PhD projects offered by Dr Kirill Cherednichenko
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Subject areas: Mathematics, Applied Mathematics, Materials Science (within Physical Sciences)

I. Project title: Scattering theory for periodic graphs and related problems for metamaterials

This project is offering an opportunity to make an exciting mathematical contribution in a subject that is a focal point of physics and materials research at the moment. It concerns understanding how properties of periodic media at the micro-level result in a specific kind of the macroscopic behaviour, and is stemming from a set of mathematical approaches and techniques that are often referred to as "mathematical theory of homogenisation''. The mathematical challenge here is to understand, by means of an asymptotic analysis, how solutions to parametrised families of partial differential equations from a special class (equations with periodic rapidly oscillating coefficients) behave as the parameter in the equation (the period of oscillations) goes to zero.

The subject of homogenisation is about 40 years old, but it continues to offer interesting problems, which require new analytical tools. One of these is related to a recent discovery of the role of high-contrast composite materials in such physical phenomena as super-resolution and cloaking (or "invisibility"): it turns out that high-contrast periodic media is a convenient prototype of a "metamaterial", although the precise mathematical explanation of this relationship is still outstanding. It may be said that mathematical analysis should help to ask the right kind of questions about this new kind of materials, in particular what properties one should expect from them and how they can be manufactured (and, ultimately, define what a metamaterial is). The "classical" theory of homogenisation is unable to address these questions in an efficient way, due to its being limited to non-contrasting composite media.

A similar set of observations can be made about periodic media that have "thin" structure, i.e. a structure with dimensions that are small even on the microscale. It has recently been shown mathematically that high-contrast media and thin-structure media respond in similar ways to external electromagnetic fields, which by itself is an example of the power of mathematical analysis in solving problems of modern physics research.

In addition to linking analysis of families of parametrised PDEs to physics and materials science, the project will make an advance on the interface between two areas of mathematical analysis, namely scattering theory and homogenisation. It is an exciting recent discovery that sophisticated tools from operator theory can be used to study problems of homogenisation theory that have been out of reach of earlier approaches. On the order hand, scattering theory in its own right is set to benefit from the analysis of problems in materials science as opposed to questions of quantum mechanics (described by the Schrödinger equation).

The project would be particularly suitable for someone who would like to know more about analytical techniques for partial differential equations, related integral functionals, real and functional analysis including measure theory, but also to benefit from seeing the results of their research being applied to exciting problems of physics, materials science and engineering.

II. Project title: Parameter-dependent multi-scale methods and lattice sums for partially degenerate families of PDE

The project revolves around numerical implementation the latest developments in the analysis families of degenerate second-order partial differential equations (PDE) with rapidly oscillating coefficients, e.g. those with high contrast between the values of the coefficients in the different parts of the periodicity cell. Recent work on operator-norm asymptotics for the resolvents of the associated operators (Cherednichenko, Cooper), has indicated that in order to capture the behaviour of the spectra of such problems one may have to consider more advanced versions of the existing two-scale approaches by including a dependence on the Floquet-Bloch quasimomentum or its appropriate modification.

A related set of questions concerns the behaviour of the numerical schemes in the study of photonics crystals and "arrow fibres". Here the method of lattice sums used in the moderate-contrast case needs to be evaluated and appropriately modified in order to achieve convergence rates of the order of the period of the composite.

Literature:

1. Bensoussan, A., Lions, J.-L., and Papanicolaou, G. C., 1978. *Asymptotic Analysis for Periodic Structures,* North-Holland.

2. Hoang, V. H., Schwab, C., 2004. High-dimensional finite elements for elliptic problems with multiple scales. *Multiscale Model. Simul*. **3**(1), 168--194.

3. Kamotski, I. V., Smyshlyaev, V. P. , 2013. Two-scale homogenization for a class of partially degenerating PDE systems, *Preprint arXiv:1309.4579*

4. Cherednichenko, K, Cooper, S., 2015. Resolvent estimates for high-contrast homogenisation problems. *Archive for Rational Mechanics and Analysis;* DOI: 10.1007/s00205-015-0916-4

5. McPhedran, R. C., Nicorovici, N. A., Botten, L. C., 1997. The TEM mode and homogenization of doubly periodic structures, *Journal of Electromagnetic Waves and Applications*, **11**(7), 981--1012; DOI: 10.1163/156939397X00378

6. Bakhvalov, N. S., Panasenko, G. P. 1984. *Homogenisation: Averaging of Processes in Periodic Media,* Kluwer Academic Publishers.

7. Jikov, V. V., Kozlov, S. M., Oleinik, O. A., 1994. *Homogenization of differential operators and integral functionals*, Springer.

III. Project title: New examples of multiscale effective models for the system of Maxwell equations in periodic composite media

The work on the project will consist of three parts, united by the common theme of analysing the effect of resonant properties of Maxwell equations of electromagnetism in periodic media on their effective behaviour:

A. Mathematical analysis of the homogenised behaviour of:

1) Thin periodic structures (frameworks) filled by a perfect conductor, for various scalings between their thickness and period, following the methodologes developed in earlier literature (Refs. 1,2 in the list below in the context of elasticity; Ref. 3 in the case of a scalar equation).

2) Periodic composites consisting of a thin optically dense framework with voids (filled by vacuum or a material with similar optical properties).

B. Analysis of the spectral properties (wave propagation versus wave localisation) for thin structures in the context of Maxwell equations, using the convergence statements proved in Part A.

C. Analysis of a threshold effect near the bottom of the inner gap in the spectrum (see Refs. 4,5 for the scalar case) for the periodic Maxwell system. This will include the derivation of the corresponding system of effective equations, proving the resolvent convergence estimates, and analysing the effect of the microgeometry of the composite on its threshold properties.

<u>*Literature:*</u>

1. Zhikov, V. V., 2002. Averaging of problems in the theory of elasticity on singular structures. *Izv. Math.* **66**(2), 299--365

2. Zhikov, V. V., Pastukhova, S. E., 2003. Averaging of problems in the theory of elasticity on periodic grids of critical thickness. *Sb. Math* **194**(5-6), 697--732.

3. Cherednichenko, K. D., Cooper, S., 2015. Resolvent estimates for high-contrast homogenisation problems. *Archive for Rational Mechanics and Analysis*, DOI: 10.1007/s00205-015-0916-4.

4. Birman, M. Sh., 2004 On the averaging procedure for periodic operators in a neighborhood of an edge of an internal gap. *St. Petersburg Math. J.* **15**(4), 507--513.

5. Birman, M. Sh., Suslina, T. A., 2006. Homogenization of a multidimensional periodic elliptic operator in a neighborhood of an edge of an inner gap. *J. Math. Sci. (N. Y.)* **136** (2), 3682--3690.