\mathcal{A}_{\rm hom}^{(\tau)}$ on $L^2(0,l_2)\oplus\mathbb{C}$, defined as follows. The domain of $\mathcal{A}_{\rm hom}^{(\tau)}$ is

$$\operatorname{dom}\mathcal{A}_{\rm hom}^{(\tau)}=\biggl\{(u_1,u_2,\beta)^\top\in L^2[0,l_1]\oplus L^2(0,l_2)\oplus\mathbb{C}^1:\ u_j\in W^{2,2}(0,l^{(j)}),j=1,2;\quad u_2\big|_0=\overline{\xi_2^{(\tau)}}u_2\big|_{l_2}=\overline{\xi_1^{(\tau)}}u_1\big|_0=u_1\big|_{l_1}=\frac{\beta}{\sqrt{l_3}}\biggr\}.$$

On $\operatorname{dom}\mathcal{A}_{\rm hom}^{(\tau)}$ the action of the operator is set by

$$\mathcal{A}_{\rm hom}^{(\tau)}\begin{pmatrix}u_1\\ u_2\\ \beta\end{pmatrix}=\begin{pmatrix}a_1^2\biggl(\dfrac{1}{\mathrm{i}}\dfrac{d}{dx}+\tau\biggr)^2\\ a_2^2\biggl(\dfrac{1}{\mathrm{i}}\dfrac{d}{dx}+\tau\biggr)^2\\ -\dfrac{1}{\sqrt{l_3}}\Bigl(a_2^2\partial^{(\tau)}u_2\big|_0-a_2^2\overline{\xi_2^{(\tau)}}\partial^{(\tau)}u_2\big|_{l_2}+a_1^2\overline{\xi_1^{(\tau)}}\partial^{(\tau)}u_1\big|_0-a_1^2\partial^{(\tau)}u_1\big|_{l_1}\Bigr)\end{pmatrix},$$

where

$$\xi_2^{(\tau)}=\exp(-\mathrm{i}l_2\tau),\qquad \xi_1^{(\tau)}=\exp\bigl(-\mathrm{i}(l_2+l_3)\tau\bigr),\qquad \partial^{(\tau)}u_j:=\biggl(\frac{d}{dx}+i\tau\biggr)u_j,\quad j=1,2.$$

Theorem 6.3. *The resolvent* $(A_\varepsilon^{(\tau)} - z)^{-1}$ *admits the following estimate in the uniform operator norm topology:*

$$(A_\varepsilon^{(\tau)} - z)^{-1} - \Psi^*(\mathcal{A}_{\rm hom}^{(\tau)} - z)^{-1}\Psi = O(\varepsilon^2),$$

where Ψ *is a partial isometry from* H *to* $H_{\rm hom}$*. This estimate is uniform in* $\tau \in [-\pi, \pi)$*.*

7 Reduction to the stiff component

In this section we continue the study of the three examples, for which in Section 6 we constructed the resolvent asymptotics, in view to obtain equivalent time-dispersive formulations on the real line. In order to achieve this, we first introduce the orthogonal projection $\mathfrak{P}$ of $H_{\rm hom}$ onto $H_{\rm hom} \ominus H_{\rm soft}$, the latter space being $\mathbb{C}^1$ in all three cases. Following this, we determine the corresponding Schur-Frobenius complement $\mathfrak{P}(\mathcal{A}_{\rm hom}^{(\tau)} - z)^{-1}\mathfrak{P}$, see [14, p. 416].

7.1 Examples (0) and (1)

Due to the fact that the soft component in each of these examples consists of only one edge, we shall consider Examples (0) and (1) of Section 6 simultaneously. To this end, we set

$$\Gamma_\tau \begin{pmatrix} u \\ \beta \end{pmatrix} = -\partial^{(\tau)} u\big|_0 + \overline{w_\tau}\partial^{(\tau)} u\big|_{l_2} + \left(\frac{\sigma\tau}{\varepsilon}\right)^2 \frac{\beta}{\rho},$$

where w_τ, σ and ρ depend on the particular case, *cf.* Theorems 6.1, 6.2. The problem of calculating $\mathfrak{P}(\mathcal{A}_{\rm hom}^{(\tau)} - z)^{-1}\mathfrak{P}$ consists in determining β that solves

$$-\left(\frac{d}{dx} + {\rm i}\tau\right)^2 u - zu = 0, \tag{12}$$

$$\begin{pmatrix} u \\ \beta \end{pmatrix} \in {\rm dom}\, \mathcal{A}_{\rm hom}^{(\tau)}, \quad \frac{1}{\rho}\Gamma_\tau \begin{pmatrix} u \\ \beta \end{pmatrix} - z\beta = \delta. \tag{13}$$

In order to exclude u from (12)–(13), we represent it as a sum of two functions: one of them is a solution to the related inhomogeneous Dirichlet problem, while the other takes care of the boundary condition. More precisely, consider the solution v to the problem

$$-\left(\frac{d}{dx} + {\rm i}\tau\right)^2 v = 0, \qquad v(0) = 1, \qquad v(l_2) = w_\tau,$$

i.e.

$$v(x) = \Big\{1 + (l_2)^{-1}\Big(w_\tau \exp({\rm i}\tau l_2) - 1\Big)x\Big\}\exp(-{\rm i}\tau x), \qquad x \in (0, l_2). \tag{14}$$

The function

$$\widetilde{u} := u - \frac{\beta}{\rho}v$$

satisfies

$$-\left(\frac{d}{dx} + {\rm i}\tau\right)^2 \widetilde{u} - z\widetilde{u} = \frac{z\beta}{\rho}v, \qquad \widetilde{u}(0) = \widetilde{u}(l_2) = 0.$$

Equivalently, one has

$$\widetilde{u} = \frac{z\beta}{\rho}(A_{\rm D} - zI)^{-1}v,$$

where $A_{\rm D}$ is the Dirichlet operator in $L^2(0, l_2)$ associated with the differential expression

$$-\left(\frac{d}{dx} + {\rm i}\tau\right)^2.$$

We now write the "boundary" part of the system (12)–(13) as

$$K(\tau,z)\beta - z\beta = \delta, \tag{15}$$

where

$$K(\tau,z) := \frac{1}{\rho^2}\left\{ z\Gamma_\tau \begin{pmatrix} (A_{\mathrm{D}} - zI)^{-1}v \\ 0 \end{pmatrix} + \Gamma_\tau \begin{pmatrix} v \\ \rho \end{pmatrix} \right\}. \tag{16}$$

Thus $\mathfrak{P}(\mathcal{A}^{(\tau)}_{\mathrm{hom}} - z)^{-1}\mathfrak{P}$ is the operator of multiplication in $\mathbb{C}^1$ by $(K(\tau,z) - z)^{-1}$.

The formula (16) shown, in particular, that the dispersion function K is singular only at eigenvalues of the Dirichlet Laplacian on the soft component. It allows to compute K in terms of the spectral decomposition of A_{D}, *cf.* [18]. In order to see this, we represent the action of the resolvent $(A_{\mathrm{D}} - zI)^{-1}$ as a series in terms of the normalised eigenfunctions

$$\varphi_j(x) = \sqrt{\frac{2}{l_2}}\exp(-\mathrm{i}\tau x)\sin\frac{\pi j x}{l_2}, \qquad x \in (0, l_2), \qquad j = 1,2,3,\dots, \tag{17}$$

of the operator A_{D}, which yields

$$K(\tau,z) := \frac{1}{\rho^2}\left\{ z\sum_{j=1}^{\infty}\frac{\langle v, \varphi_j\rangle}{\mu_j - z}\Gamma_\tau \begin{pmatrix} \varphi_j \\ 0 \end{pmatrix} + \Gamma_\tau \begin{pmatrix} v \\ \rho \end{pmatrix} \right\}.$$

where $\mu_j = (\pi j/l_2)^2$, $j = 1,2,3,\dots$, are the corresponding eigenvalues and v is defined in (14). In each case we consider the problem (12)–(13), where operator Γ_τ depends on the specific example at hand.

Exercise 7.1. *Show that in the case of Example (0), the set of values z such that for some $\tau \in [-\pi, \pi)]$ one has $K(\tau, z) = z$ coincides with the set of of k^2 such that fors some τ the equation (10) holds.*

7.2 Example (2)

Here we define

$$\Gamma_\tau \begin{pmatrix} u \\ \beta \end{pmatrix} = -a_1^2\Big(-\partial^{(\tau)}u_1\big|_{l_1} + \overline{\xi_1^{(\tau)}}\partial^{(\tau)}u_1\big|_0\Big) + a_2^2\Big(-\partial^{(\tau)}u_2\big|_0 + \overline{\xi_2^{(\tau)}}\partial^{(\tau)}u_2\big|_{l_2}\Big),$$

$$\xi_1^{(\tau)} := \exp\big(-\mathrm{i}(l_2 + l_3)\tau\big), \qquad \xi_2^{(\tau)} := \exp(-\mathrm{i}l_2\tau),$$

where u_1 and u_2 are the restrictions of the function u to the edges $(0, l_1)$ and $(0, l_2)$, respectively, and the resolvent problem for $A^{(\tau)}_{\mathrm{hom}}$ is given by (*cf.* (12)–(13))

$$-a_1^2\Big(\frac{d}{dx} + \mathrm{i}\tau\Big)^2 u_1 - zu_1 = 0, \tag{18}$$

$$-a_2^2\Big(\frac{d}{dx} + \mathrm{i}\tau\Big)^2 u_2 - zu_2 = 0, \tag{19}$$

$$\begin{pmatrix} u \\ \beta \end{pmatrix} \in \operatorname{dom}\mathcal{A}^{(\tau)}_{\mathrm{hom}}, \quad \frac{1}{\rho}\Gamma_\tau\begin{pmatrix} u \\ \beta \end{pmatrix} - z\beta = \delta, \tag{20}$$

where $\rho = \sqrt{l_3}$. Following the strategy of Section 7.1, we consider the functions v_j, $j = 1,2$ that satisfy appropriate Dirichlet problems:

$$-\Big(\frac{d}{dx} + \mathrm{i}\tau\Big)^2 v_1 = 0, \qquad v_1(0) = \xi_1^{(\tau)}, \quad v_1(l_1) = 1,$$

$$-\left(\frac{d}{dx}+\mathrm{i}\tau\right)^2 v_2=0, \qquad v_1(0)=1, \qquad v_2(l_2)=\xi_2^{(\tau)},$$

i.e.

$$v_1(x)=\xi_1^{(\tau)}\bigl\{1+(l_1)^{-1}\bigl(\exp(\mathrm{i}\tau)-1\bigr)x\bigr\}\exp(-\mathrm{i}\tau x), \quad x\in(0,l_1), \qquad v_2(x)=\exp(-\mathrm{i}\tau x), \quad x\in(0,l_2).$$

As in Section 7.1, we infer that

$$\widetilde{u}=\frac{z\beta}{\rho}\sum_{n=1}^{2}\chi^{(n)}(A_{\mathrm{D}}^{(n)}-zI)^{-1}v_n,$$

where $A_{\mathrm{D}}^{(n)}$, $n=1,2$, are the Dirichlet operators in $L^2(0,l^{(n)})$, $n=1,2$ associated with the differential expression

$$-a_n^2\left(\frac{d}{dx}+\mathrm{i}\tau\right)^2, \quad n=1,2,$$

and $\chi^{(n)}$, $n=1,2$, are the characteristic functions of the edges $(0,l^{(n)})$, $n=1,2$. Therefore we can write the "boundary" part of the resolvent equation (18)–(20) as

$$K(\tau,z)\beta-z\beta=\delta,$$

where

$$\begin{aligned}K(\tau,z)&:=\frac{1}{\rho^2}\sum_{n=1}^{2}\left\{z\Gamma_\tau\begin{pmatrix}\chi^{(n)}(A_{\mathrm{D}}^{(j)}-zI)^{-1}v_n\\ 0\end{pmatrix}+\Gamma_\tau\begin{pmatrix}\chi^{(n)}v_n\\ \rho\end{pmatrix}\right\}\\ &=\frac{1}{\rho^2}\sum_{n=1}^{2}\left\{z\sum_{j=1}^{\infty}\frac{\langle v_n,\varphi_j^{(n)}\rangle}{\mu_j^{(n)}-z}\Gamma_\tau\begin{pmatrix}\chi^{(n)}\varphi_j^{(n)}\\ 0\end{pmatrix}+\Gamma_\tau\begin{pmatrix}\chi^{(n)}v_n\\ \rho\end{pmatrix}\right\}.\end{aligned}$$

Here (*cf.* (17))

$$\varphi_j^{(n)}(x)=\sqrt{\frac{2}{l^{(n)}}}\exp(-\mathrm{i}\tau x)\sin\frac{\pi jx}{l^{(n)}}, \qquad x\in(0,l^{(n)}), \qquad \mu_j^{(n)}=\left(\frac{\pi j}{l^{(n)}}\right), \qquad j=1,2,3,\dots, \qquad n=1,2.$$

7.3 Expressions for the dispersion functions (and answer to Exercise 7.1)

For each $\tau\in[-\pi,\pi)$, the action of $\mathfrak{P}(\mathcal{A}_{\mathrm{hom}}^{(\tau)}-z)^{-1}\mathfrak{P}$ is the multiplication by $(K(\tau,z)-z)^{-1}$, where

$$\text{Example (0)}: \qquad K(\tau,z)=\frac{2\sqrt{z}\bigl(\cos(l_2\sqrt{z})-\cos\tau\bigr)}{l_1\sin(l_2\sqrt{z})}, \tag{21}$$

$$\text{Example (1)}: \qquad K(\tau,z)=\frac{1}{l_1+l_3}\left\{2\sqrt{z}\left(\cot(l_2\sqrt{z})-\frac{\Re\theta(\tau)}{\sin(l_2\sqrt{z})}\right)+\left(\frac{\sigma\tau}{\varepsilon}\right)^2\right\}, \tag{22}$$

$$\text{Example (2)}: \qquad K(\tau,z)=\frac{2\sqrt{z}}{l_3}\left\{a_1^2\frac{\cos(l_1\sqrt{z})-\cos\tau}{\sin(l_1\sqrt{z})}-a_2^2\tan\left(\frac{l_2\sqrt{z}}{2}\right)\right\}, \tag{23}$$

$$\theta(\tau):=\left|\frac{a_1^2}{l_1}\mathrm{e}^{-i\tau}+\frac{a_3^2}{l_3}\right|^{-1}\left(\frac{a_1^2}{l_1}\mathrm{e}^{-i\tau}+\frac{a_3^2}{l_3}\right), \qquad \sigma^2:=\left(\frac{l_1}{a_1^2}+\frac{l_3}{a_3^2}\right)^{-1}.$$

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A Effective macroscopic problems on the real line

Here we shall interpret the Schur-Frobenius complements constructed in the previous section as a result of applying the Gelfand transform (see Section 3) to a one-dimensional homogeneous medium. To this end, we unitarily immerse the L^2 space of functions of t into the L^2 space of functions of t and x, corresponding to the stiff component of the original medium, by the formula

$$\beta(t) \mapsto \beta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x),$$

where L is the length of the stiff component, *i.e.* $L = l_1$ in Examples (0), $L = l_1 + l_3$ in Example (1), $L = l_3$ in Example (2), and write the effective problem (15) in the form

$$K(\varepsilon t, z)\beta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x) - z\beta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x) = \delta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x), \qquad t \in [-\pi/\varepsilon, \pi/\varepsilon), \quad x \in (0, \varepsilon L), \tag{24}$$

The solution operator for (24), namely

$$\delta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x) \mapsto \beta(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x) \ \text{ such that (24) holds,}$$

is the composition of a projection operator in $L^2\big((-\pi/\varepsilon, \pi/\varepsilon)\times(0, \varepsilon L)\big)$ onto constants in x and multiplication by the function $\big(K(\varepsilon t, z) - z\big)^{-1}$, as follows:

$$\big(K(\varepsilon t, z) - z\big)^{-1}\Big\langle \cdot, \frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x)\Big\rangle\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x), \tag{25}$$

for all z such that $K(\varepsilon t, z) - z$ is invertible. The sought representation on $\mathbb{R}$ is the Schur-Frobenius complement obtained by sandwiching the operator (25) with the Gelfand transform

$$\mathcal{G}F(x,t) = \sqrt{\frac{\varepsilon}{2\pi}}\sum_{n\in\mathbb{Z}} F(x + n\varepsilon)\exp\big(-\mathrm{i}(x + n\varepsilon)t\big), \qquad x \in (0, \varepsilon), \quad t \in [-\pi/\varepsilon, \pi/\varepsilon), \qquad F \in L^2(\mathbb{R}),$$

and its inverse

$$\mathcal{G}^*u(x) = \sqrt{\frac{\varepsilon}{2\pi}}\int_{-\pi/\varepsilon}^{\pi/\varepsilon} u(x,t)\exp(\mathrm{i}xt)dt, \quad x \in \mathbb{R}, \qquad u \in L^2\big((-\pi/\varepsilon, \pi/\varepsilon)\times(0, \varepsilon)\big),$$

so that the overall operator is given by

$$\mathcal{G}^*\bigg\{\big(K(\varepsilon t, z) - z\big)^{-1}\Big\langle \mathcal{G}\,\cdot, \frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x)\Big\rangle\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x)\bigg\}.$$

In constructing the above operator we assume that the operator given by (25) has been extended by zero to the soft component of the medium.

This results in the mapping

$$\begin{aligned} F \mapsto \Psi_K^\varepsilon F :=& \sqrt{\frac{\varepsilon}{2\pi}}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\big(K(\varepsilon t, z) - z\big)^{-1}\Big\langle \mathcal{G}F, \frac{1}{\sqrt{\varepsilon L}}\mathbb{1}\Big\rangle(t)\frac{1}{\sqrt{\varepsilon L}}\mathbb{1}(x)\exp(\mathrm{i}tx)dt \\ =& \frac{1}{L\sqrt{2\pi}}\int_{-\pi/\varepsilon}^{\pi/\varepsilon}\big(K(\varepsilon t, z) - z\big)^{-1}\widehat{F}(t)\exp(\mathrm{i}tx)dt, \end{aligned} \tag{26}$$

whose inverse yields the required effective problem on $\mathbb{R}$. Here

$$\widehat{F}(t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} F(x)\exp(-\mathrm{i}xt)dx, \qquad t \in \mathbb{R},$$

is the Fourier transform of the function F.

By applying Theorems 6.1, 6.2, 6.3, we arrive at the following statement.

Theorem A.1. *The direct integral of Schur-Frobenius complements*

$$\oplus \int_{\pi/\varepsilon}^{\pi/\varepsilon} P_{\text{stiff}}(A_\varepsilon^{(t)} - z)^{-1} P_{\text{stiff}} dt \tag{27}$$

is $O(\varepsilon^r)$-close, in the uniform operator-norm topology, to an operator unitary equivalent to the pseudo-differential operator defined by (26). Here $r = 1$ in Example (1) and $r = 2$ in Examples (0) and (2).

The direct integral (27) is the composition of the original resolvent family $(A_\varepsilon - z)^{-1}$ applied to functions supported by the stiff component of $\mathbb{G}_{\text{per}}$ and the orthogonal projection onto the same stiff component

On the basis of the above theorem, we will now explicitly characterise the effective time-dispersive medium in each of the examples.

A.1 Example (0)

Notice that, by (26), for $U := \Psi_K^\varepsilon F$ one has

$$\frac{1}{2}\Big(U(x+\varepsilon) + U(x-\varepsilon)\Big) = \frac{1}{l_1\sqrt{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \frac{\cos(\varepsilon t)}{K(\varepsilon t, z) - z} \widehat{F}(t) \exp(\mathrm{i}tx) dt, \tag{28}$$

and since in Example (0) we have, see (21),

$$K(\tau, z) = \frac{2\sqrt{z}}{l_1} \cot(l_2\sqrt{z}) - \frac{2\sqrt{z}}{l_1 \sin(l_2\sqrt{z})} \cos\tau,$$

we obtain

$$\frac{2\sqrt{z}}{l_1} \cot(l_2\sqrt{z}) U(x) - \frac{1}{2}\Big(U(x+\varepsilon) + U(x-\varepsilon)\Big) \frac{2\sqrt{z}}{l_1 \sin(l_2\sqrt{z})} - zU(x)$$
$$= \frac{1}{l_1\sqrt{2\pi}} \int_{-\pi/\varepsilon}^{\pi/\varepsilon} \widehat{F}(t) \exp(\mathrm{i}tx) dt \sim \frac{1}{l_1} F(x), \quad \varepsilon \to 0.$$

It follows that the asymptotic form of the equation on the function U is

$$-\frac{\sqrt{z}}{\sin(l_2\sqrt{z})} \Delta_\varepsilon U - \left\{ l_1 z + 2\sqrt{z} \tan\left(\frac{l_2\sqrt{z}}{2}\right) \right\} U = F,$$

where

$$\Delta_\varepsilon U := U(\cdot + \varepsilon) + U(\cdot - \varepsilon) - 2U, \qquad \varepsilon > 0. \tag{29}$$

is the difference Laplace operator. Clearly, by a unitary rescaling of the independent variable we obtain an ε-independent limit problem.

A.2 Example (1)

Lemma A.2. *One has the estimate*

$$\|\Psi_K^\varepsilon - \Psi_K^0\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = O(\varepsilon^2), \quad \varepsilon \to 0,$$

where (cf. 26)

$$\Psi_K^0 := \frac{1}{(l_1 + l_3)\sqrt{2\pi}} \int_{-\infty}^{\infty} \big(K(\varepsilon t, z) - z\big)^{-1} \widehat{F}(t) \exp(\mathrm{i}tx) dt,$$

with $K(\tau, z)$ defined by the formula (22) for all values of τ.

Proof. The proof is standard, see *e.g.* [8]. $\square$

It follows from $\Re\theta(\tau) = \Re\theta(0) + O(\tau^2) = 1 + O(\tau^2)$ that

$$K(\varepsilon t, z) = \widetilde{K}(t, z) + O(\varepsilon^2 t^2), \qquad \widetilde{K}(t, z) := \frac{1}{l_1 + l_3}\left\{(\sigma t)^2 - 2\sqrt{z}\tan\left(\frac{l_2\sqrt{z}}{2}\right)\right\}, \qquad t \in [-\pi/\varepsilon, \pi/\varepsilon),$$

from which we infer

$$\big(K(\varepsilon t, z) - z\big)^{-1} = \big(\widetilde{K}(t, z) - z\big)^{-1} + O(\varepsilon^2),$$

and hence we obtain the following statement.

Lemma A.3. *The following estimate holds:*

$$\|\Psi^0_K - \Psi^0_{\widetilde{K}}\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} = O(\varepsilon^2), \quad \varepsilon \to 0.$$

Therefore, for $U := \Psi^0_{\widetilde{K}} F$, we obtain

$$-\sigma^2\, U''(x) - \left\{(l_1 + l_3)z + 2\sqrt{z}\tan\left(\frac{l_2\sqrt{z}}{2}\right)\right\}U(x) = F(x), \qquad x \in \mathbb{R}. \tag{30}$$

Theorem A.4. *In the case of Example (1) the effective time-dispersive formulation on the real line is provided by the formula (30) with an error bound of order $O(\varepsilon^2)$.*

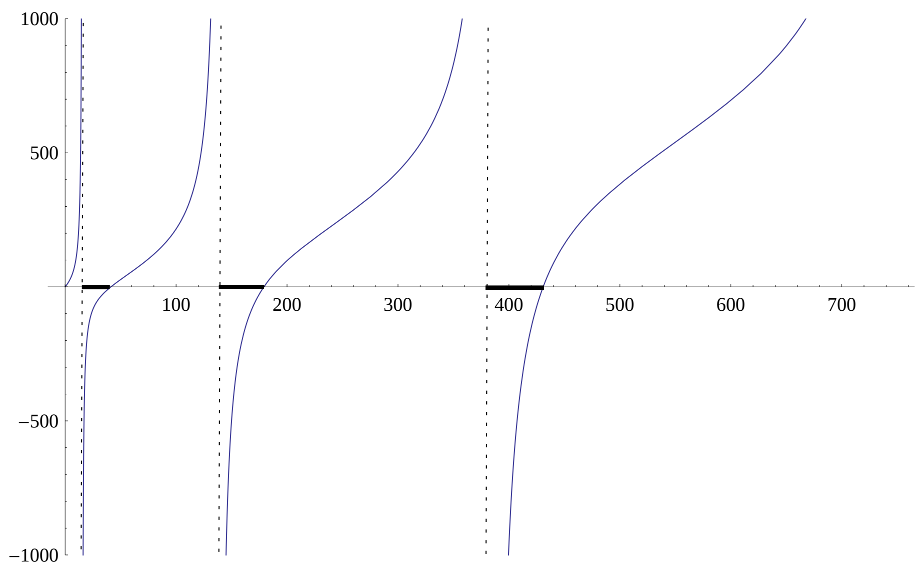

Figure 8: Dispersion function. The plot of the dispersion function on the right-hand side of (30), for $l_1 + l_3 = 1 - l_2 = 0.2$, *cf.* the analogous plot in [18]. The spectral gaps are highlighted in bold.

Remark A.5. *1. Note that in Examples (0) and (2) the effective time-dispersive formulation is given by a difference equation, whereas in Example (1) — by a differential one. The reason for this is the global connectedness of the stiff component (cf. [18]) in Example (1), which leads, see (23) to a nonuniform in ε dependence of the kernel $K(\tau, z)$ on τ.*

2. We point out that even in the case of globally connected stiff component one could end up in a situation where Theorem A.4 yields a rate of convergence lower than $O(\varepsilon^2)$, see [3] for further details. In this case, a "corrected" result can still be obtained, with the rate of convergence $O(\varepsilon^2)$, by replacing (30) *by a differential equation with non-local perturbation.*

A.3 Example (2)

By analogy with Example (0), we use the formula (28) and note that, in view of (23), we have

$$K(\tau, z) = \frac{2\sqrt{z}}{l_3}\left\{a_1^2\tan(l_1\sqrt{z}) - a_2^2\tan\left(\frac{l_2\sqrt{z}}{2}\right)\right\} - \frac{2a_1^2\sqrt{z}}{l_3\sin(l_1\sqrt{z})}\cos\tau.$$

It follows that the time-dispersive effective formulation on $U := \Psi_K^\varepsilon F$ has the form

$$-\frac{a_1^2\sqrt{z}}{\sin(l_1\sqrt{z})}\Delta_\varepsilon U - \left\{l_3 z + 2\sqrt{z}\left(a_1^2\tan\left(\frac{l_1\sqrt{z}}{2}\right) + a_2^2\tan\left(\frac{l_2\sqrt{z}}{2}\right)\right)\right\}U = F,$$

where the difference Laplacian Δ_ε is defined by (29).

Remark A.6. *1. A version of Theorem A.4 seems to be impossible to obtain in Examples (0) and (2), due to the fact that there is no suitable counterpart of Lemma A.2 available. In these cases the convergence to the described effective medium holds, albeit without explicit control of the order of the remainder term.*

2. The effective formulation (30) is precisely the one yielded by the approach of [18]. We note that in the cited paper the stated result involves only two-scale convergence with no estimate on the error term. In contrast, our approach provides norm-resolvent convergence, with an order-explicit error estimate.